Dissolving a Partnership Dynamically*

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July 21, 2016

Abstract

In financial disputes arising from divorce, inheritance, or the dissolution of a partnership, frequently the need arises to assign ownership of an indivisible item to one member of a group. This paper introduces and analyzes a dynamic auction for simply and efficiently allocating an item when participants are privately informed of their values. In the auction, the price rises continuously. A bidder who drops out of the auction, in return for surrendering his claim to the item, obtains compensation equal to the difference between the price at which he drops and the preceding drop price. When only one bidder remains, that bidder wins the item and pays the compensations of his rivals. We characterize the unique equilibrium with risk-neutral and CARA risk averse bidders. We show that dropout prices are decreasing as bidders become more risk averse. Each bidder's equilibrium payoff is at least $1/N$-th of his value for the item.

*We are grateful for comments from an anonymous referee and seminar particpants at the Australian National University, National University of Singapore, NYU Abu Dhabi, University of Alabama, University of Melbourne, University of New South Wales, University of Texas Dallas, University of Washington, Vanderbilt University, and the 2015 meetings of the SAET. Wooders is grateful for financial support from the Australian Research Council's *Discovery Projects* funding scheme (project number DP140103566).

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1 Introduction

In financial disputes arising from divorce, inheritance, or the dissolution of a partnership, frequently the need arises to assign ownership of an indivisible item to one member of a group. This paper introduces and analyzes a dynamic auction for simply and efficiently resolving such disputes.

The canonical example of a division mechanism is divide and choose. In addition to helping children split pieces of cake, this procedure is widely used in a variety of other practical settings. A version of divide and choose called a "Texas Shoot-Out" is a commonly used exit mechanism found in two-person equal-share partnership contracts.[1] In this mechanism, the owner who wants to dissolve the partnership names a price and the other owner is compelled to either purchase his partner's share or sell his own share at the named price.

Divide and choose is simple and fair. Parents (lawyers) can explain the procedure to their children (clients) without difficulty. Moreover, whether a participant is the divider or the chooser, they can guarantee themselves at least half of their value for the object by following a simple strategy. In a Texas Shoot-Out, for example, an owner who names a price that leaves him indifferent to whether his partner buys or sells is guaranteed to receive half of his value for the partnership. Likewise, his partner, by simply taking the best deal, either selling or buying at the proposed price, cannot leave with less than fifty percent of her value for the partnership.

Despite these properties, divide and choose has several flaws which limit its applicability and attractiveness. First, the procedure does not easily scale to more than two participants. Second, it does not treat the participants symmetrically: there is an advantage to being the divider when information is complete and to being the chooser when information is incomplete. Finally, when information is incomplete, then divide and choose is not efficient.[2]

[1] Brooks, Landeo, and Spier (2010) detail the popularity of this exit mechanism and examine why Texas Shoot-Outs are rarely triggered in real-world contracts.

[2] In a complete information environment, these issues have been well studied. Crawford

We present a dynamic auction which avoids the negative features of divide and choose while retaining its many attractive properties. In the auction, the price, starting from zero, rises continuously. Bidders may drop out at any point. A bidder who drops out surrenders his claim to the item and, in return, receives compensation from the (eventual) winner equal to the difference between the price at which he drops and the price at which the prior bidder dropped. The auction ends when exactly one bidder remains. That bidder wins the item and compensates the other bidders. Thus in an auction with N bidders, if $\{p_k\}_{k=1}^{N-1}$ is the sequence of dropout prices, then the compensation of the k-th bidder to drop is $p_k - p_{k-1}$, where $p_0 = 0$, and the winner's total payment is $p_{N-1} = \Sigma_{k=1}^{N-1}(p_k - p_{k-1})$.[3] Hereafter, we refer to this auction as the compensation auction.

In our setting, a strategy for a bidder is a sequence of bid functions, where the k-th bid function identifies the price at which the bidder drops out as a function of his value and the $k-1$ prior dropout prices. In the symmetric independent private values setting we provide necessary and sufficient conditions for a sequence of bid functions to be a symmetric Bayes Nash equilibrium in increasing and differentiable strategies. We characterize the unique such equilibrium when bidders are risk neutral and when they are CARA risk averse; in equilibrium the compensation auction efficiently dissolves partnerships. We show that equilibrium dropout prices are decreasing as bidders become more risk averse. Equilibrium is also interim proportional, i.e., each bidder's equilibrium expected payoff is equal to at least $1/N$-th of

(1979) shows that auctioning off the divider role in divide and choose can correct the asymmetry of the procedure, and Demange (1984) offers a procedure for N players that is fair and efficient. In an incomplete information environment, de Frutos and Kittsteiner (2008) show how bidding to be the chooser can restore efficiency to a Texas Shoot-Out.

[3] The auction can equivalantly be framed as follows: At the beginning of each round, compensation is set zero and then increased continuously until one of the participants agrees to take this compensation in return for giving up his claim to the item. This participant exits, and the process is repeated, until only one participant remains. The last participant is awarded the item and pays each of the others their individualized compensation.

his value for the item, and thus it is individual rational for each bidder to participate in the auction if $1/N$-th of his value is his disagreement payoff.

Related Literature

In an independent private values setting, Cramton, Gibbons, and Klemperer (1987) identify necessary and sufficient conditions for a N-bidder partnership to be efficiently dissolvable when bidders are risk neutral, and they identify a static bidding game that dissolves it. They show that only equal partnerships are dissolvable as the number of bidders grows large. When bidders do have equal ownership shares, they show that partnerships are dissolvable by simple $k+1$ auctions.[4] A mechanism is *simple* in McAfee's (1992) sense if it can be described without reference to the players' utility functions or the distribution of their values. Loertscher and Wasser (2015) characterize the optimal dissolution mechanism for arbitrary initial ownerships, when the objective is to maximize a weighted sum of revenue and social surplus.

To our knowledge, McAfee (1992) is the only paper to study the dissolution of partnerships when the participants are risk averse. It characterizes the equilibrium bid functions of several simple mechanisms when there are $N = 2$ CARA risk averse bidders: the Winner's bid auction, the Loser's bid auction, and the Texas Shootout (which he calls the Cake Cutting Mechanism).[5] Morgan (2004) considers fairness in dissolving a two-person partnership in a

[4] In a $k+1$ auction, bids are simultaneous, the item is transferred to the highest bidder, and he pays each of the other bidders a price equal to

$$\frac{1}{N}\left[kb_s + (1-k)b_f\right],$$

where b_s is the the second highest bid, b_f is the highest bid, and $k \in [0, 1]$. This mechanism is also studied in Guth and van Damme (1986). de Frutos (2000) studies the $k = 0$ and $k = 1$ versions of this auction when bidders' values are drawn from asymmetric distributions. A similar family of auctions is considered by Lengwiler and Wolfstetter (2005).

[5] In the Winner's Bid auction the high bidder wins and pays half his own bid to the loser, while in the Loser's Bid auction he pays half the losing bid to the loser. The Loser's Bid auction is strategically equivalent to the two-player version of our compensation auction.

common value framework. Athanassoglou, Brams, and Sethuraman (2008) consider the problem of dissolving a partnership when the objective of the bidders is to minimize maximum regret.

The present paper is the first to propose and analyze a dynamic procedure for dissolving a partnership with $N > 2$ bidders. Abundant experimental evidence suggests that dynamic mechanisms perform more reliably than static ones, e.g., English ascending bid auctions achieve efficient allocations far more reliably than second-price sealed-bid auctions, despite being strategically equivalent.[6] The prior literature has imposed the restriction that either bidders be risk neutral or there only be two bidders. We dispense with both restrictions.[7]

We address the efficient allocation of an indivisible object. However, the dynamic auction we propose is inspired by the early cake cutting literature which concerned the division of a divisible item.[8] In the classical cake cutting problem, N individuals are interested in dividing a heterogeneous cake. Assume that the cake is rectangular and of unit width, where $t = 0$ and $t = 1$ correspond to the left and right edge, respectively. Dubins and Spanier (1961) describe one solution to this problem: A referee holds a knife at the left edge of the cake (i.e., $t = 0$) and slowly moves it rightward across the cake, keeping it parallel to the left edge. At any time, any of the participants can call out "cut." If the first participant calls cut at t_1 then he takes the piece to the left of the knife, i.e., $[0, t_1)$, and exits. The knife now continues moving rightward until a second participant calls cut at some t_2, and he receives $[t_1, t_2)$ and exits. This continues until the $N-1$-st participant calls cut and takes the piece $[t_{N-2}, t_{N-1}]$. The last participant receives the remainder

[6] See Kagel (1995) for a discussion of several such studies in his well-known survey of auction experiments.

[7] See Moldovanu (2002) for a survey of the literature on dissolving a partnership.

[8] Steinhaus (1948), Dubins and Spanier (1961), and Kuhn (1967) are early examples. See Brams and Taylor (1996) or Robertson and Webb (1998) for a textbook treatment of the subject. Chen, Lai, Parkes, and Procaccia (2013) is a more recent contribution.

$[t_{N-1}, 1]$.

A participant who calls "cut" whenever his value for the piece of cake to the left of the knife is $1/N$-th of his value for whole cake is easily verified to obtain a piece no smaller that $1/N$-th (in his own estimation), independent of when the other participants call cut. If pieces of cake are viewed as compensation, then the Dubins and Spanier procedure is similar to our auction: In each round, compensation (money or cake) is continuously increased until one participant agrees to take the compensation and give up his right to continue. The process continues until a single participant remains, who wins the cake or the item, and who compensates the other participants (with either money or compensatory pieces of the cake). The two procedures are not identical, and we focus on equilibrium behavior rather than fair division.

2 The Model

A single indivisible item is to be allocated to one of $N \geq 2$ bidders. The bidders' values for the item are independently and identically distributed according to cumulative distribution function F with support $[0, \bar{x}]$, where $\bar{x} < \infty$ and $f \equiv F'$ is continuous and positive on $[0, \bar{x}]$. Let $X_1, \ldots, X_N$ be N independent draws from F, and let $Z_1^{(N)}, \ldots, Z_N^{(N)}$ be a rearrangement of the X_i's such that $Z_1^{(N)} \leq Z_2^{(N)} \leq \ldots \leq Z_N^{(N)}$, and let $G_k^{(N)}$ denote the *c.d.f.* of $Z_k^{(N)}$, i.e., $G_k^{(N)}$ is the distribution of the k-th lowest of N draws. It is easy to verify that the conditional density of $Z_{k+1}^{(N)}$ given $Z_1^{(N)} = z_1, \ldots, Z_k^{(N)} = z_k$ is

$$g^{(N)}_{Z_{k+1}^{(N)}|Z_1^{(N)},\ldots,Z_k^{(N)}}(z_{k+1}|z_1, \ldots, z_k) = (N-k) f(z_{k+1}) \frac{[1 - F(z_{k+1})]^{N-(k+1)}}{[1 - F(z_k)]^{N-k}}$$

if $0 \leq z_1 \leq \ldots \leq z_{k+1}$ and is zero otherwise.[9] As the conditional distribution of $Z_{k+1}^{(N)}$ given $Z_1^{(N)}, \ldots, Z_k^{(N)}$ depends only on $Z_k^{(N)}$, we simply denote it by

[9]See Claim 1 of the Supplemental Appendix for the derivation of this density.

$G_{k+1}^{(N)}(z_{k+1}|z_k)$ rather than the more cumbersome $G^{(N)}_{Z^{(N)}_{k+1}|Z^{(N)}_1,\dots,Z^{(N)}_k}(z_{k+1}|Z_1^{(N)} = z_1, \dots, Z_k^{(N)} = z_k)$, and likewise we write $g_{k+1}^{(N)}(z_{k+1}|z_k)$ for the conditional density. Define

$$\lambda_k^N(z) \equiv g_{k+1}^{(N)}(z|z) = (N-k)\frac{f(z)}{1-F(z)},$$

to be the instantaneous probability that one of $N-k$ bidders has a value of z conditional on the k-th lowest value being z.

In the auction, the price starts at 0 and rises continuously until $N-1$ of the bidders drop out. The remaining bidder wins the item. A bidder may drop out at any point as the price ascends, dropping out is irrevocable, and dropout prices are publicly observed. Let $p_0 = 0$ and suppose $p_1 \le p_2 \le \dots \le p_{N-1}$ is the sequence of $N-1$ dropout prices.[10] The winner pays compensation of $p_k - p_{k-1}$ to the k-th bidder to drop, for each $k \in \{1, \dots, N-1\}$. We say that the k-th bidder has dropped at "round" k. Thus if a bidder whose value is x wins the auction, then his total payment is $p_{N-1} = \sum_{k=1}^{N-1}(p_k - p_{k-1})$ and his payoff is $u(x_i - p_{N-1})$. The payoff of the k-th bidder to drop is $u(p_k - p_{k-1})$. We assume that $u' > 0$ and $u'' \le 0$.

A strategy is a list of $N-1$ functions $\beta = (\beta_1, \dots, \beta_{N-1})$, where $\beta_k(x; p_1, \dots, p_{k-1})$ gives the dropout price in the k-th round of a bidder whose value is x, when $k-1$ bidders have previously dropped out at prices $p_1 \le p_2 \le \dots \le p_{k-1}$. Since a strategy must call for a feasible dropout price, we require that $\beta_k(x; p_1, \dots, p_{k-1}) \ge p_{k-1}$ for each k and $p_1, \dots, p_{k-1}$. Sometimes we refer to a bidder's dropout price simply as his bid.

[10] In the event that several bidders drop at the same price, then one randomly selected bidder drops, the rest remain, and the auction resumes.

3 Equilibrium Bidding Strategies

Proposition 1(i) identifies necessary conditions for β to be a symmetric equilibrium in strictly increasing and differentiable strategies. Proposition 1(ii) establishes that any solution to this system of differential equations is an equilibrium. The remainder of this section establishes existence and uniqueness of equilibrium in two important special cases – (i) risk neutral bidders and (ii) bidders with constant absolute risk aversion.

Proposition 1: *(i) Any symmetric equilibrium β, in increasing and differentiable bidding strategies, satisfies the following system of differential equations:*

$$\begin{aligned} & u'(\beta_{N-1}(x;\mathbf{p}_{N-2}) - p_{N-2})\beta'_{N-1}(x;\mathbf{p}_{N-2}) \\ = \; & [u(\beta_{N-1}(x;\mathbf{p}_{N-2}) - p_{N-2}) - u(x - \beta_{N-1}(x;\mathbf{p}_{N-2}))]\lambda^N_{N-1}(x) \end{aligned} \tag{1}$$

and, for $k \in \{1, \ldots, N-2\}$, that

$$\begin{aligned} & u'(\beta_k(x;\mathbf{p}_{k-1}) - p_{k-1})\beta'_k(x;\mathbf{p}_{k-1}) \\ = \; & \begin{bmatrix} u(\beta_k(x;\mathbf{p}_{k-1}) - p_{k-1}) \\ -u(\beta_{k+1}(x;\mathbf{p}_{k-1}, \beta_k(x;\mathbf{p}_{k-1})) - \beta_k(x;\mathbf{p}_{k-1})) \end{bmatrix} \lambda^N_k(x), \end{aligned} \tag{2}$$

where

$$\lambda^N_k(x) = (N-k)\frac{f(x)}{1-F(x)}.$$

(ii) If $\beta = (\beta_1, \ldots, \beta_{N-1})$ is a solution to the system of differential equations in (i), then it is an equilibrium.

Risk Neutral Bidders

Proposition 2 characterizes equilibrium when bidders are risk neutral. It shows that in round k a bidder whose value is x sets a drop price equal to a weighted average of the dropout price observed in round $k-1$ and the expectation of the second highest value conditional on x being between the k-th and the $k-1$-st lowest values.

Proposition 2: *Suppose that bidders are risk neutral. The unique symmetric equilibrium in increasing and differentiable strategies is given, for* $k = 1, \ldots, N-1$, *by*

$$\beta_k^0(x; \mathbf{p}_{k-1}) = \frac{N-k}{N-k+1} p_{k-1} + \frac{1}{N-k+1} E\left[Z_{N-1}^{(N)} | Z_k^{(N)} > x > Z_{k-1}^{(N)} \right]. \quad (3)$$

According to Proposition 2, equilibrium dropout prices in round k are determined by a bidder's value and the round $k-1$ dropout price, but do not depend on dropout prices in rounds prior to $k-1$.

Let N, k, N', and k', be integers such that $N - k = N' - k' \geq 1$, but otherwise be arbitrary. It is straightforward to verify that[11]

$$E[Z_{N'-1}^{(N')} | Z_{k'}^{(N')} > x > Z_{k'-1}^{(N')}] = E[Z_{N-1}^{(N)} | Z_{k-1}^{(N)} > x > Z_{k-1}^{(N)}] \ \forall x \in [0, \bar{x}].$$

In other words, the expectation of the second highest of N' draws, conditional on x being between the k'-th and $k'-1$-st lowest draws, is the same as the expectation of the second highest of N draws, conditional on x being between the k-th and $k-1$-st lowest draws. Corollary 1 follows immediately from Proposition 2. In stating the corollary it is useful to write $\beta_{k,N}^0(x; \mathbf{p}_k)$ for the equilibrium bid function in round k of an auction with N bidders.

Corollary 1: *If* $N' - k' = N - k$ *and bidders are risk neutral, then the equilibrium bid function in round* k' *of an auction with* N' *bidders is the same as the equilibrium bid function in round* k *of an auction with* N *bidders. Equilibrium bids depend on only the number of rounds remaining in the auction and the last observed dropout price. In particular,* $\beta_{k',N'}^0(x; \mathbf{p}_{k'}) = \beta_{k,N}^0(x; \mathbf{p}_k)$ whenever $p_{k'} = p_k$.

[11] See Claims 2 and 3 of the Supplemental Appendix.

Corollary 1 identifies an intuitive property of equilibrium, but it depends on the uniqueness of equilibrium. If there are multiple equilibria, then one might make a selection based on the number of bidders.

Example 1: Suppose $N = 3$, bidders are risk neutral, and values are distributed $U[0,1]$. Equilibrium drop out prices in round 1 are given by

$$\beta_1^0(x) = \frac{1}{6}x + \frac{1}{6},$$

and in round 2 are given by

$$\beta_2^0(x; p_1) = \frac{1}{3}x + \frac{1}{6} + \frac{1}{2}p_1.$$

By Corollary 1, the equilibrium bid function in round 3 of an auction with 4 bidders is $\beta_3^0(x; \mathbf{p}_2) = \frac{1}{3}x + \frac{1}{6} + \frac{1}{2}p_2$.

CARA Bidders

Proposition 3 characterizes equilibrium when bidders have constant absolute risk aversion (CARA), i.e., their utility functions are given by

$$u^\alpha(x) = \frac{1 - e^{-\alpha x}}{\alpha},$$

where $\alpha > 0$ is their index of risk aversion. Note that $\lim_{\alpha \to 0} u^\alpha(x) = x$, i.e., bidders are risk neutral in the limit as α approaches zero. Denote by β_k^α the equilibrium bid function in round k when bidders have CARA index of risk aversion α.

Proposition 3: *Suppose that bidders are CARA risk averse with index of risk aversion $\alpha > 0$. The unique symmetric equilibrium in increasing and differentiable strategies is given, for $k = 1, \ldots, N-1$, by*

$$\beta_k^\alpha(x; \mathbf{p}_{k-1}) = \frac{N-k}{N-k+1} p_{k-1} - \frac{N-k}{(N-k+1)\,\alpha} \ln\left(J_k^\alpha(x)\right), \tag{4}$$

where

$$J^{\alpha}_{N-1}(x) = E[e^{-\alpha Z^{(N)}_{N-1}} | Z^{(N)}_{N-1} > x > Z^{(N)}_{N-2}]$$

and, for $k < N-1$, $J^{\alpha}_k(x)$ *is defined recursively as*

$$J^{\alpha}_k(x) = E\left[\left(J^{\alpha}_{k+1}(Z^{(N)}_k)\right)^{\frac{N-k-1}{N-k}} | Z^{(N)}_k > x > Z^{(N)}_{k-1}\right].$$

Example 2: Suppose $N = 3$, bidders are CARA risk averse with index of risk aversion α, and values are distributed $U[0,1]$. Equilibrium drop out prices in round 1 are given by

$$\beta^{\alpha}_1(x) = -\frac{2}{3\alpha} \ln \left(\frac{\int_x^1 \left(\frac{\int_z^1 e^{-\alpha t} 2(1-t)dt}{(1-z)^2} \right)^{\frac{1}{2}} 3(1-z)^2 dz}{(1-x)^3} \right),$$

and in round 2 are given by

$$\beta^{\alpha}_2(x; p_1) = \frac{1}{2} p_1 - \frac{1}{2\alpha} \ln \left(\frac{\int_x^1 e^{-\alpha z} 2(1-z)dz}{(1-x)^2} \right).$$

Figure 1 (below) shows the equilibrium bid functions for $\alpha = 10$. The round 2 bid function is shown under the assumption that the first bidder drops at a bid of .2, which reveals (in equilibrium) his value is $z_1 = (\beta^{10}_1)^{-1}(.2) \approx .35154$. Since this value is the lower bound of the set of buyer types remaining in the auction, the figure shows $\beta_2^{10}(x; 1/5)$ for $x \geq z_1$.

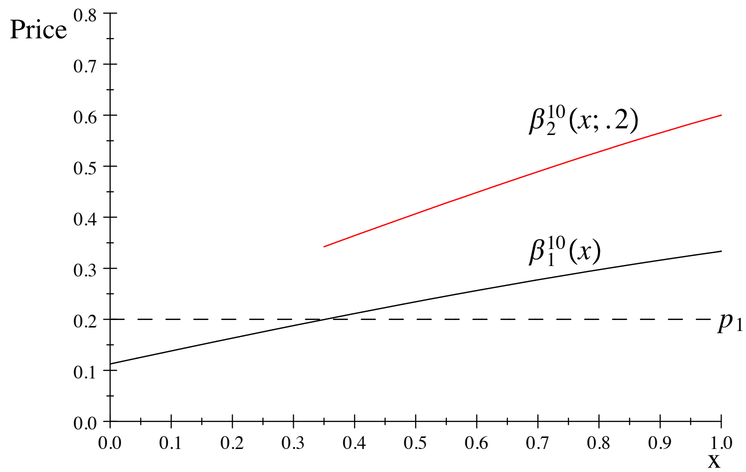

Figure 1: Equilibrium bids by round, for $N = 3, U[0, 1]$, and $CARA$ $(\alpha = 10)$.

Proposition 4 establishes tight upper and lower bounds for the dropout prices of CARA risk averse bidders.

Proposition 4: *Suppose that bidders are CARA risk averse with index of risk aversion $\alpha > 0$. Then for each $k = 1, \ldots, N-1$ and $\mathbf{p}_{k-1}$ we have that*

$$\beta_k^0(x; \mathbf{p}_{k-1}) > \beta_k^\alpha(x; \mathbf{p}_{k-1}) > \frac{x - p_{k-1}}{N - k + 1} + p_{k-1} \text{ for } x < \bar{x},$$

i.e., CARA risk averse bidders demand less compensation than risk neutral bidders, but always demand compensation of at least $(x - p_{k-1})/(N - k + 1)$.

Proposition 5 establishes the intuitive result that CARA bidders drop out at lower prices as they become more risk averse.

Proposition 5: *Suppose that bidders are CARA risk averse with index of risk aversion α. Dropout prices decrease as bidders become more risk averse,* *i.e., $\tilde{\alpha} > \alpha$ implies, for $k = 1, \ldots, N-1$, that*

$$\beta_k^{\alpha}(x; \mathbf{p}_{k-1}) > \beta_k^{\tilde{\alpha}}(x; \mathbf{p}_{k-1}) \ \forall k \in \{1, \ldots, N-1\}, \forall x \in [0, \bar{x}), \forall \mathbf{p}_{k-1},$$

except for bidders with the highest possible value $\bar{x}$, for whom the dropout price does not depend α.

If a bidder with value x_i follows the strategy of dropping out whenever his compensation reaches x_i/N, then he *guarantees* himself a payoff of at least $u(x_i/N)$. In particular, regardless of the strategies and values of the other bidders, he either drops at some stage k and obtains compensation of exactly x_i/N or he wins the item at a price no more than $(N-1)x_i/N$. Since a bidder's equilibrium strategy must give him at least this payoff, the compensation auction is said to be interim proportional.[12]

Corollary 2: *The compensation auction is interim proportional, i.e., the equilibrium payoff of a bidder with value x is at least $u(x/N)$.*

Proposition 6 studies the limit properties of equilibrium bid functions. P6.1 shows that as α approaches zero, the CARA bid function approaches the risk neutral bid function. P6.2 shows that as CARA bidders become infinitely risk averse, equilibrium bids approach the (linear) lower bound identified in Proposition 4.[13]

Proposition 6: *Suppose that bidders are CARA risk averse with index of risk aversion α. Then for each $k = 1, \ldots, N-1$ and $\mathbf{p}_{k-1}$ we have*
P6.1: $\lim_{\alpha \to 0^+} \beta_k^{\alpha}(x; \mathbf{p}_{k-1}) = \beta_k^{0}(x; \mathbf{p}_{k-1})$ *for $x \leq \bar{x}$.*
P6.2: $\lim_{\alpha \to \infty} \beta_k^{\alpha}(x; \mathbf{p}_{k-1}) = \frac{x - p_{k-1}}{N-k+1} + p_{k-1}$ *for $x \leq \bar{x}$.*

[12] A bidder's payoff need not be ex-post proportional.

[13] We are grateful to an anonymous referee for suggesting Proposition 6.1 and providing a proof.

Figure 2 below illustrates these results. It shows equilibrium bids in round 1 for $\alpha = 0$, 10, 100, and ∞ when $N = 3$ and values are distributed $U[0, 1]$. As α approaches infinity, $\lim_{\alpha\to\infty} \beta_1^{\alpha}(x) = x/3$.

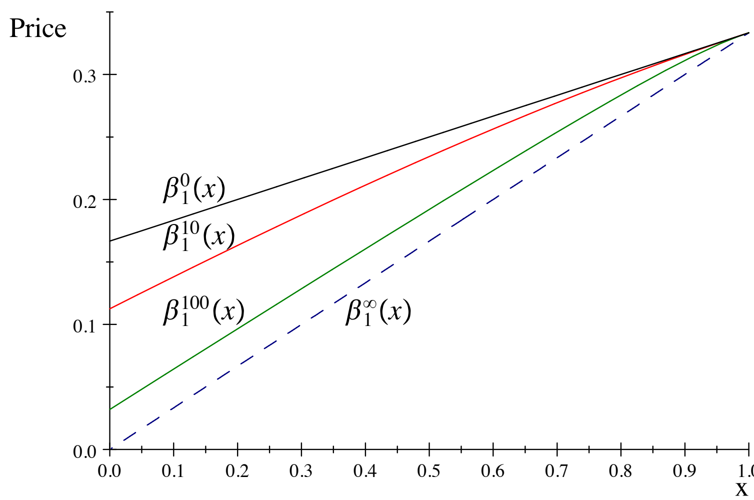

Figure 2: Round 1 equilibrium bids for $N = 3$, $U[0, 1]$, and $\alpha = 0, 10, 100,$ and ∞.

4 Discussion

Compensation auctions can also be used to allocate an indivisible undesirable item (e.g., a waste dump or a nuclear power plant) or an indivisible costly task or chore (e.g., an administrative position). An allocation mechanism in such a setting must determine which of the N players is to accept the undesirable item or complete the chore and how the other players are going to compensate him. We consider the problem of allocating a chore.

The key to employing the compensation auction (which is defined for a "good") is to make the chore desirable. Suppose each bidder's cost of completing the chore is independently and identically distributed according to cumulative distribution function F with support $[0, \bar{c}]$. In order to make the chore desirable, each of the N bidders contributes $\bar{c}/N$ into a pot which will be awarded to the bidder assigned to complete the chore. Thus, if bidder i with cost c_i undertakes the chore, then he receives a total payoff of $v_i = \bar{c} - c_i \geq 0$. The compensation auction can be used to allocate the chore to a bidder and to determine the compensations (which can be viewed as rebates of $\bar{c}/N$) that the winner provides to the remaining bidders.

The auction operates as before: The price, starting from zero, rises continuously and a bidder may drop out at any point. A bidder who drops out surrenders the opportunity to do the chore but, in return, receives compensation from the winner equal to the difference between the price at which he drops and the price at which the prior bidder dropped. The auction ends when exactly one bidder remains. Since the auction is interim proportional (see Proposition 7), then bidder i's equilibrium payoff is at least $v_i/N = (\bar{c} - c_i)/N$. Thus bidder i's payoff, net of his contribution $\bar{c}/N$, is at least

$$\frac{\bar{c} - c_i}{N} - \frac{\bar{c}}{N} = -\frac{c_i}{N}.$$

In other words, each bidder's payoff is equal to at least $1/N$-th of his cost of undertaking the chore. Furthermore, since the auction is ex-post efficient, the chore is allocated to the bidder for whom the cost of completing the chore is smallest.

5 Appendix

Lemma 0 found in McAfee (1992) is not directly applicable to our paper since the payoff function may not be C^2 for all x and y. However, the following simple extension plays the same role and can be applied in our setting.

Lemma 0: *Suppose an agent of type x who reports y receives profits equal to*

$$\pi(x, y) = \begin{cases} \pi^H(x, y) & \text{if } y \geq x \\ \pi^L(x, y) & \text{if } y \leq x \end{cases}.$$

Further suppose that for all x we have

$$\frac{\partial}{\partial y}\pi(x,x) = \frac{\partial}{\partial y}\pi^H(x,x) = \frac{\partial}{\partial y}\pi^L(x,x) = 0 \tag{5}$$

and that

$$\begin{aligned} &\tfrac{\partial^2}{\partial x \partial y}\pi^H(x,y) \geq 0 \quad \text{for } y > x \\ &\tfrac{\partial^2}{\partial x \partial y}\pi^L(x,y) \geq 0 \quad \text{for } y < x. \end{aligned} \tag{6}$$

Then π is maximized over y at $y = x$.

Proof of Lemma 0: First, from (5), we have that $\frac{\partial}{\partial y}\pi(x,x) = \frac{\partial}{\partial y}\pi^H(x,x) = \frac{\partial}{\partial y}\pi^L(x,x) = 0$ for all x. Second, since (5) and (6) if $y > x$, then

$$\frac{\partial}{\partial y}\pi^H(x,y) \leq 0$$

and if $y < x$, then

$$\frac{\partial}{\partial y}\pi^L(x,y) \geq 0.$$

Hence, we have established that: (i) if $y < x$, then $\frac{\partial}{\partial y}\pi(x,y) = \frac{\partial}{\partial y}\pi^L(x,y) \geq 0$; (ii) if $y = x$, then $\frac{\partial}{\partial y}\pi(x,x) = 0$; and (iii), if $y > x$, then $\frac{\partial}{\partial y}\pi(x,y) = \frac{\partial}{\partial y}\pi^H(x,y) \leq 0$. Therefore π is maximized over y at $y = x$. $\square$

Proof of Proposition 1: Let $\beta = (\beta_1, \ldots, \beta_{N-1})$ be a symmetric equilibrium in increasing and differentiable strategies. Since equilibrium is in increasing strategies, the sequence of dropout prices $(p_1, \ldots, p_{k-1})$ at round k reveals the $k-1$ lowest values $(z_1, \ldots, z_{k-1})$. In the proof it is convenient to write the round k equilibrium bid as a function of the prior dropout values rather than as a function of the prior dropout prices. In particular, we write $\beta_k(x|\mathbf{z}_{k-1})$ rather than $\beta_k(x; \mathbf{p}_{k-1})$.

For each $k < N$, let $\pi_k(y, x|\mathbf{z}_{k-1})$ be the expected payoff to a bidder with value x who in round k deviates from equilibrium and bids as though his value is y (i.e., he bids $\beta_k(y|\mathbf{z}_{k-1})$), when $\mathbf{z}_{k-1}$ is the profile of values of the

$k-1$ bidders to drop so far. In this case we will sometimes say the bidder "bids y". Let

$$\Pi_k(x|\mathbf{z}_{k-1}) = \pi_k(x, x|\mathbf{z}_{k-1})$$

be the equilibrium payoff of a bidder in round k when his value is x and $\mathbf{z}_{k-1}$ is the profile of values of the $k-1$ bidders to drop in prior rounds.

Consider the following two-part claim for round k:

(a) For each $\mathbf{z}_{k-1}$:

- (a.i) β_k satisfies the differential equation given in Proposition 1(i).
- (a.ii) if $x \geq z_{k-1}$ then $x \in \arg\max_y \pi_k(y, x|\mathbf{z}_{k-1})$, i.e., it is optimal for each bidder to follow β_k in round k; if $x < z_{k-1}$ then $z_{k-1} \in \arg\max_y \pi_k(y, x|\mathbf{z}_{k-1})$.

(b) For each $\mathbf{z}_{k-1}$:

$$\frac{d\Pi_k(x|\mathbf{z}_{k-1})}{dx} \geq 0\,.$$

We prove by induction that the claim is true for each $k \in \{1, \ldots, N-1\}$, thereby establishing Proposition 1.

We first show the claim is true for round $N-1$. Let $\mathbf{z}_{N-2}$ be arbitrary. Consider an active bidder in the k-th round whose value is x but who bids as though it is $y \geq z_{N-2}$. There are two cases to consider: $x \geq z_{N-2}$ and $x < z_{N-2}$.

Suppose that $x \geq z_{N-2}$. With a bid of $y \geq z_{N-2}$, the bidder wins and obtains $x - \beta_{N-1}(z_{N-1}|\mathbf{z}_{N-2})$ if $y > z_{N-1}$, and he obtains compensation $\beta_{N-1}(y|\mathbf{z}_{N-2}) - p_{N-2}$ if $y < z_{N-1}$, where $p_{N-2} = \beta_{N-2}(z_{N-2}|\mathbf{z}_{N-3})$. Hence

$$\begin{aligned}\pi_{N-1}(y, x|\mathbf{z}_{N-2}) = & \int_{z_{N-2}}^{y} u(x - \beta_{N-1}(z_{N-1}|\mathbf{z}_{N-2}))g_{N-1}^{(N-1)}(z_{N-1}|z_{N-2})dz_{N-1} \\ & + \int_{y}^{\bar{x}} u(\beta_{N-1}(y|\mathbf{z}_{N-2}) - p_{N-2})g_{N-1}^{(N-1)}(z_{N-1}|z_{N-2})dz_{N-1}.\end{aligned}$$

Differentiating with respect to y yields $\partial\pi_{N-1}(y, x|\mathbf{z}_{N-2})/\partial y =$

$$\begin{aligned} & [u(x - \beta_{N-1}(y|\mathbf{z}_{N-2})) - u(\beta_{N-1}(y|\mathbf{z}_{N-2}) - p_{N-2})] g_{N-1}^{(N-1)}(y|z_{N-2}) \\ + \quad & u'(\beta_{N-1}(y|\mathbf{z}_{N-2}) - p_{N-2})\beta'_{N-1}(y|\mathbf{z}_{N-2})(1 - G_{N-1}^{(N-1)}(y|z_{N-2})). \end{aligned} \tag{7}$$

A necessary condition for β to be an equilibrium is that $\partial\pi_{N-1}(y, x|\mathbf{z}_{N-2})/\partial y|_{y=x} = 0$, i.e.,

$$\begin{aligned} & u'(\beta_{N-1}(x|\mathbf{z}_{N-2}) - p_{N-2})\beta'_{N-1}(x|\mathbf{z}_{N-2}) \\ = \quad & [u(\beta_{N-1}(x|\mathbf{z}_{N-2}) - p_{N-2}) - u(x - \beta_{N-1}(x|\mathbf{z}_{N-2}))]\lambda_{N-1}^{N}(x), \end{aligned} \tag{8}$$

where

$$\lambda_{N-1}^{N}(x) = \frac{g_{N-1}^{(N-1)}(x|z_{N-2})}{1 - G_{N-1}^{(N-1)}(x|z_{N-2})} = \frac{f(x)}{1 - F(x)}.$$

Alternatively, since types can be inferred from dropout prices, we can write the necessary condition as

$$\begin{aligned} & u'(\beta_{N-1}(x; \mathbf{p}_{N-2}) - p_{N-2})\beta'_{N-1}(x; \mathbf{p}_{N-2}) \\ = \quad & [u(\beta_{N-1}(x; \mathbf{p}_{N-2}) - p_{N-2}) - u(x - \beta_{N-1}(x; \mathbf{p}_{N-2}))]\lambda_{N-1}^{N}(x), \end{aligned}$$

which establishes (a.i) for $k = N - 1$.

The necessary condition holds for all x and, in particular, it holds for $x = y$, i.e.,

$$\begin{aligned} & u'(\beta_{N-1}(y|\mathbf{z}_{N-2}) - p_{N-2})\beta'_{N-1}(y|\mathbf{z}_{N-2}) \\ = \quad & [u(\beta_{N-1}(y|\mathbf{z}_{N-2}) - p_{N-2}) - u(y - \beta_{N-1}(y|\mathbf{z}_{N-2}))]\lambda_{N-1}^{N}(y). \end{aligned} \tag{9}$$

Substituting (9) into (7) and simplifying yields

$$\frac{\partial\pi_{N-1}(y, x|\mathbf{z}_{N-2})}{\partial y} = \left[u(x - \beta_{N-1}(y|\mathbf{z}_{N-2})) - u(y - \beta_{N-1}(y|\mathbf{z}_{N-2}))\right] g_{N-1}^{(N-1)}(y|z_{N-2}).$$

Clearly, $\partial\pi_{N-1}(y,x|\mathbf{z}_{N-2})/\partial y|_{y=x}=0$. Moreover, for $y \geq z_{N-2}$ we have

$$\frac{\partial^2\pi_{N-1}(y,x|\mathbf{z}_{N-2})}{\partial y\partial x}=u'(x-\beta_{N-1}(y|\mathbf{z}_{N-2}))g_{N-1}^{(N-1)}(y|z_{N-2})\geq 0,$$

where the inequality holds since $u'>0$ and $g_{N-1}^{(N-1)}(y|z_{N-2})\geq 0$. Hence, if $x \geq z_{N-2}$ then $x \in \arg\max_y \pi_{N-1}(y,x|\mathbf{z}_{k-1})$ by Lemma 0.

Suppose $x < z_{N-2}$. It is clearly never optimal for a bidder to bid as though his type were less than z_{N-2}, i.e., bid less than $\beta_{N-1}(z_{N-2}|\mathbf{z}_{N-2})$, since bidding z_{N-2} yields a greater compensation. For $y \geq z_{N-2}$ we have

$$\frac{\partial\pi_{N-1}(y,x|\mathbf{z}_{N-2})}{\partial y}=[u(x-\beta_{N-1}(y|\mathbf{z}_{N-2}))-u(y-\beta_{N-1}(y|\mathbf{z}_{N-2}))]g_{N-1}^{(N-1)}(y|z_{N-2})<0,$$

and thus $z_{N-2}\in\arg\max_y \pi_{N-1}(y,x|\mathbf{z}_{N-2})$. Hence (a.ii) is true for $k=N-1$.

To prove (b), note that

$$\begin{aligned}\frac{d\Pi_{N-1}(x|\mathbf{z}_{N-2})}{dx} &= \left.\frac{\partial\pi_{N-1}(y,x|\mathbf{z}_{N-2})}{\partial y}\right|_{y=x}+\left.\frac{\partial\pi_{N-1}(y,x|\mathbf{z}_{N-2})}{\partial x}\right|_{y=x}\\ &= \int_{z_{N-2}}^{x} u'(x-\beta_{N-1}(z_{N-1}|\mathbf{z}_{N-2}))g_{N-1}^{(N-1)}(z_{N-1}|z_{N-2})dz_{N-1}\\ &\geq 0,\end{aligned}$$

where the second equality holds since $\partial\pi_{N-1}(y,x|\mathbf{z}_{N-2})/\partial y|_{y=x}=0$. Hence (b) holds for $k=N-1$.

Assume the claim is true for rounds $k+1$ through $N-1$. We show that the claim is true for round k. Let $\mathbf{z}_{k-1}$ be arbitrary. If $x<z_{k-1}$ then clearly $z_{k-1}\in\arg\max_y \pi_k(y,x|\mathbf{z}_{k-1})$. Suppose $x\geq z_{k-1}$. Consider an active bidder in the k-th round whose value is x and who bids as though his value is $y \geq z_{k-1}$. We need to distinguish between two cases: (i) $y\in[z_{k-1},x]$ and (ii) $y>x$ since his payoff function differs in each case. (A bid below z_{k-1} is not optimal.) In what follows, we denote the payoff to a bid of y as $\pi_k^L(y,x|\mathbf{z}_{k-1})$

if $y \in [z_{k-1}, x]$ and as $\pi_k^H(y, x|\mathbf{z}_{k-1})$ if $y \geq x$.

Case (i): Consider a bid $y \in [z_{k-1}, x]$. If $z_k \in [z_{k-1}, y]$ the bidder continues to round $k+1$ where, by the induction hypothesis, he optimally bids x and he has an expected payoff of $\Pi_{k+1}(x|\mathbf{z}_{k-1}, z_k)$. If $z_k \geq y$ he obtains compensation of $\beta_k(y|\mathbf{z}_{k-1}) - p_{k-1}$ in round k, where $p_{k-1} = \beta_{k-1}(z_{k-1}|\mathbf{z}_{k-2})$. Hence his payoff is

$$\begin{aligned}\pi_k^L(y, x|\mathbf{z}_{k-1}) = \quad & \int_{z_{k-1}}^{y} \Pi_{k+1}(x|\mathbf{z}_{k-1}, z_k) g_k^{(N-1)}(z_k|z_{k-1}) dz_k \\ & + \int_{y}^{\bar{x}} u\left(\beta_k(y|\mathbf{z}_{k-1}) - p_{k-1}\right) g_k^{(N-1)}(z_k|z_{k-1}) dz_k.\end{aligned}$$

Differentiating with respect to y yields

$$\begin{aligned}\frac{\partial \pi_k^L(y, x|\mathbf{z}_{k-1})}{\partial y} = \quad & [\Pi_{k+1}(x|\mathbf{z}_{k-1}, y) - u\left(\beta_k(y|\mathbf{z}_{k-1}) - p_{k-1}\right)] g_k^{(N-1)}(y|z_{k-1}) \\ & + u'\left(\beta_k(y|\mathbf{z}_{k-1}) - p_{k-1}\right) \beta_k'(y|\mathbf{z}_{k-1})(1 - G_k^{(N-1)}(y|z_{k-1})).\end{aligned}$$

A necessary condition for equilibrium is that $\partial \pi_k^L(y, x|\mathbf{z}_{k-1})/\partial y|_{y=x} \geq 0$, i.e.,

$$\begin{aligned}& [\Pi_{k+1}(x|\mathbf{z}_{k-1}, x) - u\left(\beta_k(x|\mathbf{z}_{k-1}) - p_{k-1}\right)] g_k^{(N-1)}(x|z_{k-1}) \\ & + u'\left(\beta_k(x|\mathbf{z}_{k-1}) - p_{k-1}\right) \beta_k'(x|\mathbf{z}_{k-1})(1 - G_k^{(N-1)}(x|z_{k-1})) \geq 0.\end{aligned}$$

Since

$$\Pi_{k+1}(x|\mathbf{z}_{k-1}, x) = u(\beta_{k+1}(x|\mathbf{z}_{k-1}, x) - \beta_k(x|\mathbf{z}_{k-1})),$$

the necessary condition can be written as

$$\begin{aligned}& u'\left(\beta_k(x|\mathbf{z}_{k-1}) - p_{k-1}\right) \beta_k'(x|\mathbf{z}_{k-1}) \\ \geq \quad & [u\left(\beta_k(x|\mathbf{z}_{k-1}) - p_{k-1}\right) - u(\beta_{k+1}(x|\mathbf{z}_{k-1}, x) - \beta_k(x|\mathbf{z}_{k-1}))] \lambda_k^N(x),\end{aligned} \tag{10}$$

where

$$\lambda_k^N(x) = \frac{g_k^{(N-1)}(x|z_{k-1})}{1 - G_k^{(N-1)}(x|z_{k-1})} = (N-k)\frac{f(x)}{1 - F(x)}.$$

For $y \in [z_{k-1}, x]$ we have

$$\frac{\partial^2 \pi_k^L(y, x|\mathbf{z}_{k-1})}{\partial y \partial x} = \frac{d}{dx}\Pi_{k+1}(x|\mathbf{z}_{k-1}, y) g_k^{(N-1)}(y|z_{k-1}) \geq 0,$$

where the inequality follows since (b) is true for round $k+1$ by the induction hypothesis.

Case (ii): Consider a bid $y \geq x$. If $z_k \in [z_{k-1}, x]$, then the bidder continues to round $k+1$ and, by the induction hypothesis, he bids x and obtains $\Pi_{k+1}(x|\mathbf{z}_{k-1}, z_k)$. If $z_k \in [x, y]$, then he continues to round $k+1$ and, by the induction hypothesis, he bids z_k and obtains compensation $\beta_{k+1}(z_k|\mathbf{z}_{k-1}, z_k) - \beta_k(z_k|\mathbf{z}_{k-1})$. If $z_k > y$ then in round k he wins compensation $\beta_k(y|\mathbf{z}_{k-1}) - p_{k-1}$. Thus his payoff at round k is

$$\begin{aligned} \pi_k^H(y, x|\mathbf{z}_{k-1}) = \quad & \int_{z_{k-1}}^{x} \Pi_{k+1}(x|\mathbf{z}_{k-1}, z_k) g_k^{(N-1)}(z_k|z_{k-1}) dz_k \\ & + \int_x^y u(\beta_{k+1}(z_k|\mathbf{z}_{k-1}, z_k) - \beta_k(z_k|\mathbf{z}_{k-1})) g_k^{(N-1)}(z_k|z_{k-1}) dz_k, \\ & + \int_y^{\bar{x}} u\left(\beta_k(y|\mathbf{z}_{k-1}) - p_{k-1}\right) g_k^{(N-1)}(z_k|z_{k-1}) dz_k. \end{aligned}$$

Differentiating with respect to y yields

$$\begin{aligned} \frac{\partial \pi_k^H(y, x|\mathbf{z}_{k-1})}{\partial y} \quad = \quad & [u(\beta_{k+1}(y|\mathbf{z}_{k-1}, y) - \beta_k(y|\mathbf{z}_{k-1})) - u\left(\beta_k(y|\mathbf{z}_{k-1}) - p_{k-1}\right)] g_k^{(N-1)}(y|z_{k-1}) \\ & + u'\left(\beta_k(y|\mathbf{z}_{k-1}) - p_{k-1}\right) \beta_k'(y|\mathbf{z}_{k-1})(1 - G_k^{(N-1)}(y|z_{k-1})). \end{aligned}$$

A necessary condition for equilibrium is that $\partial \pi_k^H(y, x|\mathbf{z}_{k-1})/\partial y|_{y=x} \leq 0$, i.e.,

$$\begin{aligned} & u'\left(\beta_k(x|\mathbf{z}_{k-1}) - p_{k-1}\right) \beta_k'(x|\mathbf{z}_{k-1}) \\ \leq \quad & [u\left(\beta_k(x|\mathbf{z}_{k-1}) - p_{k-1}\right) - u(\beta_{k+1}(x|\mathbf{z}_{k-1}, x) - \beta_k(x|\mathbf{z}_{k-1}))] \lambda_k^N(x). \end{aligned} \tag{11}$$

Equations (10) and (11) imply that

$$\begin{aligned} & u'\left(\beta_k(x|\mathbf{z}_{k-1}) - p_{k-1}\right)\beta_k'(x|\mathbf{z}_{k-1}) \qquad (12) \\ = \; & [u\left(\beta_k(x|\mathbf{z}_{k-1}) - p_{k-1}\right) - u(\beta_{k+1}(x|\mathbf{z}_{k-1}, x) - \beta_k(x|\mathbf{z}_{k-1}))]\lambda_k^N(x). \end{aligned}$$

Since the bid functions are increasing, we can replace $\mathbf{z}_{k-1}$ with $\mathbf{p}_{k-1}$ and replace $\beta_{k+1}(x|\mathbf{z}_{k-1}, x)$ with $\beta_{k+1}(x; \mathbf{p}_{k-1}, \beta_k(x; \mathbf{p}_{k-1}))$, writing the first order condition as

$$\begin{aligned} & u'\left(\beta_k(x; \mathbf{p}_{k-1}) - p_{k-1}\right)\beta_k'(x; \mathbf{p}_{k-1}) \\ = \; & [u\left(\beta_k(x; \mathbf{p}_{k-1}) - p_{k-1}\right) - u(\beta_{k+1}(x; \mathbf{p}_{k-1}, \beta_k(x; \mathbf{p}_{k-1})) - \beta_k(x; \mathbf{p}_{k-1}))]\lambda_k^N(x), \end{aligned}$$

which establishes (a.i) for round k.

Equation (12) holds for all x and, in particular, it holds for $x = y$, i.e.,

$$\begin{aligned} & u'\left(\beta_k(y|\mathbf{z}_{k-1}) - p_{k-1}\right)\beta_k'(y|\mathbf{z}_{k-1}) \\ = \; & [u\left(\beta_k(y|\mathbf{z}_{k-1}) - p_{k-1}\right) - u(\beta_{k+1}(y|\mathbf{z}_{k-1}, y) - \beta_k(y|\mathbf{z}_{k-1}))]\lambda_k^N(y). \end{aligned}$$

Substituting this expression into the expression for $\partial\pi_k^H(y, x|\mathbf{z}_{k-1})/\partial y$ yields

$$\frac{\partial\pi_k^H(y, x|\mathbf{z}_{k-1})}{\partial y} = 0 \text{ for } y \geq x.$$

Furthermore,

$$\frac{\partial^2\pi_k^H(y, x|\mathbf{z}_{k-1})}{\partial y \partial x} = 0 \text{ for } y \geq x.$$

We have shown that

$$\left.\frac{\partial\pi_k^L(y, x|\mathbf{z}_{k-1})}{\partial y}\right|_{y=x} = \left.\frac{\partial\pi_k^H(y, x|\mathbf{z}_{k-1})}{\partial y}\right|_{y=x} = 0$$

and

$$\frac{\partial^2 \pi_k^L(y,x|\mathbf{z}_{k-1})}{\partial y \partial x} \geq 0 \text{ for } y \in [z_{k-1}, x] \text{ and } \frac{\partial^2 \pi_k^H(y,x|\mathbf{z}_{k-1})}{\partial y \partial x} \geq 0 \text{ for } y \geq x.$$

Hence (a.ii) is true for round k by Lemma 0.

To establish (b) is true for round k, observe that

$$\begin{aligned} \Pi_k(x|\mathbf{z}_{k-1}) &= \int_{z_{k-1}}^{x} \Pi_{k+1}(x|\mathbf{z}_{k-1}, z_k) g_k^{(N-1)}(z_k|z_{k-1}) dz_k \\ &\quad + \int_{x}^{\bar{x}} u\left(\beta_k(x|\mathbf{z}_{k-1}) - p_{k-1}\right) g_k^{(N-1)}(z_k|z_{k-1}) dz_k. \end{aligned}$$

Differentiating and simplifying yields

$$\frac{d\Pi_k(x|\mathbf{z}_{k-1})}{dx} = \int_{z_{k-1}}^{x} \frac{d}{dx} \Pi_{k+1}(x|\mathbf{z}_{k-1}, z_k) g_k^{(N-1)}(z_k|z_{k-1}) dz_k \geq 0,$$

where the equality follows from $\Pi_{k+1}(x|\mathbf{z}_{k-1}, x) = u(\beta_{k+1}(x|\mathbf{z}_{k-1}, x) - \beta_k(x|\mathbf{z}_{k-1}))$ and (12), and the inequality follows since $d\Pi_{k+1}(x|\mathbf{z}_{k-1}, z_k)/dx \geq 0$ by the induction hypothesis. □

Proof of Proposition 2: The proof is symmetric to the proof of Proposition 3, and is therefore omitted. □

Proof of Proposition 3: We first solve for the round $N-1$ bid function. When $u(x) = (1 - e^{-\alpha x})/\alpha$, then (1) yields the differential equation

$$-\alpha e^{-\alpha\left(2\beta_{N-1}^{\alpha}(x;\mathbf{p}_{N-2}) - p_{N-2}\right)} \frac{d\beta_{N-1}^{\alpha}(x;\mathbf{p}_{N-2})}{dx} = \left(e^{-\alpha\left(2\beta_{N-1}^{\alpha}(x;\mathbf{p}_{N-2}) - p_{N-2}\right)} - e^{-\alpha x}\right) \lambda_{N-1}^{N}(x).$$

Multiply both sides by $2(1 - F(x))^2$, this equation can be written as

$$\frac{d}{dx}\left(e^{-\alpha\left(2\beta_{N-1}^{\alpha}(x;\mathbf{p}_{N-2}) - p_{N-2}\right)} (1 - F(x))^2\right) = -2e^{-\alpha x} f(x)(1 - F(x)).$$

By the Fundamental Theorem of Calculus

$$e^{-\alpha\left(2\beta^{\alpha}_{N-1}(x;\mathbf{p}_{N-2})-p_{N-2}\right)}(1-F(x))^2 = -\int_0^x e^{-\alpha z}2f(z)(1-F(z))dz + C,$$

where C is the constant of integration. Since the left hand side of this equation is zero when $x = \bar{x}$, then

$$C = \int_0^{\bar{x}} e^{-\alpha z}2(1-F(z))f(z)dz.$$

and therefore the equation can be written as

$$e^{-\alpha\left(2\beta^{\alpha}_{N-1}(x;\mathbf{p}_{N-2})-p_{N-2}\right)}(1-F(x))^2 = \int_x^{\bar{x}} e^{-\alpha z}2(1-F(z))f(z)dz.$$

Solving yields

$$\beta^{\alpha}_{N-1}(x;\mathbf{p}_{N-2}) = \frac{1}{2}p_{N-2} - \frac{1}{2\alpha}\ln\left[\int_x^{\bar{x}} e^{-\alpha z}\frac{2(1-F(z))f(z)}{(1-F(x))^2}dz\right],$$

which, by Claim 4 in the Supplemental Appendix, can be written as[14]

$$\beta^{\alpha}_{N-1}(x;\mathbf{p}_{N-2}) = \frac{1}{2}p_{N-2} - \frac{1}{2\alpha}\ln\left(E\left[e^{-\alpha Z^{(N)}_{N-1}}|Z^{(N)}_{N-1} > x > Z^{(N)}_{N-2}\right]\right).$$

Finally, by the definition of J^{α}_{N-1}, we can write

$$\beta^{\alpha}_{N-1}(x;\mathbf{p}_{N-2}) = \frac{1}{2}p_{N-2} - \frac{1}{2\alpha}\ln\left(J^{\alpha}_{N-1}(x)\right).$$

Next, we solve for the round k bid function when $k < N-1$. Assume that in round $k+1$ bidders follow the bid function

$$\beta^{\alpha}_{k+1}(x;\mathbf{p}_k) = \frac{N-k-1}{N-k}p_k - \frac{N-k-1}{(N-k)\,\alpha}\ln\left(J^{\alpha}_{k+1}(x)\right).$$

[14] See Claim 2 of the Supplemental Appendix for the conditional density $g^{(N)}_{N-1}(z_{N-1}|Z^{(N)}_k > x > Z^{(N)}_{k-1})$.

Then

$$\beta^{\alpha}_{k+1}(x;\beta^{\alpha}_{k}(x;\mathbf{p}_{k-1}),\mathbf{p}_{k-1}) = \frac{N-k-1}{N-k}\beta^{\alpha}_{k}(x;\mathbf{p}_{k-1}) - \frac{N-k-1}{(N-k)\,\alpha}\ln\left(J^{\alpha}_{k+1}(x)\right).$$

and thus $\beta^{\alpha}_{k+1}(x;\beta^{\alpha}_{k}(x;\mathbf{p}_{k-1}),\mathbf{p}_{k-1}) - \beta^{\alpha}_{k}(x;\mathbf{p}_{k-1})$ equals

$$-\frac{N-k-1}{(N-k)\,\alpha}\ln\left(J^{\alpha}_{k+1}(x)\right) - \frac{1}{N-k}\beta^{\alpha}_{k}(x;\mathbf{p}_{k-1}).$$

For round $k < N$, by equation (2) we have

$$\begin{aligned} &-\alpha e^{-\alpha\left(\frac{N-k+1}{N-k}\beta^{\alpha}_{k}(x;\mathbf{p}_{k-1})-p_{k-1}\right)}\frac{d\beta^{\alpha}_{k}(x;\mathbf{p}_{k-1})}{dx} \\ &= \left[e^{-\alpha\left(\frac{N-k+1}{N-k}\beta^{\alpha}_{k}(x;\mathbf{p}_{k-1})-p_{k-1}\right)} - J^{\alpha}_{k+1}(x)^{\frac{N-k-1}{N-k}}\right](N-k)\frac{f(x)}{1-F(x)}. \end{aligned}$$

Multiplying both sides of this equation by $\frac{N-k+1}{N-k}\left(1-F(x)\right)^{N-k+1}$ yields

$$\begin{aligned} &\frac{d}{dx}\left(e^{-\alpha\left(\frac{N-k+1}{N-k}\beta^{\alpha}_{k}(x;\mathbf{p}_{k-1})-p_{k-1}\right)}\left(1-F(x)\right)^{N-k+1}\right) \\ &= -J^{\alpha}_{k+1}(x)^{\frac{N-k-1}{N-k}}\left(N-k+1\right)\left(1-F(x)\right)^{N-k}f(x). \end{aligned}$$

Applying the Fundamental Theorem of Calculus, we obtain

$$\begin{aligned} &e^{-\alpha\left(\frac{N-k+1}{N-k}\beta^{\alpha}_{k}(x;\mathbf{p}_{k-1})-p_{k-1}\right)}\left(1-F(x)\right)^{N-k+1} \\ &= -\int_0^x J^{\alpha}_{k+1}(z)^{\frac{N-k-1}{N-k}}\left(N-k+1\right)\left(1-F(z)\right)^{N-k}f(z)dz + C, \end{aligned} \tag{13}$$

where C is an arbitrary constant.

Since the LHS of (13) is zero when $x = \bar{x}$, then

$$C = \int_0^{x} J^{\alpha}_{k+1}(z)^{\frac{N-k-1}{N-k}}\left(N-k+1\right)\left(1-F(z)\right)^{N-k}f(z)dz.$$

Hence
$$e^{-\alpha\left(\frac{N-k+1}{N-k}\beta_k^\alpha(x;\mathbf{p}_{k-1})-p_{k-1}\right)}(1-F(x))^{N-k+1}$$

$$=\int_x^{\bar{x}} J_{k+1}^\alpha(z)^{\frac{N-k-1}{N-k}}\,(N-k+1)\,(1-F(z))^{N-k}f(z)dz.$$

Hence $\beta_k^\alpha(x;\mathbf{p}_{k-1})$ equals

$$\frac{N-k}{N-k+1}p_{k-1}-\frac{N-k}{(N-k+1)\,\alpha}\ln\left(\int_x^{\bar{x}}\left(J_{k+1}^\alpha(z)\right)^{\frac{N-k-1}{N-k}}\frac{(N-k+1)\,(1-F(z))^{N-k}\,f(z)}{(1-F(x))^{N-k+1}}dz\right).$$

By Claim 4 of the Supplemental Appendix, we can write

$$\beta_k^\alpha(x;\mathbf{p}_{k-1})=\frac{N-k}{N-k+1}p_{k-1}-\frac{N-k}{(N-k+1)\,\alpha}\ln\left(E\left[\left(J_{k+1}^\alpha(Z_k^{(N)})\right)^{\frac{N-k-1}{N-k}}|Z_k^{(N)}>x>Z_{k-1}^{(N)}\right]\right),$$

and hence by the definition of J_k^α we have

$$\beta_k^\alpha(x;\mathbf{p}_{k-1})=\frac{N-k}{N-k+1}p_{k-1}-\frac{N-k}{(N-k+1)\,\alpha}\ln\left(J_k^\alpha(x)\right),$$

which is the desired result. $\square$

Let
$$H_{k+1}^0(x)=\frac{1}{N-k}E[Z_{N-1}^{(N)}|Z_{k+1}^{(N)}>x>Z_k^{(N)}].$$
The conditional density for this expectation is given by

$$g_{N-1}^{(N)}(t|Z_{k+1}^{(N)}>x>Z_k^{(N)})=\frac{(N-k)\,(N-k-1)\,[F(t)-F(x)]^{N-k-2}[1-F(t)]f(t)}{(1-F(x))^{N-k}}.$$

The following lemma is useful in proving Proposition 4.

Lemma A:

$$\int_x^{\bar{x}}H_{k+1}^0(t)\frac{(N-k+1)[1-F(t)]^{N-k}f(t)}{[1-F(x)]^{N-k+1}}dt=\frac{1}{N-k}E\left[Z_{N-1}^{(N)}|Z_k^{(N)}>x>Z_{k-1}^{(N)}\right]$$

Proof: We have

$$\int_x^{\bar{x}} H_{k+1}^0(t) \frac{(N-k+1)[1-F(t)]^{N-k} f(t)}{[1-F(x)]^{N-k+1}} dt$$
$$= \int_x^{\bar{x}} \int_t^{\bar{x}} q \frac{(N-k+1)(N-k-1) [F(q)-F(t)]^{N-k-2} [1-F(q)] f(q) f(t)}{[1-F(x)]^{N-k+1}} dq dt.$$

Changing the order of integration, we can write this as

$$\int_x^{\bar{x}} \int_x^{q} q \frac{(N-k+1)(N-k-1) [F(q)-F(t)]^{N-k-2} [1-F(q)] f(q) f(t)}{[1-F(x)]^{N-k+1}} dt dq,$$

or

$$\int_x^{\bar{x}} q \frac{(N-k+1)(N-k-1)[1-F(q)] f(q)}{[1-F(x)]^{N-k+1}} \left(\int_x^q [F(q)-F(t)]^{N-k-2} f(t) dt \right) dq.$$

Since

$$\int_x^q [F(q)-F(t)]^{N-k-2} f(t) dt = - \frac{[F(q)-F(t)]^{N-k-1}}{N-k-1} \Bigg|_{t=x}^{t=q} = \frac{[F(q)-F(x)]^{N-k-1}}{N-k-1},$$

the above expression is equal to

$$\frac{1}{N-k} \int_x^{\bar{x}} q \frac{(N-k+1)(N-k) [F(q)-F(x)]^{N-k-1} [1-F(q)] f(q)}{[1-F(x)]^{N-k+1}} dq$$

which is just $\frac{1}{N-k} E\left[Z_{N-1}^{(N)} | Z_k^{(N)} > x > Z_{k-1}^{(N)} \right]$. $\square$

Proof of Proposition 4: Part (i). We first show that for each $k = 1, \ldots, N-1$ and $\mathbf{p}_{k-1}$ that $\beta_k^0(x; \mathbf{p}_{k-1}) \geq \beta_k^\alpha(x; \mathbf{p}_{k-1})$ for $x < \bar{x}$. The proof is by induction.

For $k = N-1$, since e^x is a convex function, then by Jensen's Inequality, for $x < \bar{x}$ we have

$$e^{E[-\alpha Z_{N-1}^{(N)} | Z_{N-1}^{(N)} > x > Z_{N-2}^{(N)}]} < E[e^{-\alpha Z_{N-1}^{(N)}} | Z_{N-1}^{(N)} > x > Z_{N-2}^{(N)}].$$

Noting that the RHS is $J_{N-1}^{\alpha}(x)$, taking the log of both sides, and then multiplying through by $-1/(2\alpha)$ yields

$$\frac{1}{2} E[Z_{N-1}^{(N)} | Z_{N-1}^{(N)} > x > Z_{N-2}^{(N)}] > -\frac{1}{2\alpha} \ln(J_{N-1}^{\alpha}(x))$$

Adding $\frac{1}{2} p_{N-2}$ to both sides yields the result $\beta_{N-1}^{0}(x; \mathbf{p}_{N-2}) > \beta_{N-1}^{\alpha}(x; \mathbf{p}_{N-2})$ for $x < \bar{x}$.

For $k \leq N-1$, define

$$H_k^0(x) = \frac{1}{N-k+1} E[Z_{N-1}^{(N)} | Z_k^{(N)} > x > Z_{k-1}^{(N)}],$$

and

$$H_k^{\alpha}(x) = -\frac{N-k}{(N-k+1)\alpha} \ln(J_k^{\alpha}(x)) = -\frac{1}{\alpha} \ln\left(J_k^{\alpha}(x)^{\frac{N-k}{N-k+1}}\right),$$

where $J_k^{\alpha}(x)$ is defined in Proposition 3. We have that

$$e^{-\alpha H_k^{\alpha}(x)} = J_k^{\alpha}(x)^{\frac{N-k}{N-k+1}}.$$

We established above that $H_{N-1}^{0}(x) > H_{N-1}^{\alpha}(x)$, i.e.,

$$\frac{1}{2} E[Z_{N-1}^{(N)} | Z_{N-1}^{(N)} > x > Z_{N-2}^{(N)}] > -\frac{1}{2\alpha} \ln(E[e^{-\alpha Z_{N-1}^{(N)}} | Z_{N-1}^{(N)} > x > Z_{N-2}^{(N)}]).$$

Assume for $k < N-2$ that $H_{k+1}^{0}(x) > H_{k+1}^{\alpha}(x)$ for $x < \bar{x}$. We show that $H_k^0(x) > H_k^{\alpha}(x)$ for $x < \bar{x}$. Since $-\alpha H_{k+1}^{0}(x) < -\alpha H_{k+1}^{\alpha}(x)$ and e^x is increasing, then

$$e^{-\alpha H_{k+1}^{0}(x)} < e^{-\alpha H_{k+1}^{\alpha}(x)} \text{ for } x < \bar{x},$$

or

$$e^{-\alpha H^0_{k+1}(x)} < J^{\alpha}_{k+1}(x)^{\frac{N-k-1}{N-k}} \text{ for } x < \bar{x},$$

Thus

$$E[e^{-\alpha H^0_{k+1}(Z^{(N)}_k)}|Z^{(N)}_k > x > Z^{(N)}_{k-1}] < E[J^{\alpha}_{k+1}(Z^{(N)}_k)^{\frac{N-k-1}{N-k}}|Z^{(N)}_k > x > Z^{(N)}_{k-1}]. \tag{14}$$

The right hand side is $J^{\alpha}_k(x)$. Consider the left hand side. Since e^x is convex, then

$$e^{E[-\alpha H^0_{k+1}(Z^{(N)}_k)|Z^{(N)}_k > x > Z^{(N)}_{k-1}]} < E[e^{-\alpha H^0_{k+1}(Z^{(N)}_k)}|Z^{(N)}_k > x > Z^{(N)}_{k-1}].$$

This inequality and (14) imply

$$e^{E[-\alpha H^0_{k+1}(Z^{(N)}_k)|Z^{(N)}_k > x > Z^{(N)}_{k-1}]} < J^{\alpha}_k(x).$$

Taking logs of both sides of this inequality yields

$$E[-\alpha H^0_{k+1}(Z^{(N)}_k)|Z^{(N)}_k > x > Z^{(N)}_{k-1}] < \ln\left(J^{\alpha}_k(x)\right).$$

Multiplying both sides by $-\frac{N-k}{(N-k+1)\alpha}$ yields

$$\int_x^{\bar{x}} H^0_{k+1}(z)\frac{(N-k)[1-F(z)]^{N-k}f(z)}{(1-F(x))^{N-k+1}}dz > -\frac{N-k}{(N-k+1)\alpha}\ln\left(J^{\alpha}_k(x)\right).$$

By Lemma A, the LHS can be written as

$$\frac{1}{N-k+1}E\left[Z^{(N)}_{N-1}|Z^{(N)}_k > x > Z^{(N)}_{k-1}\right].$$

Hence

$$\frac{1}{N-k+1}E\left[Z^{(N)}_{N-1}|Z^{(N)}_k > x > Z^{(N)}_{k-1}\right] > -\frac{N-k}{(N-k+1)\alpha}\ln\left(J^{\alpha}_k(x)\right).$$

Adding $\frac{N-k}{N-k+1}p_{k-1}$ to both sides yields $\beta_k^0(x;\mathbf{p}_{k-1}) > \beta_k^\alpha(x;\mathbf{p}_{k-1})$ for $x < \bar{x}$. This proves Part (i).

Part (ii). We now show that for each $k = 1, \ldots, N-1$ and $\mathbf{p}_{k-1}$ that

$$\beta_k^\alpha(x;\mathbf{p}_{k-1}) > \frac{1}{N-k+1}x + \frac{N-k}{N-k+1}p_{k-1} \text{ for } x < \bar{x}.$$

The proof is by induction.

Since $e^{-\alpha z} < e^{-\alpha x}$ for $z \in (x, \bar{x}]$ then

$$J_{N-1}^\alpha(x) = E[e^{-\alpha Z_{N-1}^{(N)}} | Z_{N-1}^{(N)} > x > Z_{N-2}^{(N)}] < e^{-\alpha x}.$$

Taking logs of both sides and rearranging yields

$$-\frac{1}{2\alpha}\ln(J_{N-1}^\alpha(x)) > \frac{1}{2}x,$$

i.e., $H_{N-1}^\alpha(x) > x/2$ for $x < \bar{x}$. Adding $\frac{1}{2}p_{N-2}$ to both sides yields $\beta_{N-1}^\alpha(x;\mathbf{p}_{k-1}) > x/2 + p_{N-2}/2$ for $x < \bar{x}$.

Assume for $k < N-2$ that $H_{k+1}^\alpha(x) > 1/(N-k)$ for $x < \bar{x}$. We show that

$$H_k^\alpha(x) > \frac{1}{N-k+1}x \text{ for } x < \bar{x}.$$

Since $H_{k+1}^\alpha(x)$ is increasing, then for $z > x$ we have $H_{k+1}^\alpha(z) > H_{k+1}^\alpha(x) > x/(N-k)$ or $-\alpha H_{k+1}^\alpha(z) < -\alpha H_{k+1}^\alpha(x) < -\alpha x/(N-k)$ and thus

$$e^{-\alpha H_{k+1}^\alpha(z)} = J_{k+1}^\alpha(z)^{\frac{N-k-1}{N-k}} < e^{-\alpha H_{k+1}^\alpha(x)} < e^{-\alpha \frac{x}{N-k}}.$$

Hence

$$E[J_{k+1}^\alpha(Z_k^{(N)})^{\frac{N-k-1}{N-k}} | Z_k^{(N)} > x > Z_k^{(N)}] < e^{-\alpha \frac{x}{N-k}}.$$

Taking logs of both sides yields

$$\ln(E[J_{k+1}^\alpha(Z_k^{(N)})^{\frac{N-k-1}{N-k}} | Z_k^{(N)} > x > Z_k^{(N)}]) < -\alpha\frac{x}{N-k},$$

i.e.,

$$-\frac{N-k}{(N-k+1)\alpha}\ln(E[J_{k+1}^{\alpha}(Z_k^{(N)})^{\frac{N-k-1}{N-k}}|Z_k^{(N)} > x > Z_k^{(N)}]) > \frac{x}{N-k+1}.$$

Hence $H_k^{\alpha}(x) > x/(N-k+1)$ for $x < \bar{x}$. Adding $\frac{N-k}{N-k+1}p_{k-1}$ to each side gives us

$$\beta_k^{\alpha}(x;\mathbf{p}_{k-1}) > \frac{x}{N-k+1} + \frac{N-k}{N-k+1}p_{k-1} \text{ for } x < \bar{x}. \ \square$$

Proof of Proposition 5: The proof is by induction. Suppose $\tilde{\alpha} > \alpha$. Since the transformation $y = x^{\frac{\alpha}{\tilde{\alpha}}}$ is concave, then by Jensen's inequality we have that

$$\begin{aligned}&\left(E[e^{-\tilde{\alpha}Z_{N-1}^{(N)}}|Z_{N-1}^{(N)} > x > Z_{N-1}^{(N)}]\right)^{\frac{\alpha}{\tilde{\alpha}}}\\ &\geq E[\left(e^{-\tilde{\alpha}Z_{N-1}^{(N)}}\right)^{\frac{\alpha}{\tilde{\alpha}}}|Z_{N-1}^{(N)} > x > Z_{N-1}^{(N)}]\\ &= E[e^{-\alpha Z_{N-1}^{(N)}}|Z_{N-1}^{(N)} > x > Z_{N-1}^{(N)}].\end{aligned} \tag{15}$$

Next, after applying logs to both sides of (15), doing some algebraic manipulations, and adding $\frac{1}{2}p_{N-2}$ to both sides of (15) we have

$$\begin{aligned}&\tfrac{1}{2}p_{N-2} - \tfrac{1}{2\alpha}\ln E[e^{-\alpha Z_{N-1}^{(N)}}|Z_{N-1}^{(N)} > x > Z_{N-1}^{(N)}]\\ &\geq \tfrac{1}{2}p_{N-2} - \tfrac{1}{2\tilde{\alpha}}\ln E[e^{-\tilde{\alpha}Z_{N-1}^{(N)}}|Z_{N-1}^{(N)} > x > Z_{N-1}^{(N)}]\end{aligned} \tag{16}$$

Thus, we have $\beta_{N-1}^{\alpha}(x;p_{N-1}) \geq \beta_{N-1}^{\tilde{\alpha}}(x;p_{N-1})$.

(Induction Hypothesis): Suppose $\beta_{k+1}^{\alpha}(x;p_k) \geq \beta_{k+1}^{\tilde{\alpha}}(x;p_k)$. Let $H_{k+1}^{\alpha}(x)$ be the non-linear part of $\beta_{k+1}^{\alpha}(x;p_k)$ and let $H_{k+1}^{\tilde{\alpha}}(x)$ be the non-linear part of $\beta_{k+1}^{\tilde{\alpha}}(x;p_k)$.

We now consider the k-th round. As before, since the transformation

$y = x^{\frac{\alpha}{\tilde{\alpha}}}$ is concave, then by Jensen's inequality we have that

$$\begin{aligned}&\left(E[e^{-\tilde{\alpha}H_{k+1}^{\tilde{\alpha}}(Z_{N-1}^{(N)})}|Z_k^{(N)} > x > Z_{k-1}^{(N)}]\right)^{\frac{\alpha}{\tilde{\alpha}}}\\ &\geq E[e^{-\alpha H_{k+1}^{\tilde{\alpha}}(Z_{N-1}^{(N)})}|Z_k^{(N)} > x > Z_{k-1}^{(N)}].\end{aligned} \tag{17}$$

By the induction hypothesis we have that $H_{k+1}^{\alpha}(x) \geq H_{k+1}^{\tilde{\alpha}}(x)$ and therefore the RHS of (17) is greater than

$$E[e^{-\alpha H_{k+1}^{\alpha}(Z_{N-1}^{(N)})}|Z_k^{(N)} > x > Z_{k-1}^{(N)}]. \tag{18}$$

Consequently, the LHS of (17) is greater than (18) or

$$\begin{aligned}&\left(E[e^{-\tilde{\alpha}H_{k+1}^{\tilde{\alpha}}(Z_{N-1}^{(N)})}|Z_k^{(N)} > x > Z_{k-1}^{(N)}]\right)^{\frac{\alpha}{\tilde{\alpha}}}\\ &\geq E[e^{-\alpha H_{k+1}^{\alpha}(Z_{N-1}^{(N)})}|Z_k^{(N)} > x > Z_{k-1}^{(N)}].\end{aligned} \tag{19}$$

Using simple manipulations of (19) we have

$$\begin{aligned}H_k^{\alpha}(x) &= -\tfrac{N-k}{(N-k+1)\alpha} \ln E[e^{-\alpha H_{k+1}^{\alpha}(Z_{N-1}^{(N)})}|Z_k^{(N)} > x > Z_{k-1}^{(N)}]\\ &\geq -\tfrac{N-k}{(N-k+1)\tilde{\alpha}} \ln E[e^{-\tilde{\alpha}H_{k+1}^{\tilde{\alpha}}(Z_{N-1}^{(N)})}|Z_k^{(N)} > x > Z_{k-1}^{(N)}] = H_k^{\tilde{\alpha}}(x)\end{aligned}$$

and therefore that $\beta_k^{\alpha}(x; p_{k-1}) \geq \beta_k^{\tilde{\alpha}}(x; p_{k-1})$. $\square$

Before proving Proposition 6 we first establish the following two useful lemmas.

Lemma B: For each k, we have $|J_k^{\alpha}(x)| \leq 1$ for $\alpha \geq 0$ and $x \in [0, \bar{x}]$.
Proof: The proof is by induction. We first show the result for $k = N - 1$.

Since $0 \leq e^{-\alpha z} \leq 1$ for $z \in [0, \bar{x}]$, then

$$0 \leq J_{N-1}^{\alpha}(x) = E[e^{-\alpha Z_{N-1}^{(N)}} | Z_{N-1}^{(N)} > x > Z_{N-2}^{(N)}] \leq E[1 | Z_{N-1}^{(N)} > x > Z_{N-2}^{(N)}] = 1.$$

Assume that $\left|J_{k+1}^{\alpha}(x)\right| \leq 1$ for $\alpha \geq 0$ and $x \in [0, \bar{x}]$. We show that $|J_k^{\alpha}(x)| \leq 1$ for $\alpha \geq 0$ and $x \in [0, \bar{x}]$. We have

$$J_k^{\alpha}(x) = E\left[\left(J_{k+1}^{\alpha}(Z_k^{(N)})\right)^{\frac{N-k-1}{N-k}} | Z_k^{(N)} > x > Z_{k-1}^{(N)}\right] \leq E\left[1 | Z_k^{(N)} > x > Z_{k-1}^{(N)}\right] = 1.$$

This establishes the result. □

Lemma C: For each k there is an $M_k < \infty$ such that $\left|\frac{d}{d\alpha} J_k^{\alpha}(x) / J_k^{\alpha}(x)\right| \leq M_k$ for $\alpha \geq 0$ and $x \in [0, \bar{x}]$.

Proof: The proof is by induction. We first show the result for $k = N - 1$. Since

$$J_{N-1}^{\alpha}(x) = E[e^{-\alpha Z_{N-1}^{(N)}} | Z_{N-1}^{(N)} > x > Z_{N-2}^{(N)}]$$

then by Leibnitz's rule we have

$$\frac{\frac{d}{d\alpha} J_{N-1}^{\alpha}(x)}{J_{N-1}^{\alpha}(x)} = \frac{-\int_x^{\bar{x}} z e^{-\alpha z} \frac{2(1-F(z))f(z)}{(1-F(x))^2} dz}{\int_x^{\bar{x}} e^{-\alpha z} \frac{2(1-F(z))f(z)}{(1-F(x))^2} dz}.$$

Furthermore, since $z e^{-\alpha z} \geq 0$ and $z \leq \bar{x}$ then

$$0 \geq \frac{\frac{d}{d\alpha} J_{N-1}^{\alpha}(x)}{J_{N-1}^{\alpha}(x)} \geq \frac{-\bar{x} \int_x^{\bar{x}} e^{-\alpha z} \frac{2(1-F(z))f(z)}{(1-F(x))^2} dz}{\int_x^{\bar{x}} e^{-\alpha z} \frac{2(1-F(z))f(z)}{(1-F(x))^2} dz} = -\bar{x}.$$

Hence $\left|\frac{d}{d\alpha} J_{N-1}^{\alpha}(x) / J_{N-1}^{\alpha}(x)\right| \leq M_{N-1}$ for $M_{N-1} = \bar{x}$.

Assume that there is an $M_{k+1} < \infty$ such that $\left|\frac{d}{d\alpha} J_{k+1}^{\alpha}(x) / J_{k+1}^{\alpha}(x)\right| < M_{k+1}$ for $\alpha \geq 0$ and $x \in [0, \bar{x}]$. We show that there is an $M_k < \infty$ such that

$\left|\frac{d}{d\alpha}J_k^\alpha(x)/J_k^\alpha(x)\right| < M_k$. We have

$$\left|\frac{\frac{d}{d\alpha}J_k^\alpha(x)}{J_k^\alpha(x)}\right| = \left|\frac{E\left[\frac{N-k-1}{N-k}\left(J_{k+1}^\alpha(Z_k^{(N)})\right)^{\frac{N-k-1}{N-k}}\frac{\frac{d}{d\alpha}J_{k+1}^\alpha(Z_k^{(N)})}{J_{k+1}^\alpha(Z_k^{(N)})}|Z_k^{(N)} > x > Z_{k-1}^{(N)}\right]}{E\left[\left(J_{k+1}^\alpha(Z_k^{(N)})\right)^{\frac{N-k-1}{N-k}}|Z_k^{(N)} > x > Z_{k-1}^{(N)}\right]}\right|$$

Since $\left|\frac{d}{d\alpha}J_{k+1}^\alpha(x)/J_{k+1}^\alpha(x)\right| \leq M_{k+1}$, we have

$$\begin{aligned}\left|\frac{\frac{d}{d\alpha}J_k^\alpha(x)}{J_k^\alpha(x)}\right| &\leq \frac{N-k-1}{N-k}M_{k+1}\left|\frac{E\left[\left(J_{k+1}^\alpha(Z_k^{(N)})\right)^{\frac{N-k-1}{N-k}}|Z_k^{(N)} > x > Z_{k-1}^{(N)}\right]}{E\left[\left(J_{k+1}^\alpha(Z_k^{(N)})\right)^{\frac{N-k-1}{N-k}}|Z_k^{(N)} > x > Z_{k-1}^{(N)}\right]}\right| \\ &= \frac{N-k-1}{N-k}M_{k+1}.\end{aligned}$$

Thus $M_k = \frac{N-k-1}{N-k}M_{k+1} < \infty$ is such a bound. □

Proof of Proposition 6.1: We show that $\lim_{\alpha\to 0^+}\beta_k^\alpha(x;\mathbf{p}_{k-1}) = \beta_k^0(x;\mathbf{p}_{k-1})$ for $x \in [0,\bar{x}]$. This is equivalent to establishing that

$$\lim_{\alpha\to 0^+}\frac{\ln\left(J_k^\alpha(x)\right)}{\alpha} = -\frac{1}{N-k}E\left[Z_{N-1}^{(N)}|Z_k^{(N)} > x > Z_{k-1}^{(N)}\right]. \qquad (20)$$

We first show for each $k \in \{1,\ldots,N-1\}$ that $J_k^0(x) = 1$ for $x \in [0,\bar{x}]$. Clearly

$$J_{N-1}^0(x) = E[e^0|Z_{N-1}^{(N)} > x > Z_{N-2}^{(N)}] = 1.$$

Assume $J_{k+1}^0(x) = 1$ for $x \in [0,\bar{x}]$. Then

$$J_k^0(x) = E\left[\left(J_{k+1}^0(Z_k^{(N)})\right)^{\frac{N-k-1}{N-k}}|Z_k^{(N)} > x > Z_{k-1}^{(N)}\right] = 1,$$

for $x \in [0,\bar{x}]$ and, by induction, $J_k^0(x) = 1$ for each $k \in \{1,\ldots,N-1\}$. Hence $\ln(J_k^0(x)) = 0$. Since $J_k^\alpha(x)$ and $\ln(J_k^\alpha(x))$ are both continuos in α, then $\lim_{\alpha\to 0^+} J_k^\alpha(x) = 1$ and $\lim_{\alpha\to 0^+} \ln(J_k^\alpha(x)) = 0$.

Since $\lim_{\alpha\to 0^+} \ln(J_k^\alpha(x)) = 0$, then to establish (20) it is sufficient, by L'Hopital's rule, to show that

$$\lim_{\alpha\to 0^+} \frac{d}{d\alpha} \ln\left(J_k^\alpha(x)\right) = -\frac{1}{N-k} E\left[Z_{N-1}^{(N)} | Z_k^{(N)} > x > Z_{k-1}^{(N)}\right].$$

The proof proceeds by induction.

We first show the result for $k = N-1$. We have

$$\frac{d}{d\alpha} \ln\left(J_{N-1}^\alpha(x)\right) = \frac{\int_x^{\bar{x}} -ze^{-\alpha z} \frac{2(1-F(z))}{(1-F(x))^2} f(z)dz}{J_{N-1}^\alpha(x)}.$$

Since $|-ze^{-\alpha z}| \le \bar{x} < \infty$ and since $\lim_{\alpha\to 0^+} -ze^{-\alpha z} = -z$, then

$$\lim_{\alpha\to 0^+} \int_x^{\bar{x}} -ze^{-\alpha z} \frac{2(1-F(z))}{(1-F(x))^2} f(z)dz = -\int_x^{\bar{x}} z \frac{2(1-F(z))}{(1-F(x))^2} f(z)dz$$

by the Dominated Convergence Theorem. Since $\lim_{\alpha\to 0^+} J_{N-1}^\alpha(x) = 1$, then

$$\lim_{\alpha\to 0^+} \frac{d}{d\alpha} \ln\left(J_{N-1}^\alpha(x)\right) = -\int_x^{\bar{x}} z \frac{2(1-F(z))}{(1-F(x))^2} f(z)dz = -E\left[Z_{N-1}^{(N)} | Z_{N-1}^{(N)} > x > Z_{N-2}^{(N)}\right],$$

which establishes the result for $k = N-1$.

Suppose that the claim is true for $m \le k \le N-1$, i.e.,

$$\lim_{\alpha\to 0^+} \frac{d}{d\alpha} \ln\left(J_k^\alpha(x)\right) = -\frac{1}{N-k} E\left[Z_{N-1}^{(N)} | Z_k^{(N)} > x > Z_{k-1}^{(N)}\right].$$

We show that it is true for $k = m-1$. We have that

$$\lim_{\alpha\to 0^+} \frac{d}{d\alpha} \ln\left(J_{m-1}^\alpha(x)\right) = \lim_{\alpha\to 0^+} \frac{\frac{d}{d\alpha} E\left[\left(J_m^\alpha(Z_{m-1}^{(N)})\right)^{\frac{N-m}{N-m+1}} | Z_{m-1}^{(N)} > x > Z_{m-2}^{(N)}\right]}{J_{m-1}^\alpha(x)}.$$

By Leibnitz's rule we have

$$\begin{aligned}
& \frac{d}{d\alpha} E\left[\left(J_m^\alpha(Z_{m-1}^{(N)})\right)^{\frac{N-m}{N-m+1}} |Z_{m-1}^{(N)} > x > Z_{m-2}^{(N)}\right] \\
= \quad & \frac{N-m}{N-(m-1)} E\left[\left(J_m^\alpha(Z_{m-1}^{(N)})\right)^{\frac{N-m}{N-m+1}} \frac{d}{d\alpha} \ln\left(J_m^\alpha(Z_{m-1}^{(N)})\right) |Z_{m-1}^{(N)} > x > Z_{m-2}^{(N)}\right].
\end{aligned}$$

By Lemma B we have $|J_m^\alpha(x)| \leq 1$, and by Lemma C there is an $M_m < \infty$ such that $\left|\frac{d}{d\alpha} \ln\left(J_m^\alpha(x)\right)\right| \leq M_m$, and thus the product $|J_m^\alpha(x) \frac{d}{d\alpha} \ln\left(J_m^\alpha(x)\right)|$ is also bounded by M_m. Taking limits of both sides of the equality above, and applying the Dominated Convergence Theorem and the induction hypothesis yields

$$\begin{aligned}
& \lim_{\alpha \to 0^+} \frac{d}{d\alpha} E\left[\left(J_m^\alpha(Z_{m-1}^{(N)})\right)^{\frac{N-m}{N-m+1}} |Z_{m-1}^{(N)} > x > Z_{m-2}^{(N)}\right] \\
= \quad & -\frac{1}{N-(m-1)} \int_x^{\bar{x}} E\left[Z_{N-1}^{(N)} | Z_m^{(N)} > q > Z_{m-1}^{(N)}\right] g_{m-1}^{(N)}(q|Z_{m-1}^{(N)} > q > Z_{m-2}^{(N)}) dq,
\end{aligned}$$

where for notational convenience we write q rather than z_{m-1} for the variable of integration in the outer expectation. We can write

$$\int_x^{\bar{x}} E\left[Z_{N-1}^{(N)} | Z_m^{(N)} > q > Z_{m-1}^{(N)}\right] g_{m-1}^{(N)}(q|Z_{m-1}^{(N)} > q > Z_{m-2}^{(N)}) dq$$

as

$$\int_x^{\bar{x}} \int_q^{\bar{x}} t g_{N-1}^{(N)}(t|Z_m^{(N)} > q > Z_{m-1}^{(N)}) g_{m-1}^{(N)}(q|Z_{m-1}^{(N)} > q > Z_{m-2}^{(N)}) dt dq,$$

where the relevant densities are

$$g_{N-1}^{(N)}(t|Z_m^{(N)} > q > Z_{m-1}^{(N)}) = \frac{(N-m+1)(N-m)[1-F(t)][F(t)-F(q)]^{N-m-1}}{[1-F(q)]^{N-m+1}} f(t)$$

and

$$g_{m-1}^{(N)}(q|Z_{m-1}^{(N)} > x > Z_{m-2}^{(N)}) = \frac{(N-m+2)[1-F(q)]^{N-m+1}}{[1-F(x)]^{N-m+2}} f(q).$$

Hence the double integral is

$$\int_x^{\bar{x}} \int_q^{\bar{x}} t \frac{(N-m+2)(N-m+1)(N-m)[1-F(t)][F(t)-F(q)]^{N-m-1}}{[1-F(x)]^{N-m+2}} f(t) f(q) dt dq.$$

Changing the order of integration yields

$$\begin{aligned} & \int_x^{\bar{x}} \int_x^{t} t \frac{(N-m+2)(N-m+1)(N-m)[1-F(t)][F(t)-F(q)]^{N-m-1}}{[1-F(x)]^{N-m+2}} f(t) f(q) dq dt \\ = & \int_x^{\bar{x}} t \frac{(N-m+2)(N-m+1)(N-m)[1-F(t)]}{[1-F(x)]^{N-m+2}} f(t) \left[\int_x^t [F(t)-F(q)]^{N-m-1} f(q) dq \right] dt. \end{aligned}$$

Since

$$\int_x^t [F(t)-F(q)]^{N-m-1} f(q) dq = -\frac{[F(t)-F(q)]^{N-m}}{N-m}\Bigg|_{q=x}^{q=t} = \frac{[F(t)-F(x)]^{N-m}}{N-m},$$

then after substitution the integral can be written as

$$\begin{aligned} & \int_x^{\bar{x}} t \frac{(N-m+2)(N-m+1)[1-F(t)][F(t)-F(x)]^{N-m}}{[1-F(x)]^{N-m+2}} f(t) dt \\ = \;& E\left[Z_{N-1}^{(N)} | Z_{m-1}^{(N)} > x > Z_{m-2}^{(N)} \right]. \end{aligned}$$

Thus we have shown that

$$\lim_{\alpha \to 0^+} \frac{d}{d\alpha} \ln\left(J_{m-1}^{\alpha}(x)\right) = -\frac{N-m}{N-(m-1)} E\left[Z_{N-1}^{(N)} | Z_{m-1}^{(N)} > x > Z_{m-2}^{(N)} \right],$$

which completes the proof. □

Proof of Proposition 6.2: The bidding function $\beta_k^{\alpha}(x; \mathbf{p}_{k-1})$ in an arbitrary round $k \leq N-1$ can be written as

$$\beta_k^\alpha(x; \mathbf{p}_{k-1}) = \frac{N-k}{N-k+1} p_{k-1} - \frac{1}{\alpha} \ln \left(J_k^\alpha(x)^{\frac{N-k}{N-k+1}} \right).$$

By the definition of $J_k^\alpha(x)$ we have

$$J_k^\alpha(x)^{\frac{N-k}{N-k+1}} = \left(E\left[\left(J_{k+1}^\alpha(Z_k^{(N)}) \right)^{\frac{N-k-1}{N-k}} | Z_k^{(N)} > x > Z_{k-1}^{(N)}] \right] \right)^{\frac{N-k}{N-k+1}}.$$

Since $y^{\frac{N-k}{N-k+1}}$ is concave, then by Jensen's inequality

$$\begin{aligned} J_k^\alpha(x)^{\frac{N-k}{N-k+1}} &\geq E\left[\left(\left(J_{k+1}^\alpha(Z_k^{(N)}) \right)^{\frac{N-k-1}{N-k}} \right)^{\frac{N-k}{N-k+1}} | Z_k^{(N)} > x > Z_{k-1}^{(N)}] \right] \\ &= E\left[\left(J_{k+1}^\alpha(Z_k^{(N)}) \right)^{\frac{N-k-1}{N-k+1}} | Z_k^{(N)} > x > Z_{k-1}^{(N)}] \right] \\ &= \int_x^{\bar{x}} J_{k+1}^\alpha(z_k)^{\frac{N-k-1}{N-k+1}} \frac{(N-k+1)(1-F(z_k))^{N-k}}{(1-F(x))^{N-k+1}} dF(z_k). \end{aligned}$$

Likewise, since $y^{\frac{N-k-1}{N-k+1}}$ is concave, repeating the same argument yields

$$\begin{aligned} J_{k+1}^\alpha(z_k)^{\frac{N-k-1}{N-k+1}} &\geq E\left[\left(J_{k+2}^\alpha(Z_{k+1}^{(N)}) \right)^{\frac{N-k-2}{N-k+1}} | Z_{k+1}^{(N)} > x > Z_k^{(N)}] \right] \\ &= \int_{z_k}^{\bar{x}} J_{k+2}^\alpha(z_{k+1})^{\frac{N-k-2}{N-k+1}} \frac{(N-k)(1-F(z_{k+1}))^{N-k-1}}{(1-F(z_k))^{N-k}} dF(z_{k+1}). \end{aligned}$$

Substituting this expression into the prior one, and simplifying yields

$$J_k^\alpha(x)^{\frac{N-k}{N-k+1}} \geq \frac{(N-k+1)!}{(N-k-1)!} \int_x^{\bar{x}} \int_{z_k}^{\bar{x}} J_{k+2}^\alpha(z_{k+1})^{\frac{N-k-2}{N-k+1}} \frac{(1-F(z_{k+1}))^{N-k-1}}{(1-F(x))^{N-k+1}} dF(z_{k+1}) dF(z_k).$$

Changing the order of integration, the right hand side is

$$\begin{aligned}
&\frac{(N-k+1)!}{(N-k-1)!}\int_x^{\bar{x}}\int_x^{z_{k+1}} J^{\alpha}_{k+2}(z_{k+1})^{\frac{N-k-2}{N-k+1}}\frac{(1-F(z_{k+1}))^{N-k-1}}{(1-F(x))^{N-k+1}}dF(z_k)dF(z_{k+1}) \\
= \ &\frac{(N-k+1)!}{(N-k-1)!}\int_x^{\bar{x}} J^{\alpha}_{k+2}(z_{k+1})^{\frac{N-k-2}{N-k+1}}\frac{[F(z_{k+1})-F(x)](1-F(z_{k+1}))^{N-k-1}}{(1-F(x))^{N-k+1}}dF(z_{k+1}).
\end{aligned}$$

This last integral is just an expectation, taken with respect to the density of $Z^{(N)}_{k+1}$ conditional on $Z^{(N)}_k > x > Z^{(N)}_{k-1}$. Thus

$$J^{\alpha}_k(x)^{\frac{N-k}{N-k+1}} \geq E[J^{\alpha}_{k+2}(Z^{(N)}_{k+1})^{\frac{N-k-2}{N-k+1}}|Z^{(N)}_k > x > Z^{(N)}_{k-1}].$$

Continuing in this fashion, we obtain

$$J^{\alpha}_k(x)^{\frac{N-k}{N-k+1}} \geq E[J^{\alpha}_{N-1}(Z^{(N)}_{N-2})^{\frac{1}{N-k+1}}|Z^{(N)}_k > x > Z^{(N)}_{k-1}]. \tag{21}$$

Since

$$f_{N-2}(z_{N-2}|Z^{(N)}_k > x > Z^{(N)}_{k-1}) = \frac{(N-k+1)!}{(N-k-2)!2!}\frac{[F(z_{N-2})-F(x)]^{N-k-2}[1-F(z_{N-2})]^2}{[1-F(x)]^{N-k+1}}f(z_{N-2}),$$

the right hand side of (21) can be written as

$$\int_x^{\bar{x}} J^{\alpha}_{N-1}(z_{N-2})^{\frac{1}{N-k+1}}\frac{(N-k+1)!}{(N-k-2)!2!}\frac{[F(z_{N-2})-F(x)]^{N-k-2}[1-F(z_{N-2})]^2}{[1-F(x)]^{N-k+1}}dF(z_{N-2}). \tag{22}$$

By a now-standard argument, we have

$$\begin{aligned}
J^{\alpha}_{N-1}(x)^{\frac{1}{N-k+1}} &= \left(E[e^{-\alpha Z^{(N)}_{N-1}}|Z^{(N)}_{N-1} > x > Z^{(N)}_{N-2}]\right)^{\frac{1}{N-k+1}} \\
&\geq E[e^{-\frac{\alpha}{N-k+1}Z^{(N)}_{N-1}}|Z^{(N)}_{N-1} > x > Z^{(N)}_{N-2}] \\
&= \int_x^{\bar{x}} e^{-\frac{\alpha}{N-k+1}z_{N-1}}\frac{2(1-F(z_{N-1}))}{[1-F(x)]^2}dF(z_{N-1}).
\end{aligned}$$

Replacing $J_{N-1}^{\alpha}(z_{N-2})^{\frac{1}{N-k+1}}$ in (22) with the right hand side of this expression with $x = z_{N-2}$ yields

$$\frac{(N-k+1)!}{(N-k-2)!}\int_x^{\bar{x}}\int_{z_{N-2}}^{\bar{x}} e^{-\frac{\alpha}{N-k+1}z_{N-1}}\frac{(1-F(z_{N-1}))\left[F(z_{N-2})-F(x)\right]^{N-k-2}}{\left[1-F(x)\right]^{N-k+1}}dF(z_{N-1})dF(z_{N-2}).$$

Changing the order of integration, this expression can be written as

$$\frac{(N-k+1)!}{(N-k-2)!}\int_x^{\bar{x}}\int_{x}^{z_{N-1}} e^{-\frac{\alpha}{N-k+1}z_{N-1}}\frac{(1-F(z_{N-1}))\left[F(z_{N-2})-F(x)\right]^{N-k-2}}{\left[1-F(x)\right]^{N-k+1}}dF(z_{N-2})dF(z_{N-1}).$$

Since

$$\begin{aligned}\int_x^{z_{N-1}}\left[F(z_{N-2})-F(x)\right]^{N-k-2}f(z_{N-2})dz_{N-2} &= \frac{1}{N-k-1}\left[F(z_{N-2})-F(x)\right]^{N-k-1}\Big|_x^{z_{N-1}} \\ &= \frac{1}{N-k-1}\left[F(z_{N-1})-F(x)\right]^{N-k-1},\end{aligned}$$

the expression further simplifies to

$$\begin{aligned}&\frac{(N-k+1)!}{(N-k-1)!}\int_x^{\bar{x}} e^{-\frac{\alpha}{N-k+1}z_{N-1}}\frac{(1-F(z_{N-1}))\left[F(z_{N-1})-F(x)\right]^{N-k-1}}{\left[1-F(x)\right]^{N-k+1}}dF(z_{N-1}) \\ =\ & E[e^{-\frac{\alpha}{N-k+1}Z_{N-1}^{(N)}}|Z_k^{(N)}>x>Z_{k-1}^{(N)}].\end{aligned}$$

Thus we have established that

$$\frac{1}{\alpha}\ln(J_k^{\alpha}(x)^{\frac{N-k}{N-k+1}}) \geq \frac{1}{\alpha}\ln\left(E[e^{-\frac{\alpha}{N-k+1}Z_{N-1}^{(N)}}|Z_k^{(N)}>x>Z_{k-1}^{(N)}]\right).$$

The round k equilibrium bidding function therefore is bounded above by

$$\beta_k^{\alpha}(x;\mathbf{p}_{k-1}) \leq \frac{N-k}{N-k+1}p_{k-1} - \frac{1}{\alpha}\ln\left(E[e^{-\frac{\alpha}{N-k+1}Z_{N-1}^{(N)}}|Z_k^{(N)}>x>Z_{k-1}^{(N)}]\right).$$

We show that $\lim_{\alpha\to\infty} -\frac{1}{\alpha}\ln(E[e^{-\frac{\alpha}{N-k+1}Z_{N-1}^{(N)}}|Z_k^{(N)}>x>Z_{k-1}^{(N)}]) = \frac{x}{N-k+1}$, i.e.,

$$\lim_{\alpha\to\infty} -\frac{1}{\alpha}\ln\left(\int_{x}^{\bar{x}} e^{-\frac{\alpha}{N-k+1}t}h(t)dt\right) = \frac{x}{N-k+1},$$

where

$$h(t) = \frac{(N-k+1)!}{(N-k-1)!}\frac{[1-F(t)]\,[F(t)-F(x)]^{N-k-1}}{(1-F(x))^{N-k+1}}f(t).$$

Then $\lim_{\alpha\to\infty}\beta_k^{\alpha}(x;\mathbf{p}_{k-1}) \leq \frac{N-k}{N-k+1}p_{k-1} + \frac{x}{N-k+1}$. By Proposition 4, $\beta_k^{\alpha}(x;\mathbf{p}_{k-1}) \geq \frac{N-k}{N-k+1}p_{k-1} + \frac{x}{N-k+1}$ for $\alpha > 0$. Hence we have

$$\lim_{\alpha\to\infty}\beta_k^{\alpha}(x;\mathbf{p}_{k-1}) = \frac{N-k}{N-k+1}p_{k-1} + \frac{x}{N-k+1}.$$

We now establish the above limit. Applying l'Hopital's rule, this limit equals

$$\lim_{\alpha\to\infty}\frac{1}{N-k+1}\frac{\int_{x}^{\bar{x}} te^{-\frac{\alpha}{N-k+1}t}h(t)dt}{\int_{x}^{\bar{x}} e^{-\frac{\alpha}{N-k+1}t}h(t)dt}.$$

It is sufficient to show that

$$\lim_{\alpha\to\infty}\frac{\int_{x}^{\bar{x}} te^{-\alpha t}h(t)dt}{\int_{x}^{\bar{x}} e^{-\alpha t}h(t)dt} = x.$$

Clearly

$$\frac{\int_{x}^{\bar{x}} te^{-\alpha t}h(t)dt}{\int_{x}^{\bar{x}} e^{-\alpha t}h(t)dt} \geq \frac{x\int_{x}^{\bar{x}} e^{-\alpha t}h(t)dt}{\int_{x}^{\bar{x}} e^{-\alpha t}h(t)dt} = x.$$

Also, for any $\Delta > 0$ small

$$\begin{aligned}
\lim_{\alpha\to\infty} \frac{\int\limits_x^{\bar{x}} te^{-\alpha t} h(t)dt}{\int\limits_x^{\bar{x}} e^{-\alpha t} h(t)dt} &\leq \lim_{\alpha\to\infty} \frac{\int\limits_x^{x+\Delta} te^{-\alpha t} h(t)dt + \int\limits_{x+\Delta}^{\bar{x}} te^{-\alpha t} h(t)dt}{\int\limits_x^{x+\Delta} e^{-\alpha t} h(t)dt} \\
&\leq \lim_{\alpha\to\infty} \frac{(x+\Delta) \int\limits_x^{x+\Delta} e^{-\alpha t} h(t)dt + e^{-\alpha(x+\Delta)} \int\limits_{x+\Delta}^{\bar{x}} \bar{x} h(t)dt}{\int\limits_x^{x+\Delta} e^{-\alpha t} h(t)dt} \\
&= x + \Delta + \lim_{\alpha\to\infty} \frac{\int\limits_{x+\Delta}^{\bar{x}} \bar{x} h(t)dt}{\int\limits_x^{x+\Delta} e^{\alpha(x+\Delta-t)} h(t)dt}.
\end{aligned}$$

Since $h(t) > 0$ for $t \in [x, x+\Delta]$, then $\lim_{\alpha\to\infty} \int\limits_x^{x+\Delta} e^{\alpha(x+\Delta-t)} h(t)dt = \infty$ for any $\Delta > 0$. Hence we have shown that for any $\Delta > 0$ we have

$$x \leq \lim_{\alpha\to\infty} \frac{\int\limits_x^{\bar{x}} te^{-\alpha t} h(t)dt}{\int\limits_x^{\bar{x}} e^{-\alpha t} h(t)dt} \leq x + \Delta.$$

Thus

$$\lim_{\alpha\to\infty} \frac{\int\limits_x^{\bar{x}} te^{-\alpha t} h(t)dt}{\int\limits_x^{\bar{x}} e^{-\alpha t} h(t)dt} = x. \ \square$$

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