Mimic Martingales in Sequential Auctions

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Abstract

In the equilibrium of a game, no player has an incentive to unilaterally deviate from equilibrium play. At the same time, players may have no positive incentive to follow equilibrium when every other player follows equilibrium, e.g., as in a mixed-strategy Nash equilibrium. This paper concerns the incentives of players to follow equilibrium in sequential auctions and bargaining games in which winning bids/compensations are disclosed. It shows for theses games that a player obtains his equilibrium payoff for a large class of strategies different from his equilibrium strategy. These deviations from equilibrium, while costless to the player, harm the seller in an auction. These results suggest that it may be difficult for players to learn to play equilibrium and, if reached, for play to remain at equilibrium. For the auction designer, disclosing winning bids may be harmful to the seller.

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1 Introduction

> *A dealer flips over cards in a standard well-shuffled deck, one-by-one, starting from the top. Before turning over each card, he ask you whether you want to bet that the card is red. If you bet, then you win \$1 if the card is red, you lose \$1 if it is black, and the game ends. If you choose not to bet, then the dealer will show you the card, discard it, and go to the next card. You must bet once. If you have not bet prior to the last card remaining in the deck, then you are obligated to bet that the last card is red.*[1] – **Polya's Red or Black Game**

Should you play this game? Sure! The game is fair – You can always bet immediately and receive fair odds. Can you find a betting strategy that does better than zero in expectation? Can you find a betting strategy that does worse? Interestingly, the answer to both of these questions is "no." Since the fraction of red cards in the deck as cards are turned over forms a martingale, it turns out that any betting strategy you follow is optimal and provides you with an expected payoff of zero.[2]

This paper is about incentives to play equilibrium in sequential auctions and sequential bargaining games. While, by definition, no bidder has a profitable unilateral deviation in an equilibrium, this does not imply a bidder is made worse off by deviating. We show a robust feature of sequential auctions is that bidders have weak incentives to play equilibrium when every other bidder plays equilibrium. We establish, in particular, that there is a Red or Black game embedded in the equilibrium of common sequential auctions, and we identity a large class of non-equilibrium strategies that all yield the

[1] This game is a simple variant of Polya's Urn Problem where marbles are taken from an urn without replacement. See Fuller (1968).

[2] An introduction to martingales is provided in the next section.

same expected payoff to a bidder as his equilibrium strategy.

We first study the sequential Dutch auction of K units in which each bidder demands a single unit. In the auction, units are sold one at a time, with the winning bid revealed at each round. Proposition 1 identifies the unique symmetric equilibrium in increasing and differentiable strategies, both when units sold are homogeneous (as most commonly studied) and when they are heterogenous.

In a sequential Dutch auction there are many ways in which a bidder might deviate from equilibrium. We focus on a class of deviations that we call "mimic" deviations. Consider a bidder at round t for which the price has reached his equilibrium bid. If he obeys equilibrium, then he purchases a unit at that price and he exits the auction. Suppose instead that he does not purchase a unit, he observes the winning bid at round t, and he bids optimally at round $t+1$. We show that the optimal bid at round $t+1$ is to bid as through his value were the value of the round t winner, i.e., he "mimics" the round $t+1$ equilibrium bid of the prior-round winner. We call this a 1-round mimic deviation. Mimic deviations can be of any length – in an m-round mimic deviation the bidder does not purchase a unit at rounds t through $t+m-1$, and in round $t+m$ he mimics the round $t+m$ bid of the prior round winner. Proposition 2 shows that, when units are homogeneous, then the sequence of payoffs obtained from mimic deviations of different lengths forms a martingale. An immediate consequence of this result is that a bidder obtains his equilibrium expected payoff by following any mimic deviation. When units are heterogeneous, the sequence of payoffs obtained from mimic deviations of different lengths forms a supermartingale, i.e., expected payoffs decline.

Sequential first and second-price sealed-bid auctions of homogenous units share the feature that bidders have a wide variety of non-equilibrium strategies which yield their equilibrium payoff. Mimic deviations in sequential sealed-bid auctions have a slightly different structure. Consider again a bidder in round t. In an m-round mimic deviation, the bidder refrains from bidding until round $t+m$. In round $t+m$, if the value of the prior-round winner is below his own value then, as before, he mimics the round $t+m$ equilibrium bid of the prior-round winner. Otherwise, he bids according to equilibrium. Proposition 3 shows that sequence of payoffs obtained from mimic deviations of different lengths in a sequential first-price sealed-bid auction forms a martingale.

Players also have flat incentives to play equilibrium in sequential bargaining games. Consider, for example, the problem of dissolving a partnership or allocating an item among the members of a bidding ring. A bargaining game has to determine which partner or ring member wins the partnership or item, and how much compensation the winner pays to each of the other partners or ring members. Van Essen and Wooders (2016) study a sequential compensation auction for dissolving partnerships. Proposition 4 shows that the sequence of payoffs obtained by mimic deviations in this auction forms a martingale.

Our results thus establish that sequences of mimic deviations in common multi-unit auctions or in dynamic bargaining problems yield martingales in payoffs. A powerful result, known as the Martingale Stopping Theorem, allows us to substantially expand the set of strategic deviations that are costless to a bidder.[3] It tells us that *any* rule that a bidder uses to start, continue, or stop a mimic deviation yields the same expected payoff to the bidder. The bidder's decision might be randomized or it might depend on the history of prices realized in prior rounds of a multi-unit auction or the compensations paid to exiting partners in an auction for dissolving partnerships.

[3] Ross (1996) is a standard reference on stochastic processes. Chapter 6 in this text presents results concerning martingales. The Martingale Stopping Theorem (p. 300), in particular, plays a central role in our strategic interpretation of the martingale results.

In the Online Appendix, we study more complex generalized mimic deviations in sequential Dutch auctions in which a bidder, rather than withdrawing entirely from the auction in a round, shades his bid below its equilibrium value. We show that a sequence of generalized mimic deviations of this kind also yields a martingale.[4] Taken together, our results establish that bidders have flat incentives to play equilibrium in sequential auctions, when the winning bid at each round is disclosed.

Related Literature

Flat Payoffs and the "Metric Wars"

Equilibrium in auctions and games is often characterized by the solution to a system of first order conditions, with each player's payoff flat at their (unique) best response bid/action. The shape of payoffs in the neighborhood of equilibrium is important since it affects the incentives of players to learn and play equilibrium. In the early experimental auction literature, this issue lead to a controversy known as the "metrics wars." Harrison (1989) argued that the deviations from Bayes Nash equilibrium risk-neutral bidding reported by Cox, Smith, and Walker (1983) were inconsequential in monetary terms, with subjects forgoing in expectation merely pennies relative to their equilibrium earnings, and thus these deviations did not warrant rejection of the risk-neutral bidding model. Kagel and Roth (1992) provides evidence supporting this view, showing for first-price sealed-bid auctions that the difference of observed and equilibrium bids, as a proportion of equilibrium bids, is far larger for bidders with low values, whose payoffs are less sensitive to their bids given their low probability of winning in equilibrium.[5]

[4] We study mimic deviations in which a bidder withdraws from bidding for one or more rounds, rather than these more general deviations, in order to simplify the exposition.

[5] See Table 1 of Kagel and Roth (1992), which reports data from Dyer, Kagel, and Levin (1989) and Battalio, Kogut, and Meyer (1990).

Our results show that payoffs in sequential auctions are not just flat at a bidder's equilibrium bid, but are in fact constant for all bids less than the equilibrium bid, so long as the bidder bids optimally at subsequent rounds. In this respect, equilibrium resembles mixed-strategy Nash equilibrium in normal form games, which has the property that a player's payoff is constant for all strategies which have the same support as the player's equilibrium mixture. In a mixed-strategy Nash equilibrium, a player has no positive incentive to mix in the equilibrium proportions. And, indeed, mixed-strategy Nash equilibrium tends to perform poorly in laboratory tests: subjects fail to mix in the equilibrium proportions and they exhibit serial correlation in their action choices.[6] Likewise, for the same reasons we might expect equilibrium to perform poorly in sequential auctions.

Price Transparency

When selling multiple units of a good via auction the seller has several decisions to make: What auction format to use? Should there be a reservation price? And, if the seller chooses a sequential auction, then what information should be revealed after each unit is sold? Cason, Kannan, and Siebert (2011) studies, both theoretically and experimentally, the consequences of disclosing only the winning bid versus disclosing all bids.[7] Bergemann and Horner (2018) considers an infinite sequence of first-price sealed-bid auctions in which each bidder's private value is constant across auctions. It characterizes equilibrium for three information disclosure regimes: no bids are observed, all bids are observed, and only the winner's bid is observed. Both papers concern how the information disclosure regime affects equilibrium bidding.

[6] See Chapter 3 of Camerer (2003) for a very nice survey of the experimental literature on mixed-strategy equilibrium.

[7] They consider a two-round first-price sealed-bid procurement auction. Each bidder is privately informed of his cost of supplying a unit, and this cost is the same at each round.

The present paper studies equilibrium in sequential auctions when bidders have single-unit demands and winning bids are disclosed. Unlike the papers above, equilibrium bids in a sequential first-price sealed-bid auction are the same, whether or not the winning bid is disclosed at the end of each round. The disclosure rule, however, affects the bidders' incentives to follow equilibrium: If the winning bid is not disclosed, then a bidder who deviates from equilibrium obtains less than his equilibrium payoff. In the present paper we show that if the winning bid is disclosed, then a bidder has a wide class of deviations from equilibrium for which he obtains his equilibrium payoff. Such deviations, while harmless to the deviating bidder, reduce seller revenue. To our knowledge, this consideration in auction design has not been studied.

Martingales in Sequential Auctions

In an independent private values setting, where multiple identical items are sold sequentially and bidders have unit demands, Weber (1983) established that the sequence of equilibrium prices in sequential first and second-price sealed-bid auctions is a martingale (also see Milgrom and Weber (2000)). In other words, conditional on the price of the last unit sold (and all prior prices), the expected equilibrium price of the next unit equals the price of the last unit – on average, prices have no drift, either up or down. This result obtains when all bidders play equilibrium.

In this paper, by contrast, we consider the payoff consequences to a single bidder who deviates from equilibrium in a sequential auction, and examine the sequence of random payoffs the bidder could obtain by different length mimic deviations. In the standard selling environment, the sequence of deviation payoffs obtained in the sequential Dutch auction is a supermartingale when units are heterogeneous and a martingale when units are homogeneous. The same approach extends to bargaining games as well: the sequence of mimic deviations in the dynamic partnership dissolution auction forms a martingale. The key strategic implication of these martingale results is a player's incentives to follow equilibrium are flat.

Declining Price Anomaly

The "no-drift" feature of equilibrium prices in first and second-price sequential auctions is testable, but empirical studies tend to reject this hypothesis. Ashenfelter (1989) and McAfee and Vincent (1993), for example, find that prices decline in sequential auctions of homogenous lots of wine. This empirical regularity is known as the declining price anomaly.

Several models have been proposed to explain the anomaly. McAffee and Vincent (1993), Mezzeti (2011), and Hu and Zou (2015) examine the role of risk aversion in creating declining prices. Ghosh and Liu (2021) provide an explanation based on ambiguity aversion. One can show that when units are *heterogeneous* and bidders are risk neutral, then equilibrium prices in sequential Dutch auctions form a supermartingale – i.e., they decline in expectation.

2 Martingales – A Primer

In this section we provide basic definitions and results for martingales that are used in the paper. We illustrate the main ideas using the Red or Black game.

A stochastic process is a sequence of random variables. We will be interested in a type of stochastic processes known as a martingale.

Definition: Suppose $\{\Pi_t\}_{t=1}^T$ is a stochastic process. Then $\{\Pi_t\}_{t=1}^T$ forms a *martingale* if $E\left[|\Pi_t|\right] < \infty$ for all t, and

$$E\left[\Pi_{t+1}|\Pi_1, \ldots, \Pi_t\right] = \Pi_t.$$

Example 0 shows that the sequence $W_1, \ldots, W_{52}$ of proportions of red cards in the Red or Black game is a martingale.

Example 0: In the Red or Black game, let W_t be the proportion of red cards at time t before the t-th card is flipped. This defines a sequence of random variables, i.e., a stochastic process, $W_1, \ldots, W_{52}$, where $W_1 = 1/2$. We show that the sequence $W_1, \ldots, W_{52}$ is a martingale, i.e., for each $t \in \{1, \ldots, 51\}$, we have

$$E\left[W_{t+1} | W_1, \ldots, W_t\right] = W_t.$$

Let w_t denote a realization of W_t, where $w_t = r/(r+b)$ and r is the number of red cards and b is the number of black cards before the t-th card is flipped. Then

$$W_{t+1} = \begin{cases} \frac{r-1}{r-1+b} & \text{if } W_t = \frac{r}{r+b} \text{ and a red card is drawn at time } t \\ \\ \frac{r}{r+b-1} & \text{if } W_t = \frac{r}{r+b} \text{ and a black card is drawn at time } t, \end{cases}$$

and so the distribution of W_{t+1} is determined by the realized value of W_t, i.e.,

$$E\left[W_{t+1} | W_1, \ldots, W_t\right] = E\left[W_{t+1} | W_t\right].$$

The expected proportion of red cards before the $t+1$-st card is flipped, given that $W_t = w_t$, is

$$\begin{aligned} E\left[W_{t+1} | W_t = w_t\right] &= \left(\frac{r-1}{r+b-1}\right)\left(\frac{r}{r+b}\right) + \left(\frac{r}{r+b-1}\right)\left(\frac{b}{r+b}\right) \\ &= \frac{r}{r+b} \\ &= w_t, \end{aligned}$$

and so

$$E\left[W_{t+1} | W_1, \ldots, W_t\right] = E\left[W_{t+1} | W_t\right] = W_t.$$

This establishes that the sequence of proportions of red cards forms a martingale. $\triangle$

Definition: The positive integer-valued random variable S is said to be a *random time* for the process $\{\Pi_t\}_{t=1}^T$ if the event $\{S = t\}$ is determined by

$\Pi_1, \ldots, \Pi_t$. Furthermore, if $\sum_{t=1}^{T} \Pr(\{S = t\}) = 1$, then we say S defines a *stopping time*.

In the paper we will exploit the Martingale Stopping Theorem which, for completeness, we state now.

Martingale Stopping Theorem: Let $\{\Pi_t\}_{t=1}^{T}$ be a martingale and suppose S is a stopping time for this process. Then $E[\Pi_S] = E[\Pi_1]$.

A betting strategy σ in the Red or Black game identifies a probability of betting at each t as a function of the history of cards flipped. More formally, betting strategy is a sequence $\{\sigma_t\}_{t=1}^{51}$ such that $\sigma_t : \{r, b\}^{t-1} \to \triangle\{bet, wait\}$ for each t, and $\sigma_{52}(\circ) = bet$. A betting strategy σ identifies a stopping time.

We are interested in the probability of winning as a function of σ. Let W^σ be the random variable which is the fraction of red cards at the time the player bets/stops, when following σ. Since $W_1, \ldots, W_{52}$ is a martingale and the rules of the game require that the player bet once, then the *Martingale Stopping Theorem* implies that $E[W^\sigma] = W_1 = 1/2$ for *any* betting strategy σ. Thus any betting strategy leads to a 1/2 probability of winning (and an expected payoff of zero).

The lesson of this example is that we don't need to spend any time learning a good way to play the Red or Black game. Every betting strategy is optimal! While this is good news for the gambler, it is concerning from a forecaster's viewpoint since it tells us that there is no reason for us to expect the play of one type of betting strategy over another. We shall see that there is a Red or Black game embedded in the equilibrium of common sequential auctions. In particular, there are many bidding strategies which give a bidder his equilibrium payoff.

3 Mimic Martingales in Sequential Auctions

We study sequential first-price and second-price auctions with N risk-neutral bidders, each with unit demand, and K units, where $K < N$. Bidders' values are independently and identically distributed according to cumulative distribution function F with support $[0, \bar{x}]$, where $\bar{x} < \infty$ and $f \equiv F'$ is continuous and positive on $[0, \bar{x}]$. A bidder with value x has utility $\alpha_i x$ for unit i, where α_i is the "inherent" value of unit i. Order the units so that $\alpha_1 \geq \ldots \geq \alpha_K$. This setting includes the special cases when units are homogeneous, i.e., $\alpha_1 = \cdots = \alpha_K = 1$, and when units are fully heterogeneous, i.e., $\alpha_1 > \alpha_2 > \cdots > \alpha_K$. Auctions of the later kind arise when there is a common ranking by the bidders of the items for sale. For example, when allocating priority service all the bidders prefer higher priority, being served sooner rather than later; when selling ad positions on a webpage, all the advertisers prefer their ads to appear higher on the webpage than lower.

Let $X_1, \ldots, X_N$ be N independent draws from F, and let the order statistics $Z_1^{(N)}, \ldots, Z_N^{(N)}$ be a rearrangement of the X_i's such that $Z_1^{(N)} \leq Z_2^{(N)} \leq \ldots \leq Z_N^{(N)}$. We write $z_j^{(N)}$ for the realized value of $Z_j^{(N)}$, sometimes suppressing N if it is obvious from the context.

We consider first the sequential Dutch auction in which units are sold one at a time, from best (unit 1) to worst (unit K). In the t-th round of the auction, the price descends continuously from $\alpha_t \bar{x}$. At any time, a bidder can accept the price. If a bidder accepts a price p_t, then he wins unit t, he pays p_t, and he exits the auction. The auction begins for the best remaining unit, and proceeds in this fashion until all the units are sold. A bidder with value x who wins at round t obtains a payoff of $\alpha_t x - p_t$. The sale price at each round is observed by all bidders.[8] When units are homogenous, then

[8] This auction is ancient. A description of the basic rules of the auction appears in Herodotus' Histories (430 BC) as a method for the determination of dowries.

the auction reduces to the standard sequential Dutch auction.

Proposition 1 provides the risk neutral equilibrium bidding strategies for the sequential Dutch auction. When units are homogenous, the equilibrium bid function is well known, e.g., page 291 of Krishna (2010).

Proposition 1: *The unique symmetric equilibrium in increasing and differentiable bidding strategies for the sequential Dutch auction is given by* $\beta(x) = (\beta_t(x))_{t=1}^K$ *where, for each* $t = 1, \ldots, K$, *we have*

$$\beta_t(x) = E\left[(\alpha_t - \alpha_{t+1})\, Z_{N-t}^{(N-1)} + \beta_{t+1}(Z_{N-t}^{(N-1)}) | Z_{N-t}^{(N-1)} < x < Z_{N-t+1}^{(N-1)}\right],$$

and where $\beta_{K+1}(x) \equiv 0$ *and* $\alpha_{K+1} \equiv 0$. *Equivalently,*

$$\beta_t(x) = \sum_{j=t}^{K} E\left[(\alpha_j - \alpha_{j+1})\, Z_{N-j}^{(N-1)} | Z_{N-t}^{(N-1)} < x < Z_{N-t+1}^{(N-1)}\right].$$

Note that β is also the equilibrium of the first-price sealed-bid auction since the auction is strategically equivalent to the sequential Dutch auction.

Mimic Deviations in The Sequential Dutch Auction

Consider a bidder with value x, in round $t < K$, at the moment the price reaches his equilibrium bid. The bidder has a choice. If he plays equilibrium and accepts the price, then he wins an item and obtains a payoff of $\Pi_t(x) = \alpha_t x - \beta_t(x)$.

The bidder can instead deviate from equilibrium. We consider the specific deviation in which the bidder does not accept the price, he observes the m prices realized in rounds $t, \ldots, t+m-1$, for $m \leq K - t$, and then he bids optimally at round $t + m$. Since the equilibrium bid function is increasing, from the price $p_t = \beta_t(Z_{N-t}^{(N-1)})$ he infers the value $z_{N-t}^{(N-1)}$ of the bidder winning in the t-th round, and so on, until, from the price $p_{t+m-1} = \beta_{t+m-1}(Z_{N-(t+m-1)}^{(N-1)})$ he infers $z_{N-(t+m-1)}^{(N-1)}$. The deviating bidder's optimal bid at round $t + m$ is $\beta_{t+m}(z_{N-(t+m-1)}^{(N-1)})$, i.e., he mimics the round $t + m$

equilibrium bid of the previous-round's winner. We call such a deviation an "m-round mimic" deviation.[9] By following this deviation he wins an item and obtains a payoff of $\alpha_{t+m}x - \beta_{t+m}(z^{(N-1)}_{N-(t+m-1)})$.

Let $\Pi^m_t(x)$ denote the random variable representing the payoff at round t to a bidder with value x from a m-round mimic deviation that starts at round t, where $\Pi^0_t(x) \equiv \Pi_t(x)$ is the bidder's equilibrium payoff. The sequence of random variables $\Pi^m_t(x)$, for $m \in \{0, \ldots, K-t\}$, defines a stochastic process.

Proposition 2 states that when units are homogenous, then the stochastic process $\{\Pi^m_t(x)\}^{K-t}_{m=0}$ is a martingale.

Proposition 2: *In a sequential Dutch auction of K homogeneous units, for any $t \leq K$ the stochastic process $\{\Pi^m_t(x)\}^{K-t}_{m=0}$ generated by a sequence of m-round mimic deviations by a bidder with value x is a martingale.*

An implication of Proposition 2 is that a bidder has many non-equilibrium strategies which achieve his equilibrium payoff, when every other bidder follows equilibrium. In particular, $E[\Pi^0_t(x)] = E[\Pi^1_t(x)] = \cdots = E[\Pi^{K-t}_t(x)]$. Example 1 illustrates this result.

Example 1: Consider a sequential Dutch auction of $K = 2$ homogeneous units with $N = 3$ bidders. Values are distributed $U\,[0, 1]$. The equilibrium bidding strategy is

$$\beta_1(x) = \frac{1}{3}x$$

and

$$\beta_2(x) = \frac{1}{2}x.$$

Suppose Bidder 1's value is x and the price reaches $\frac{1}{3}x$ (and so Bidder 1 has

[9] In fact, to execute the m-round mimic deviation the bidder only needs to infer $z^{(N-1)}_{N-(t+m-1)}$ from p_{t+m-1}. However, more complicated deviations, that we describe later, may condition on the whole sequence of prices.

the highest value, i.e., $Z_3^{(3)} = x$). If he accepts the price, then his payoff is

$$x - \frac{1}{3}x = \frac{2}{3}x.$$

If, instead, he follows a 1-round mimic deviation then he observes $Z_2^{(3)}$ and he bids $\beta_2(Z_2^{(3)})$ at round 2. His expected payoff is

$$E\left[x - \frac{1}{2}Z_2^{(3)}|Z_3^{(3)} = x\right] = x - \frac{1}{2}E\left[Z_2^{(3)}|Z_3^{(3)} = x\right] = \frac{2}{3}x.$$

This is the same as his actual payoff to playing equilibrium at round 1. $\triangle$

While a mimic deviation has no effect on the payoff of the deviating bidder, it has consequences for the seller and the other bidders. Following Example 1, suppose Bidder 1 has value x and the price reaches $x/3$. If Bidder 1 obeys equilibrium, then the sales price at round 1 is $x/3$ and at round 2 is $z_2^{(3)}/2$. If instead Bidder 1 executes a 1-round mimic deviation, then the sale price at round 1 is only $z_2^{(3)}/3$. At round 2, Bidder 1 mimics the bidder with value $z_2^{(3)}$, and he wins a unit at price $z_2^{(3)}/2$. The mimic deviation, which is costless in expectation for Bidder 1, reduces seller revenue in the first round and has no effect on revenue in the second round. The bidder with value $z_2^{(3)}$ is better off since he buys a unit at price $z_2^{(3)}/3$ rather than $z_2^{(3)}/2$.

If units are heterogenous, then mimic deviations do not generate a martingale, as the next example shows.

Example 2: Consider a sequential Dutch auction of $K = 2$ heterogeneous units, where $\alpha_1 > \alpha_2$, with $N = 4$ bidders. Values are distributed $U[0,1]$. By Proposition 1, the equilibrium bidding strategy is

$$\begin{aligned} \beta_1(x) &= E\left[(\alpha_1 - \alpha_2) Z_{N-1}^{(N-1)}|Z_{N-1}^{(N-1)} < x\right] + E\left[\alpha_2 Z_{N-2}^{(N-1)}|Z_{N-1}^{(N-1)} < x\right] \\ &= \frac{3}{4}(\alpha_1 - \alpha_2) x + \frac{1}{2}\alpha_2 x, \end{aligned}$$

and

$$\beta_2(x) = E\left[\alpha_2 Z_{N-2}^{(N-1)}|Z_{N-2}^{(N-1)} < x < Z_{N-1}^{(N-1)}\right] = \frac{2}{3}\alpha_2 x.$$

Suppose Bidder 1's value is $x > 0$ and the price reaches $\beta_1(x)$ (and so $Z_4^{(4)} = x$). If he accepts the price, then his payoff is

$$\alpha_1 x - \frac{3}{4}(\alpha_1 - \alpha_2)x - \frac{1}{2}\alpha_2 x = \frac{1}{4}(\alpha_1 + \alpha_2)x.$$

If, instead, he follows a 1-round mimic deviation, then he observes $Z_3^{(4)}$ and he bids $\beta_2(Z_3^{(4)}) = \frac{2}{3}\alpha_2 Z_3^{(4)}$ at round 2. His expected payoff is

$$E\left[\alpha_2 x - \frac{2}{3}\alpha_2 Z_3^{(4)} | Z_4^{(4)} = x\right] = \alpha_2 x - \alpha_2 \frac{2}{3}(\frac{3}{4}x) = \frac{1}{2}\alpha_2 x.$$

Since $\alpha_1 > \alpha_2$, then $\frac{1}{4}(\alpha_1 + \alpha_2)x > \frac{1}{2}\alpha_2 x$ and so the 1-round mimic deviation lowers the bidder's expected payoff. $\triangle$

Remark 1: Suppose that there are H high-quality units and L low-quality units, where $\alpha_1 = \ldots = \alpha_H > \alpha_{H+1} = \ldots = \alpha_{H+L}$ and $H + L = K$. Then a bidder will have a mimic deviation at every round t except for $t = H$ and $t = H + L$, i.e., for every round except at rounds where the last unit of a given quality is sold.

Mimic Deviations in Sealed Bid Auctions

Sequential first and second-price sealed-bid auctions also have the property that sequences of mimic deviations generate martingales in payoffs. Mimic deviations in these auctions have a different structure than in the sequential Dutch auction. Let β be the equilibrium bid function in a sequential sealed-bid auction, either first or second price. A bidder with value x makes an m-round mimic deviation at round t by bidding zero at rounds t through $t + m - 1$. From the prices $p_t, \ldots, p_{t+m-1}$ he infers the values $z_{N-t}^{(N-1)}, \ldots, z_{N-(t+m-1)}^{(N-1)}$. There are two cases that govern his optimal bid at round $t+m$: If the inferred value at round $t+m-1$ is smaller than his own, i.e., if $x > z_{N-(t+m-1)}^{(N-1)}$, then at round $t + m$ the deviating bidder mimics the round $t+m$ bid of the previous round winner, i.e., he bids $\beta_{t+m}(z_{N-(t+m-1)}^{(N-1)})$.

In this case he wins a unit. Otherwise, if $x < z^{(N-1)}_{N-(t+m-1)}$, then at round $t+m$ the bidder makes his equilibrium bid $\beta_{t+m}(x)$ and he continues to follow equilibrium thereafter. Let $\{\Pi^m_t(x)\}^{K-t}_{m=0}$ be the sequence of payoffs obtained by mimic deviations of length $m \in \{0, \ldots, K-t\}$.

Proposition 3 establishes that stochastic process $\{\Pi^m_t(x)\}^{K-t}_{m=0}$ is a martingale.

Proposition 3: *In a sequential first-price or second-price sealed-bid auction of K homogeneous units, for any $t \leq K$ the stochastic process $\{\Pi^m_t(x)\}^{K-t}_{m=0}$ generated by a sequence of m-round mimic deviations by a bidder with value x is a martingale.*

The proof of Proposition 3 is provided in the Appendix.[10] The following example illustrates a mimic deviation in a sequential first-price sealed-bid auction of two homogeneous units.

Example 3: Consider a sequential first-price sealed-bid auction with $K = 2$. Let β_1 and β_2 denote, respectively, the round 1 and 2 equilibrium bid functions. The equilibrium expected payoff of a bidder with value x is

$$\Pi^0_1(x) = (x - \beta_1(x)) \Pr(x > Z^{(N-1)}_{N-1}) + (x - \beta_2(x)) \Pr(Z^{(N-1)}_{N-1} > x > Z^{(N-1)}_{N-2}).$$

Define $\Pi^1_1(x, Z^{(N-1)}_{N-1})$ to be the payoff of a 1-round mimic deviation to a bidder with value x when the highest value of a rival bidder is $Z^{(N-1)}_{N-1}$. In the deviation, the bidder bids zero in round 1. If the bidder has the highest value, then he mimics the bidder with value $Z^{(N-1)}_{N-1}$ in round 2, and he wins at round 2. If the bidder doesn't have the highest value then he bids $\beta_2(x)$ in round 2. He wins a unit if he has the second-highest value, i.e., if $x = Z^{(N)}_{N-1}$,

[10] Mimic deviations in the sequential English auction are the same as in the second-price sealed-bid auction, and a sequence of mimic deviations generates a martingale.

and otherwise he obtains a payoff of zero. Thus we have

$$\Pi_1^1(x, Z_{N-1}^{(N-1)}) = \begin{cases} 0 & \text{- if } x < Z_{N-2}^{(N-1)} \\ x - \beta_2(x) & \text{- if } Z_{N-2}^{(N-1)} < x < Z_{N-1}^{(N-1)} \\ x - \beta_2(Z_{N-1}^{(N-1)}) & \text{- if } Z_{N-1}^{(N-1)} < x. \end{cases}$$

The expected payoff of a 1-round mimic deviation is $E[\Pi_1^1(x, Z_{N-1}^{(N-1)})]$

$$\begin{aligned} &= E[x - \beta_2(Z_{N-1}^{(N-1)}) | x > Z_{N-1}^{(N-1)}] \Pr(x > Z_{N-1}^{(N-1)}) + (x - \beta_2(x)) \Pr(Z_{N-1}^{(N-1)} > x > Z_{N-2}^{(N-1)}) \\ &= E[x - \beta_2(Z_{N-1}^{(N)}) | x = Z_N^{(N)}] \Pr(x > Z_{N-1}^{(N-1)}) + (x - \beta_2(x)) \Pr(Z_{N-1}^{(N-1)} > x > Z_{N-2}^{(N-1)}) \\ &= \Pi_1^0(x), \end{aligned}$$

where the last equality follows from $E[\beta_2(Z_{N-1}^{(N)}) | x = Z_N^{(N)}] = \beta_1(x)$ by Weber's Martingale Theorem.[11] Thus, at the beginning of round 1, the expected payoff from following equilibrium, $\Pi_1^0(x)$, is equal to the expected payoff of a 1-round mimic deviation. $\triangle$

4 Mimic Martingales in Sequential Bargaining

The results in the prior section established that a bidder in a sequential auction has many strategies which yield his equilibrium payoff when all the other bidders follow equilibrium. This is not unique to selling mechanisms. Consider, for example, dissolving a partnership of N risk neutral partners. The problem is to allocate ownership of the firm to a single partner and to determine the compensation paid to each of the others.[12] As before,

[11] Weber's Margingale Theorem says that under equilibrium play the expected price in round 2, conditional on the price at round 1, is equal to the round 1 price.

[12] A similar allocation problem arises in sequential knock-out auctions used by bidding rings. See Graham, Marshall, and Richard (1990).

values for the firm are independently and identically distributed according to cumulative distribution function F with support $[0, \bar{x}]$, where $\bar{x} < \infty$ and $f \equiv F'$ is continuous and positive on $[0, \bar{x}]$.

The partnership is dissolved via a compensation auction. In the auction, the price, starting from zero, rises continuously. Partners (hereafter bidders) may drop out at any point. A bidder who drops out, surrenders his claim to the firm and, in return, receives compensation from the (eventual) winner equal to the difference between the price at which he drops and the price at which the prior bidder dropped. The auction ends when exactly one bidder remains. That bidder wins the firm and pays the other bidders their compensation. Thus in an auction with N bidders, if $\{p_t\}_{t=1}^{N-1}$ is the sequence of dropout prices, then the compensation of the t-th bidder to drop is $p_t - p_{t-1}$, where $p_0 = 0$, and the winner's total payment is $p_{N-1} = \Sigma_{t=1}^{N-1}(p_t - p_{t-1})$.

The unique symmetric equilibrium in increasing and differentiable strategies for the compensation auction is given, for $t = 1, \ldots, N-1$, by[13]

$$\beta_t(x; p_{t-1}) = \frac{N-t}{N-t+1} p_{t-1} + \frac{1}{N-t+1} E\left[Z_{N-1}^{(N)} | Z_t^{(N)} > x > Z_{t-1}^{(N)} \right].$$

In what follows, we describe mimic deviations in the compensation auction, and we show that if a bidder makes a mimic deviation then he obtains his equilibrium expected payoff. Bidders have flat incentives to follow equilibrium.

Mimic Deviations in the Compensation Auction

Consider a bidder with value x in round t at the moment the price reaches his equilibrium bid. If he plays equilibrium and accepts the price, then he obtains a payoff of $\Pi_t(x; p_{t-1}) = \beta_t(x; p_{t-1}) - p_{t-1}$ and exits the auction. The bidder can instead remain in the auction, observe the m drop out prices in rounds $t, \ldots, t+m-1$, and then bid optimally at round $t+m$. In particular,

[13] See Van Essen and Wooders (2016, Proposition 2).

he observes $p_t = \beta_t(Z_{t+1}^{(N)}; p_{t-1}), \ldots, p_{t+m-1} = \beta_{t+m-1}(Z_{t+m}^{(N)}; p_{t+m-2})$, from which he infers the values $z_{t+1}^{(N)}, \ldots, z_{t+m}^{(N)}$. The deviating bidder's optimal bid at round $t+m$ is to mimic the round $t+m$ equilibrium bid of the bidder with the highest inferred value, i.e., to bid $\beta_{t+m}(z_{t+m}^{(N)}; p_{t+m-1})$. We again call such a deviation an m-round mimic deviation.

For each m, let $\Pi_t^m(x; p_{t-1})$ be the random variable which is the bidder's payoff from the m-round mimic deviation in the compensation auction, where $\Pi_t^0(x; p_{t-1})$ is his payoff if he obeys β, evaluated at the moment he drops out.

Proposition 4: *In the compensation auction for dissolving a partnership, for any $t < N-1$, the stochastic process $\{\Pi_t^m(x; p_{t-1})\}_{m=0}^{N-1-t}$ generated by a sequence of m-round mimic deviations by a bidder with value x is a martingale.*

Example 4A illustrates Proposition 4.

Example 4A: Suppose $N = 4$ and values are distributed $U[0,1]$. The equilibrium bid functions are

$$\begin{aligned}
\beta_1(x) &= \frac{1}{10}x + \frac{3}{20}, \\
\beta_2(x; p_1) &= \frac{1}{6}x + \frac{1}{6} + \frac{2}{3}p_1, \\
\beta_3(x; p_2) &= \frac{1}{3}x + \frac{1}{6} + \frac{1}{2}p_2.
\end{aligned}$$

Consider a bidder with type $x = 1/2$ when, in round 1, the bid has reached his equilibrium drop-out price of $\beta_1(1/2) = 1/5$. The bidder knows he has the lowest type (i.e., $Z_1^{(4)} = 1/2$) and, if he obeys β, he obtains a payoff $\Pi_1^0(1/2) = 1/5$.

If he follows a 1-round mimic deviation, then he remains in the auction until the bidder with the next lowest value drops at price $p_1 = \beta_1(Z_2^{(4)})$. He infers $Z_2^{(4)}$ from p_1 and then in round 2 he bids as if his value were $Z_2^{(4)}$, i.e., he drops at the price $\beta_2(Z_2^{(4)};\beta_1(Z_2^{(4)}))$. The payoff from this deviation is the random variable

$$\begin{aligned}\Pi_1^1(Z_2^{(4)}) &= \beta_2(Z_2^{(4)};\beta_1(Z_2^{(4)}))-\beta_1(Z_2^{(4)})\\ &= \frac{1}{6}Z_2^{(4)}+\frac{1}{6}-\frac{1}{3}\left(\frac{1}{10}Z_2^{(4)}+\frac{3}{20}\right)\\ &= \frac{2}{15}Z_2^{(4)}+\frac{7}{60}.\end{aligned}$$

The bidder's expected payoff from the deviation is[14]

$$E\left[\Pi_1^1(Z_2^{(4)})\,\middle|\,Z_1^{(4)}=\frac{1}{2}\right]=\frac{2}{15}E\left[Z_2^{(4)}\,\middle|\,Z_1^{(4)}=\frac{1}{2}\right]+\frac{7}{60}=\frac{1}{5}=\Pi_1^0\left(1/2\right).$$

In other words, conditional on receiving compensation in round 1, the bidder obtains the same expected payoff from the 1-round mimic deviation as obtained from equilibrium.

Last, if he follows a 2-round mimic deviation, the bidder waits two rounds, he infers the second and third lowest values $Z_2^{(4)}$ and $Z_3^{(4)}$, respectively, from the drop prices in rounds 1 and 2, and he then mimics a bidder with type $Z_3^{(4)}$ in round 3, i.e., he drops at the price $\beta_3(Z_3^{(4)};\beta_2(Z_3^{(4)};\beta_1(Z_2^{(4)})))$. The payoff from this deviation is the random variable

$$\begin{aligned}\Pi_1^2(Z_3^{(4)},Z_2^{(4)}) &= \beta_3(Z_3^{(4)};\beta_2(Z_3^{(4)};\beta_1(Z_2^{(4)})))-\beta_2(Z_3^{(4)};\beta_1(Z_2^{(4)}))\\ &= \frac{1}{3}Z_3^{(4)}+\frac{1}{6}-\frac{1}{2}\left(\frac{1}{6}Z_3^{(4)}+\frac{1}{6}+\frac{2}{3}\left(\frac{1}{10}Z_2^{(4)}+\frac{3}{20}\right)\right)\\ &= \frac{1}{4}Z_3^{(4)}-\frac{1}{30}Z_2^{(4)}+\frac{1}{30}.\end{aligned}$$

The bidder's expected payoff from the 2-round mimic deviation, conditioning on both $Z_1^{(4)}$ and $Z_2^{(4)}$, is

$$E\left[\Pi_1^2(Z_3^{(4)},Z_2^{(4)})\,\middle|\,Z_1^{(4)}=\frac{1}{2},Z_2^{(4)}=z_2\right].$$

[14] For the calculations below we use that

$$E\left[Z_2^{(4)}|Z_1^{(4)}=\frac{1}{2}\right]=\frac{5}{8},\ E\left[Z_3^{(4)}|Z_1^{(4)}=\frac{1}{2}\right]=\frac{3}{4},\ \text{and } E\left[Z_4^{(4)}|Z_1^{(4)}=\frac{1}{2}\right]=\frac{7}{8}.$$

From the Markov property of order statistics, this is

$$E\left[\Pi_1^2(Z_3^{(4)}, Z_2^{(4)})\,\middle|\,Z_2^{(4)} = z_2\right].$$

Direct calculation yields

$$\begin{aligned}
E\left[\Pi_1^2(Z_3^{(4)}, Z_2^{(4)})|Z_2^{(4)} = z_2\right] &= \frac{1}{4}E\left[Z_3^{(4)}|Z_2^{(4)} = z_2\right] - \frac{1}{30}z_2 + \frac{1}{30} \\
&= \frac{1}{4}\left(\frac{2}{3}z_2 + \frac{1}{3}\right) - \frac{1}{30}z_2 + \frac{1}{30} \\
&= \frac{2}{15}z_2 + \frac{7}{60} \\
&= \Pi_1^1(z_2).
\end{aligned}$$

Thus, we have demonstrated that the sequence $\{\Pi_1^0,\, \Pi_1^1,\, \Pi_1^2\}$ forms a martingale. △

Building on Example 4A, Example 4B computes the expected payoff of a 1 round and 2 round mimic deviation.

Example 4B: Recall from Example 4A that the bidder's expected payoff from a 1-round mimic deviation in round 1 is

$$E\left[\Pi_1^1(Z_2^{(4)})\,\middle|\,Z_1^{(4)} = \frac{1}{2}\right] = \frac{1}{5}.$$

Using the expression for $\Pi_1^2(Z_3^{(4)}, Z_2^{(4)})$ from Example 4A, the bidder's expected payoff from a 2-round mimic deviation in round 1 is

$$E\left[\Pi_1^2(Z_3^{(4)}, Z_2^{(4)})\,\middle|\,Z_1^{(4)} = \frac{1}{2}\right] = \frac{1}{4}\left(\frac{3}{4}\right) - \frac{1}{30}\left(\frac{5}{8}\right) + \frac{1}{30} = \frac{1}{5}.$$

Thus, when the price reaches his equilibrium bid in round 1, a bidder with value 1/2 is indifferent between playing equilibrium, playing a 1-round mimic deviation, and playing a 2-round mimic deviation. △

In the Red or Black game, the Martingale Stopping Theorem implies that any betting strategy, including (possibly randomized) strategies that depend on the history of cards turned over, yields the same payoff as betting immediately. An analogous result applies to sequential Dutch auctions, sequential first- and second-price sealed-bid auctions, and the compensation auction. A bidder is indifferent between equilibrium play, any mimic deviation, and any history dependent strategy for choosing when to start, continue, and stop a mimic deviation.

Example 4C shows an application of the Martingale Stopping Theorem to the compensation auction.

Example 4C: For the compensation auction, consider a bidder with value x when at round 1 the price reaches his equilibrium bid $\beta_1(x)$. If he follows equilibrium, then he obtains compensation of $\beta_1(x)$. Proposition 4 and the Martingale Stopping Theorem imply that, for arbitrary $\varepsilon \geq 0$, he obtains his equilibrium payoff from any deviation of the form:
Round 1: Never drop. Observe the Round 1 price p_1. Then $p_1 = \beta_1(Z_2^{(4)})$ identifies $Z_2^{(4)}$.
Round 2: Drop at $\beta_2(Z_2^{(4)}; p_1)$ if $Z_2^{(4)} \geq \varepsilon + x$. Otherwise never drop, in which case $p_2 = \beta_2(Z_3^{(4)}; p_1)$ identifies $Z_3^{(4)}$.
Round 3: Bid $\beta_3(Z_3^{(4)}; p_2)$.
In this deviation, the stopping time depends on the history of prices. The bidder never drops at round 1. At round 2, he (i) mimics the round 1 winner if the round 1 winner's value exceeds his own by ε, and (ii) he does not drop otherwise. If he remains in the auction at round 3, then he mimics the round 2 winner.[15] If $x = 1/2$, for example, then the expected payoff is 1/5 for any ε. $\triangle$

[15] This deviation is formally described by the strategy $\hat{\beta}$ where $\hat{\beta}_1(x) = \bar{x}$. If $\beta_1^{-1}(p_1) < x$ then $\hat{\beta}_2(x; p_1) = \beta_2(x; p_1)$ and $\hat{\beta}_3(x; p_1, p_2) = \beta_2(x; p_1, p_2)$. If $\beta_1^{-1}(p_1) \geq x$ then $\hat{\beta}_2(x; p_1) = \beta_2(Z_2^{(4)}; p_1)$ if $Z_2^{(4)} \geq \varepsilon + x$, where $Z_2^{(4)} = \beta_1^{-1}(p_1)$, and $\hat{\beta}_2(x; p_1) = \bar{x}$ otherwise. In round 3, $\hat{\beta}_2(x; p_1, p_2) = \beta_2(Z_3^{(4)}; p_1, p_2)$, where $Z_3^{(4)} = \beta_2^{-1}(p_2; p_1)$.

A mimic deviation at round t of m rounds, where $m \leq N - 1 - t$, has the property that the bidder obtains compensation (rather than winning the firm). Any deviation in which the bidder wins the firm with positive probability will lower his expected payoff.

Example 4D: Continuing Example 4, suppose that when the price reaches his equilibrium bid, instead of following a mimic deviation a bidder with type $x = 1/2$ deviates by simply remaining in the auction until he wins the firm. Then he obtains his value minus the price at which the third bidder drops. The random payoff of this strategy is

$$\begin{aligned} & \frac{1}{2} - \beta_3(Z_4^{(4)}; \beta_2(Z_3^{(4)}; \beta_1(Z_2^{(4)}))) \\ = \quad & \frac{1}{2} - \left[\frac{1}{3}Z_4^{(4)} + \frac{1}{6} + \frac{1}{2}\left[\frac{1}{6}Z_3^{(4)} + \frac{1}{6} + \frac{2}{3}\left[\frac{1}{10}Z_2^{(4)} + \frac{3}{20}\right]\right]\right]. \end{aligned}$$

The expected payoff of this strategy, conditional on $Z_1^{(4)} = 1/2$, is only $-7/40$. $\triangle$

5 Discussion

We have studied m-round mimic deviations in which a bidder withdraws from the auction for m rounds, gathers information about his rivals' values, and then bids optimally. We have shown that bidders obtain their equilibrium payoffs from such deviations, and hence their payoffs are flat in equilibrium. Applying the Martingale Stopping Theorem establishes that there is a large class of strategies for which a bidder obtains his equilibrium payoff.

The family of strategies which yields a bidder his equilibrium payoff is even larger. Consider, for example, a bidder with value x, at round t, in a sequential Dutch auction at the moment the price reaches his equilibrium bid. The bidder can deviate by shading his bid, bidding $b_t' = \beta_t(\lambda x)$ for some $\lambda < 1$. There are two possible outcomes: He may observe a winning bid p_t at round t, where $p_t = \beta_t(y) > b_t'$, i.e., $\lambda x < y < x$. In this case, at round $t+1$ he mimics the round $t+1$ equilibrium bid of the round t winner, i.e., he bids $\beta_{t+1}(y)$. Alternatively, he may win at his bid b_t'. We call such a deviation a "generalized" mimic deviation. This deviation also yields the bidder his equilibrium payoff. In the Online Appendix we show that such a sequence of generalized mimic deviations yields a martingale in payoffs. Thus the set of deviations which yields a bidder his equilibrium payoff is considerably larger than the set obtained from standard mimic deviations.

6 Appendix

The joint density of $Z_1^{(N)}, \ldots, Z_N^{(N)}$ is

$$g_{1,\ldots,N}^{(N)}(z_1, \ldots, z_N) = N! \prod\nolimits_{i=1}^{N} f(z_i)$$

if $z_1 \leq z_2 \leq \ldots \leq z_N$ and is zero otherwise. The conditional order statistic densities which appear in the proofs can all be derived from this expression.

Proof of Proposition 1: In the K-th round (i.e., the last round), given the sales price p_{K-1} at the prior round, a bidder with value x can infer $Z_{N-K+1}^{(N-1)} = z_{N-K+1}$. His expected payoff from bidding b is therefore

$$\int_0^{\beta_K^{-1}(b;p_{K-1})} [\alpha_K x - b]\, g_{N-K}^{(N-1)}(z_{N-K}|Z_{N-K+1}^{(N-1)} = z_{N-K+1}) dz_{N-K}.$$

Differentiating with respect to b yields

$$\begin{aligned} \frac{1}{\beta_K'(\beta_K^{-1}(b;p_{K-1});p_{K-1})} [\alpha_K x - b]\, g_{N-K}^{(N-1)}(\beta_K^{-1}(b;p_{K-1})|Z_{N-K+1}^{(N-1)} &= z_{N-K+1}) \\ -G_{N-K}^{(N-1)}(\beta_K^{-1}(b;p_{K-1})|Z_{N-K+1}^{(N-1)} &= z_{N-K+1}) = 0. \end{aligned}$$

Evaluated at $b = \beta_K(x; p_{K-1})$ this expression becomes

$$\begin{aligned}\frac{1}{\beta'_K(x;p_{K-1})}\left[\alpha_K x - \beta_K(x;p_{K-1})\right] g_{N-K}^{(N-1)}(x|Z_{N-K+1}^{(N-1)} &= z_{N-K+1}) \\ -G_{N-K}^{(N-1)}(x|Z_{N-K+1}^{(N-1)} &= z_{N-K+1}) = 0.\end{aligned}$$

Re-arranging

$$\begin{aligned}\beta'_K(x;p_{K-1})G_{N-K}^{(N-1)}(x|Z_{N-K+1}^{(N-1)} &= z_{N-K+1}) + \beta_K(x;p_{K-1})g_{N-K}^{(N-1)}(x|Z_{N-K+1}^{(N-1)} = z_{N-K+1}) \\ &= \alpha_K x g_{N-K}^{(N-1)}(x|Z_{N-K+1}^{(N-1)} = z_{N-K+1})\end{aligned}$$

or

$$\frac{d}{dx}\left(\beta_K(x;p_{K-1})G_{N-K}^{(N-1)}(x|Z_{N-K+1}^{(N-1)} = z_{N-K+1})\right) = \alpha_K x g_{N-K}^{(N-1)}(x|Z_{N-K+1}^{(N-1)} = z_{N-K+1}).$$

From the Fundamental Theorem of Calculus,

$$\beta_K(x;\mathbf{p}_{K-1})G_{N-K}^{(N-1)}(x|Z_{N-K+1}^{(N-1)} = z_{N-k+1}) = \int_0^x \alpha_K z g_{N-K}^{(N-1)}(z|Z_{N-K+1}^{(N-1)} = z_{N-k+1})dz + C,$$

where C is a constant. Since the LHS of the above equation equals zero when $x = 0$, then $C = 0$. Hence

$$\begin{aligned}\beta_K(x) &= \frac{\int_0^x \alpha_K z g_{N-K}^{(N-1)}(z|Z_{N-K+1}^{(N-1)} = z_{N-K+1})dz}{G_{N-K}^{(N-1)}(x|Z_{N-K+1}^{(N-1)} = z_{N-K+1})} \\ &= \frac{\int_0^x \alpha_K z(N-K)F(z)^{N-K-1}f(z)dz}{F(x)^{N-K}} \\ &= E\left[\alpha_K Z_{N-K}^{(N-1)}|Z_{N-K}^{(N-1)} < x < Z_{N-K+1}^{(N-1)}\right].\end{aligned}$$

Likewise, given $\beta_{t+1}(x)$, the symmetric argument establishes that

$$\begin{aligned}\beta_t(x) &= \frac{\int_0^x \left[(\alpha_t - \alpha_{t+1})z + \beta_{t+1}(z)\right](N-t)F(z)^{N-t-1}f(z)dz}{F(x)^{N-t}} \\ &= E\left[(\alpha_t - \alpha_{t+1})Z_{N-t}^{(N-1)} + \beta_{t+1}(Z_{N-t}^{(N-1)})|Z_{N-t}^{(N-1)} < x < Z_{N-t+1}^{(N-1)}\right].\end{aligned}$$

We now show that $\beta_t(x)$ can be written as

$$\beta_t(x) = \sum_{j=t}^{K} E\left[(\alpha_j - \alpha_{j+1})\, Z_{N-j}^{(N-1)} | Z_{N-t}^{(N-1)} < x < Z_{N-t+1}^{(N-1)} \right].$$

The proof is by induction. The claim is trivially true for $t = K$. Suppose the claim is true for $t+1$, i.e.,

$$\begin{aligned}
\beta_{t+1}(x) &= \sum_{j=t+1}^{K} E\left[(\alpha_j - \alpha_{j+1})\, Z_{N-j}^{(N-1)} | Z_{N-t-1}^{(N-1)} < x < Z_{N-t}^{(N-1)} \right]. \\
&= \sum_{j=t+1}^{K} (\alpha_j - \alpha_{j+1}) \int_0^x q \frac{(N-t-1)! F(q)^{N-j-1} f(q) [F(x) - F(q)]^{j-t-1}}{(N-j-1)!(j-t-1)! F(x)^{N-t-1}} dq.
\end{aligned}$$

We can write $\beta_t(x)$ as

$$\begin{aligned}
\beta_t(x) &= \frac{\int_0^x (\alpha_t - \alpha_{t+1})\, z(N-t) F(z)^{N-t-1} f(z) dz}{F(x)^{N-t}} \\
&\quad + \frac{\int_0^x \beta_{t+1}(z)(N-t) F(z)^{N-t-1} f(z) dz}{F(x)^{N-t}} \\
&= \frac{\int_0^x (\alpha_t - \alpha_{t+1})\, z(N-t) F(z)^{N-t-1} f(z) dz}{F(x)^{N-t}} \\
&\quad + \sum_{j=t+1}^{K} (\alpha_j - \alpha_{j+1}) \int_0^x \int_0^z q \frac{(N-t)! F(q)^{N-j-1} f(q) [F(z) - F(q)]^{j-t-1} f(z)}{(N-j-1)!(j-t-1)! F(x)^{N-t}} dq dz
\end{aligned}$$

Changing the order of integration we have

$$\begin{aligned}
\beta_t(x) &= \frac{\int_0^x (\alpha_t - \alpha_{t+1}) z(N-t)F(z)^{N-t-1} f(z)dz}{F(x)^{N-t}} \\
&\quad + \sum_{j=t+1}^{K} (\alpha_j - \alpha_{j+1}) \int_0^x \int_q^x \frac{(N-t)! qF(q)^{N-j-1} f(q)[F(z)-F(q)]^{j-t-1} f(z)}{(N-j-1)!(j-t-1)!F(x)^{N-t}} dzdq \\
&= \frac{\int_0^x (\alpha_t - \alpha_{t+1}) z(N-t)F(z)^{N-t-1} f(z)dz}{F(x)^{N-t}} \\
&\quad + \sum_{j=t+1}^{K} (\alpha_j - \alpha_{j+1}) \int_0^x \frac{(N-t)! qF(q)^{N-j-1} f(q)[F(x)-F(q)]^{j-t}}{(N-j-1)!(j-t)!F(x)^{N-t}} dq \\
&= \sum_{j=t}^{K} E\left[(\alpha_j - \alpha_{j+1}) Z_{N-j}^{(N-1)} | Z_{N-t}^{(N-1)} < x < Z_{N-t+1}^{(N-1)}\right],
\end{aligned}$$

which establishes the result. $\square$

Proof of Proposition 2: Consider a bidder with value x and suppose, in round t, that the price reaches $p_t = \beta_t(x)$. This bidder knows that he has the t-th highest value, i.e., $Z_{N-t+1}^{(N)} = z_{N-t+1} = x$. We show that the expected payoff of an $m+1$ round mimic deviation is the same as the realized payoff of an m round mimic deviation, where $t+m < K$.

In an m-round mimic deviation, at round $t+m$ he bids as though his type is $z_{N-(t+m)+1}$, he wins an item, and he pays

$$p_{t+m} = \beta_{t+m}(z_{N-(t+m)+1}) = E\left[Z_{N-K}^{(N-1)} | Z_{N-(t+m)}^{(N-1)} < z_{N-(t+m)+1} < Z_{N-(t+m-1)}^{(N-1)}\right].$$

In an $m+1$-round mimic deviation, he wins a unit and pays

$$\beta_{t+m+1}(Z_{N-(t+m+1)+1}^{(N)}) = \beta_{t+m+1}(Z_{N-(t+m)}^{(N)}).$$

The expected price is

$$E\left[\beta_{t+m+1}(Z_{N-(t+m)}^{(N)}) | Z_{N-(t+m)+1}^{(N)} = z_{N-(t+m)+1}\right].$$

We next show that the expected price is $\beta_{t+m}(z_{N-(t+m)+1})$.

From the equilibrium bid function, for any q we have

$$\begin{aligned}\beta_{t+m+1}(q) &= E\left[Z_{N-K}^{(N-1)} | Z_{N-(t+m+1)}^{(N-1)} < q < Z_{N-(t+m)}^{(N-1)}\right] \\ &= \int_0^q z \frac{(N-(t+m)-1)! F(z)^{N-K-1} f(z) [F(q)-F(z)]^{K-(t+m)-1}}{(N-K-1)!(K-(t+m)-1)! F(q)^{N-(t+m)-1}} dz\end{aligned}$$

Hence $E\left[\beta_{t+m+1}(Z_{N-(t+m)}^{(N)}) | Z_{N-(t+m)+1}^{(N)} = z_{N-(t+m)+1}\right]$ is

$$\int_0^{z_{N-(t+m)+1}} \int_0^q z \frac{(N-(t+m)-1)! F(z)^{N-K-1} f(z) \left[F(q)-F(z)\right]^{K-(t+m)-1}}{(N-K-1)! \, (K-(t+m)-1)! F(q)^{N-(t+m)-1}}$$
$$\times \frac{(N-(t+m)) F(q)^{N-(t+m)-1} f(q)}{F(z_{N-(t+m)+1})^{N-(t+m)-1}} dz dq,$$

which can be written as

$$\int_0^{z_{N-(t+m)+1}} \int_0^q z \frac{(N-(t+m))! F(z)^{N-K-1} f(z) \left[F(q)-F(z)\right]^{K-(t+m)-1} f(q)}{(N-K-1)! \, (K-(t+m))! F(z_{N-(t+m)+1})^{N-(t+m)-1}} dz dq.$$

Reversing the order of integration, this expectation becomes

$$\begin{aligned} & \int_0^{z_{N-(t+m)+1}} \int_z^{z_{N-(t+m)+1}} z \frac{(N-(t+m))! F(z)^{N-K-1} f(z) [F(q)-F(z)]^{K-(t+m)-1} f(q)}{(N-K-1)!(K-(t+m))! F(z_{N-(t+m)+1})^{N-(t+m)-1}} dq dz \\ = & \int_0^{z_{N-(t+m)+1}} z \frac{(N-(t+m))!}{(N-K-1)!(K-(t+m))!} \frac{F(z)^{N-K-1} f(z) \left[F(z_{N-(t+m)+1})-F(z)\right]^{K-(t+m)}}{F(z_{N-(t+m)+1})^{N-(t+m)-1}} dz \\ = & \; E\left[Z_{N-K}^{(N-1)} | Z_{N-(t+m)}^{(N-1)} < z_{N-(t+m)+1} < Z_{N-(t+m-1)}^{(N-1)}\right] \\ = & \; \beta_{t+m}(z_{N-(t+m)+1}),\end{aligned}$$

which establishes the result. This implies the sequence of prices associated with a sequence of mimic deviations is a martingale. Since the payoff of a bidder with value x is his value minus the price he pays, it follows that $\{\Pi_t^m(x)\}_{m=0}^{K-t}$ is a martingale. $\square$

Proof of Proposition 3: We provide the proof for the first-price sealed-bid auction. The proof for the second-price auction is similar.

Consider an arbitrary bidder with value x, at a round t, who follows an m-round mimic deviation, where $t+m < K$. In a sealed-bid auction this entails bidding zero in rounds $t, \ldots, t+m-1$. In each round s the bidder infers the winner's value $Z_{N-s}^{(N-1)} = z_{N-s}$ from the winner's bid. At round $t+m$, if $z_{N-(t+m-1)} < x$ then he mimics $z_{N-(t+m-1)}$ and bids $\beta_{t+m}(z_{N-(t+m-1)})$. Otherwise he makes his equilibrium bid $\beta_{t+m}(x)$.

We establish the result by showing that the expected payoff of an $m+1$-round mimic deviation is the same as the realized payoff of an m-round mimic deviation. There are two cases to consider: (i) $z_{N-(t+m-1)} < x$, and (ii) $z_{N-(t+m-1)} \geq x$.

Case (i): At round $t+m$, he wins an item and he pays

$$\beta_{t+m}(z_{N-(t+m-1)}) = E\left[Z_{N-K}^{(N-1)} | Z_{N-(t+m)}^{(N-1)} < z_{N-(t+m-1)} < Z_{N-(t+m-1)}^{(N-1)}\right],$$

obtaining a payoff of $x - \beta_{t+m}(z_{N-(t+m-1)})$. Note that $Z_{N-(t+m-1)}^{(N-1)} = z_{N-(t+m-1)}$ and $z_{N-(t+m-1)} < x$ implies $Z_{N-(t+m-1)}^{(N)} = Z_{N-(t+m-1)}^{(N-1)} = z_{N-(t+m-1)} < x$.

In an $m+1$-round mimic deviation, he observes the next smallest value $Z_{N-(t+m)}^{(N)}$ and wins an item with a bid equal to $\beta_{t+m+1}(Z_{N-(t+m)}^{(N)})$. The expected price at round $t+m+1$ is

$$E\left[\beta_{t+m+1}(Z_{N-(t+m)}^{(N)}) | Z_{N-(t+m-1)}^{(N)} = z_{N-(t+m-1)}\right] = \beta_{t+m}(z_{N-(t+m-1)}),$$

where the equality was established in the proof of Proposition 2, and therefore also obtains an expected payoff of $x - \beta_{t+m}(z_{N-(t+m-1)})$.

Case (ii): At round $t+m$ he makes his equilibrium bid $\beta_{t+m}(x)$. This bid is not guaranteed to win. Let

$$\phi_k(x) = \Pr(Z_{N-k}^{(N-1)} < x < Z_{N-(k-1)}^{(N-1)} | Z_{N-(t+m-1)}^{(N-1)} = z_{N-(t+m-1)})$$

denote the probability that the bidder has the highest value of the active bidders in round $k \geq t+m$, conditional on $Z^{(N-1)}_{N-(t+m-1)} = z_{N-(t+m-1)}$ being the winner's value at round $t+m-1$. The deviating bidder's expected payoff from the m-round mimic deviation is

$$\pi_{t+m} = (x - \beta_{t+m}(x))\phi_{t+m}(x) + \sum_{k=t+m+1}^{K} (x - \beta_k(x))\, \phi_k(x).$$

Now consider the payoff of an $m+1$-round mimic deviation. In round $t+m$ of the deviation, he observes $Z^{(N-1)}_{N-(t+m)}$. If this value is smaller value than x, then in round $t+m+1$ he mimics the observed value $Z^{(N-1)}_{N-(t+m)}$ and obtains a payoff of $x - \beta_{t+m+1}(Z^{(N-1)}_{N-(t+m)})$. Otherwise, he continues to play equilibrium. His random payoff from following an $m+1$-round mimic deviation is therefore

$$\Pi_{t+m+1} = \begin{cases} x - \beta_{t+m+1}(Z^{(N-1)}_{N-(t+m)}) & \text{if } Z^{(N-1)}_{N-(t+m)} < x \\ x - \beta_k(x) & \text{if } Z^{(N-1)}_{N-k} < x < Z^{(N-1)}_{N-(k-1)}, \text{ for } k = t+m+1, ..., K \\ 0 & \text{otherwise.} \end{cases}$$

Note that if $Z^{(N-1)}_{N-(t+m)} < x$ if and only if $Z^{(N)}_{N-(t+m)} < x = Z^{(N)}_{N-(t+m-1)}$. Hence, his expected payoff is

$$\begin{aligned} & E\left[x - \beta_{t+m+1}(Z^{(N-1)}_{N-(t+m)}) | Z^{(N-1)}_{N-(t+m)} < x\right] \phi_{t+m}(x) + \sum_{k=t+m+1}^{K} (x - \beta_k(x))\, \phi_k(x) \\ = \; & E\left[x - \beta_{t+m+1}(Z^{(N)}_{N-(t+m)}) | Z^{(N)}_{N-(t+m)+1} = x\right] \phi_{t+m}(x) + \sum_{k=t+m+1}^{K} (x - \beta_k(x))\, \phi_k(x). \end{aligned}$$

Using again that

$$E\left[\beta_{t+m+1}(Z^{(N)}_{N-(t+m)}) | Z^{(N)}_{N-(t+m-1)} = x\right] = \beta_{t+m}(x),$$

the expected payoff of the $m+1$-round mimic deviation can be written as

$$(x - \beta_{t+m}(x))\phi_{t+m}(x) + \sum_{k=t+m+1}^{K} (x - \beta_k(x))\, \phi_k(x),$$

which is π_{t+m}, as desired. $\square$

Proof of Proposition 4: Consider a bidder with value x in round t at the moment the price reaches his equilibrium bid. The bidder has the t-th lowest value. Let m be such that $t+m < N-1$. If the bidder follows the m-round mimic deviation, then at round $t+m$ the bidder (i) infers $Z_1^{(N)} = z_1, \ldots, Z_{t-1}^{(N)} = z_{t-1}$, (ii) he knows his own value x is the t-th lowest, i.e., $Z_t^{(N)} = z_t = x$, and if $m > 0$, then (iii) he infers $Z_{t+1}^{(N)} = z_{t+1}, \ldots, Z_{t+m}^{(N)} = z_{t+m}$. Under the mimic deviation, he bids in round $t+m$ as though his type were z_{t+m} and drops at price $p_{t+m} = \beta_{t+m}(z_{t+m}; p_{t+m-1})$. He obtains compensation

$$\pi_t^m(z_{t+m}) = p_{t+m} - p_{t+m-1} = \frac{E\left[Z_{N-1}^{(N)} | Z_{t+m}^{(N)} > z_{t+m} > Z_{t+m-1}^{(N)}\right] - p_{t+m-1}}{N-(t+m)+1}.$$

We show that the bidder obtains the same expected compensation if, instead of dropping at round $t+m$ at price p_{t+m}, he follows the $m+1$-round mimic deviation.[16] In that case, he observes the rival with the next lowest value drop in round $t+m$ and infers his rivals' type to be $Z_{t+m+1}^{(N)}$. In round $t+m+1$ he bids as though his own type is $Z_{t+m+1}^{(N)}$ and therefore he is the next bidder to drop since all bidders of type $Z_{t+m+1}^{(N)}$ or lower have already dropped. He obtains compensation

$$\Pi_t^{m+1}(Z_{t+m+1}^{(N)}) = \beta_{t+m+1}(Z_{t+m+1}^{(N)}; \beta_{t+m}(Z_{t+m+1}^{(N)}; p_{t+m-1})) - \beta_{t+m}(Z_{t+m+1}^{(N)}; p_{t+m-1}).$$

Using the equilibrium bid function given in Section 4, if q is the realized value of $Z_{t+m+1}^{(N)}$, then in round $t+m+1$ the bidder obtains compensation

$$\pi_t^{m+1}(q) = \frac{E\left[Z_{N-1}^{(N)} | Z_{t+m+1}^{(N)} > q > Z_{t+m}^{(N)}\right] - \beta_{t+m}(q; p_{t+m-1})}{N-(t+m)},$$

[16] Since $t+m < N-1$ then $t+m+1 \leq N-1$ and the bidder drops out rather than winning.

where

$$\beta_{t+m}(q;p_{t+m-1}) = \frac{E\left[Z_{N-1}^{(N)}|Z_{t+m}^{(N)} > q > Z_{t+m-1}^{(N)}\right]}{N-(t+m)+1} + \frac{N-(t+m)}{N-(t+m)+1}p_{t+m-1}.$$

Define

$$\begin{aligned} D(q) \quad &\equiv \quad \frac{N-(t+m)+1}{N-(t+m)}E\left[Z_{N-1}^{(N)}|Z_{t+m+1}^{(N)} > q > Z_{t+m}^{(N)}\right] \\ &\quad - \frac{1}{N-(t+m)}E\left[Z_{N-1}^{(N)}|Z_{t+m}^{(N)} > q > Z_{t+m-1}^{(N)}\right]. \end{aligned}$$

Then we can write

$$\pi_t^{m+1}(q) = \frac{D(q)-p_{t+m-1}}{N-(t+m)+1}.$$

The term $E[Z_{N-1}^{(N)}|Z_{t+m+1}^{(N)} > q > Z_{t+m}^{(N)}]$ is

$$\int_q^{\bar{x}} r\frac{(N-(t+m))(N-(t+m)-1)f(r)[1-F(r)][F(r)-F(q)]^{N-(t+m)-2}}{[1-F(q)]^{N-(t+m)}}dr$$

and the term $E[Z_{N-1}^{(N)}|Z_{t+m}^{(N)} > q > Z_{t+m-1}^{(N)}]$ is

$$\int_q^{\bar{x}} r\frac{(N-(t+m)+1)(N-(t+m))f(r)[1-F(r)][F(r)-F(q)]^{N-(t+m)-1}}{[1-F(q)]^{N-(t+m)+1}}dr.$$

Thus $D(q)$ is

$$\begin{aligned} &\int_q^{\bar{x}} r\frac{(N-(t+m)+1)\,(N-(t+m)-1)f(r)[1-F(r)][F(r)-F(q)]^{N-(t+m)-2}}{[1-F(q)]^{N-(t+m)}}dr \\ &-\int_q^{\bar{x}} r\frac{(N-(t+m)+1)f(r)[1-F(r)][F(r)-F(q)]^{N-(t+m)-1}}{[1-F(q)]^{N-(t+m)+1}}dr, \end{aligned}$$

which can be written as

$$\begin{aligned} &\int_q^{\bar{x}} r\,(N-(t+m)+1)\,f(r)[1-F(r)] \\ &\times\frac{[F(r)-F(q)]^{N-(t+m)-2}\left((N-(t+m)-1)[1-F(q)]-[F(r)-F(q)]\right)}{[1-F(q)]^{N-(t+m)+1}}dr. \end{aligned}$$

Hence $E[D(Z_{t+m+1}^{(N)})|Z_{t+m}^{(N)} = z_{t+m}]$ equals

$$\int_{z_{t+m}}^{\bar{x}} \int_{q}^{\bar{x}} r\,(N-(t+m)+1)\,f(r)[1-F(r)]$$
$$\times \frac{[F(r)-F(q)]^{N-(t+m)-2}\,((N-(t+m)-1)[1-F(q)]-[F(r)-F(q)])}{[1-F(q)]^{N-(t+m)+1}}$$
$$\times \frac{(N-(t+m))f(q)[1-F(q)]^{N-(t+m)-1}}{[1-F(z_{t+m})]^{N-(t+m)}} drdq.$$

Changing the order of integration, this can be rewritten as

$$\int_{z_{t+m}}^{\bar{x}} \int_{z_{t+m}}^{r} r\,(N-(t+m)+1)\,f(r)[1-F(r)]$$
$$\times \frac{[F(r)-F(q)]^{N-(t+m)-2}\,((N-(t+m)-1)[1-F(q)]-[F(r)-F(q)])}{[1-F(q)]^{N-(t+m)+1}}$$
$$\times \frac{(N-(t+m))f(q)[1-F(q)]^{N-(t+m)-1}}{[1-F(z_{t+m})]^{N-(t+m)}} dqdr.$$

Simplifying further yields

$$\int_{z_{t+m}}^{\bar{x}} \frac{r\,(N-(t+m)+1)\,(N-(t+m))f(r)[1-F(r)]}{[1-F(z_{t+m})]^{N-(t+m)}}$$
$$\times \int_{z_{t+m}}^{r} \frac{[F(r)-F(q)]^{N-(t+m)-2} f(q)\,((N-(t+m)-1)[1-F(q)]-[F(r)-F(q)])}{[1-F(q)]^2} dqdr.$$

The inner integral

$$\int_{z_{t+m}}^{r} \frac{[F(r)-F(q)]^{N-(t+m)-2} f(q)\,((N-(t+m)-1)[1-F(q)]-[F(r)-F(q)])}{[1-F(q)]^2} dq$$

simplifies to

$$\frac{[F(r)-F(z_{t+m})]^{N-(t+m)-1}}{1-F(z_{t+m})}.$$

Thus $E[D(Z_{t+m+1}^{(N)})|Z_{t+m}^{(N)} = z_{t+m}]$ equals

$$\int_{z_{t+m}}^{\bar{x}} \frac{r\,(N-(t+m)+1)\,(N-(t+m))f(r)[1-F(r)][F(r)-F(z_{t+m})]^{N-(t+m)-1}}{[1-F(z_{t+m})]^{N-(t+m)+1}},$$

i.e.,

$$E[D(Z_{t+m+1}^{(N)})|Z_{t+m}^{(N)} = z_{t+m}] = E\left[Z_{N-1}^{(N)}|Z_{t+m}^{(N)} > z_{t+m} > Z_{t+m-1}^{(N)}\right].$$

Hence

$$\begin{aligned} E[\Pi_t^{m+1}(Z_{t+m+1}^{(N)})|Z_{t+m}^{(N)} = z_{t+m}] &= \tfrac{1}{N-(t+m)+1}\left(E\left[Z_{N-1}^{(N)}|Z_{t+m}^{(N)} > z_{t+m} > Z_{t+m-1}^{(N)}\right] - p_{t+m-1}\right) \\ &= \pi_t^m(z_{t+m}). \end{aligned}$$

This establishes that the sequence of mimic compensations $\{\Pi_t^m(x;p_{t-1})\}_{m=0}^{N-1-t}$ is a martingale. $\square$

7 Online Appendix (not for publication)

Generalized Mimic Deviations - Example

We illustrate a generalized mimic deviation in a sequential first-price sealed-bid auction with 3 bidders and 2 items. The extension to general N and K is similar.

Consider the following parametric family of generalized mimic deviations. Let $\lambda \in [0,1)$ be arbitrary. A bidder with value x deviates at round 1 with a bid $\beta_1(\lambda x)$. If the bidder does not win an item at round 1, then he observes $\beta_1(Z_2^{(2)})$ from which he infers $Z_2^{(2)}$. At round 2, he bids $\beta_2(x)$ if $Z_2^{(2)} \geq x$ and he bids $\beta_2(Z_2^{(2)})$ if $Z_2^{(2)} < x$.[17]

The payoff from a 0-round mimic deviation to a bidder with value x is

$$\Pi_{1,\lambda}^0(x) = (x - \beta_1(x))\Pr(x > Z_2^{(2)}) + (x - \beta_2(x))\Pr(Z_1^{(2)} < x < Z_2^{(2)}).$$

The payoff to a bidder with value x to a 1-round generalized mimic deviation is

[17] If $\epsilon = 0$, this mimic deviation coincides with the one in the paper.

$$\Pi^1_{1,\lambda}(x, Z_2^{(2)}) = \begin{cases} x - \beta_1(\lambda x) & \text{if } Z_2^{(2)} < \lambda x \\ x - \beta_2(Z_2^{(2)}) & \text{if } \lambda x \leq Z_2^{(2)} < x \\ x - \beta_2(x) & \text{if } Z_1^{(2)} < x \leq Z_2^{(2)} \\ 0 & \text{otherwise} \end{cases}$$

We show the sequence $\{\Pi^0_{1,\lambda}, \Pi^1_{1,\lambda}\}$ is a martingale.

The expected payoff of the 1-round generalized mimic deviation conditional on x is

$$\begin{aligned} E\left[\Pi^1_{1,\lambda}(x, Z_2^{(2)})|x\right] &= \\ (x - \beta_1(\lambda x)) \Pr(\lambda x &> Z_2^{(2)}) \\ +E[x - \beta_2(Z_2^{(2)})|x &> Z_2^{(2)} > \lambda x] \Pr(x > Z_2^{(2)} > \lambda x) \\ +(x - \beta_2(x)) \Pr(Z_2^{(2)} &> x > Z_1^{(2)}). \end{aligned}$$

The difference $\Pi^0_{1,\lambda}(x) - E[\Pi^1_1(x, Z_2^{(2)})|x]$ can be written as

$$\begin{aligned} (x - \beta_1(x)) \Pr(x &> Z_2^{(2)}) \\ -(x - \beta_1(\lambda x)) \Pr(\lambda x &> Z_2^{(2)}) \\ -E[x - \beta_2(Z_2^{(2)})|x &> Z_2^{(2)} > \lambda x] \Pr(x > Z_2^{(2)} > \lambda x). \end{aligned}$$

From the Law of Total Expectation, we have the identity

$$\begin{aligned} E[\beta_2(Z_2^{(2)})|x &> Z_2^{(2)}] \Pr(x > Z_2^{(2)}) = \\ E[\beta_2(Z_2^{(2)})|x &> Z_2^{(2)} > \lambda x] \Pr(x > Z_2^{(2)} > \lambda x) \\ +E[\beta_2(Z_2^{(2)})|\lambda x &> Z_2^{(2)}] \Pr(\lambda x > Z_2^{(2)}). \end{aligned}$$

Re-arranging the identity yields

$$\begin{aligned} E[\beta_2(Z_2^{(2)})|x &> Z_2^{(2)} > \lambda x] \Pr(x > Z_2^{(2)} > \lambda x) = \\ E[\beta_2(Z_2^{(2)})|x &> Z_2^{(2)}] \Pr(x > Z_2^{(2)}) \\ -E[\beta_2(Z_2^{(2)})|\lambda x &> Z_2^{(2)}] \Pr(\lambda x > Z_2^{(2)}). \end{aligned}$$

Using the properties of order statistics, we have

$$E[\beta_2(Z_2^{(2)})|x > Z_2^{(2)}] = E[\beta_2(Z_2^{(3)})|Z_3^{(3)} = x]$$

and

$$E[\beta_2(Z_2^{(2)})|\lambda x > Z_2^{(2)}] = E[\beta_2(Z_2^{(3)})|Z_3^{(3)} = \lambda x].$$

Finally, from the proof of Proposition 2 we have

$$E[\beta_2(Z_2^{(3)})|Z_3^{(3)} = x] = \beta_1(x)$$

and

$$E[\beta_2(Z_2^{(3)})|Z_3^{(3)} = \lambda x] = \beta_1(\lambda x).$$

Collecting all these facts, we can write the difference $\Pi_{1,\lambda}^0(x) - E[\Pi_{1,\lambda}^1(x, Z_2^{(2)})|x]$ as

$$\begin{aligned}
(x - \beta_1(x))\Pr(x &> Z_2^{(2)}) - (x - \beta_1(\lambda x))\Pr(\lambda x > Z_2^{(2)}) \\
-E[x - \beta_2(Z_2^{(2)})|x &> Z_2^{(2)} > \lambda x]\Pr(x > Z_2^{(2)} > \lambda x) \\
&= \\
-\beta_1(x)\Pr(x &> Z_2^{(2)}) + \beta_1(\lambda x)\Pr(\lambda x > Z_2^{(2)}) \\
+E[\beta_2(Z_2^{(2)})|x &> Z_2^{(2)}]\Pr(x > Z_2^{(2)}) - E[\beta_2(Z_2^{(2)})|\lambda x > Z_2^{(2)}]\Pr(\lambda x > Z_2^{(2)}) \\
&= \\
-\beta_1(x)\Pr(x &> Z_2^{(2)}) + \beta_1(\lambda x)F_2^{(2)}(\lambda x) \\
+\beta_1(x)\Pr(x &> Z_2^{(2)}) - \beta_1(\lambda x)F_2^{(2)}(\lambda x) \\
&= 0.
\end{aligned}$$

Hence the sequence $\{\Pi_{1,\lambda}^0, \Pi_{1,\lambda}^1\}$ is a martingale. □

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