Allocating Positions Fairly: Auctions and Shapley Value*

Matt Van Essen† John Wooders‡

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Abstract

We study the problem of fairly allocating heterogenous items, priorities, positions, or property rights to participants with equal claims from three perspectives: cooperative, decision theoretic, and non-cooperative. We characterize the Shapley value of the cooperative game and then introduce a class of auctions for non-cooperatively allocating positions. We show that for any auction in this class, each bidder obtains his Shapley value when every bidder follows the auction's unique maxmin perfect bidding strategy. When information is incomplete we characterize the Bayesian equilibrium of these auctions, and show that equilibrium play converges to maxmin perfect play as bidders become infinitely risk averse. The equilibrium allocations thus converges to the Shapley value allocation as bidders become risk averse. Together these results provide both decision theoretic and non-cooperative equilibrium foundations for the Shapley value in the position allocation problem.

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†Department of Economics, University of Tennessee (mvanesse@utk.edu).

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1 Introduction

This paper studies the problem of allocating heterogeneous items, priorities, positions, or rights to participants who have equal claims. Examples of this type of problem include allocating items to heirs in an estate, allocating the priority of service in a queue, allocating positions of ads on a webpage, or allocating fishing rights to different geographical areas. In our environment, participants have unit demands and a common ranking of the items/priorities/positions/rights, which we hereafter simply refer to as "positions." In particular, all participants agree that one position is the most desirable, a second position is the next most desirable, and so on.[1] Despite the common ranking of positions, participants vary in the intensity of their preferences. The problem is to find a mechanism that generates efficient, budget balanced, and fair allocations.

We study the allocation problem from three different perspectives. The first approach, from cooperative game theory, is to characterize the Shapley value allocation, for the participants' true preferences, of the transferable utility (TU) cooperative game. The second approach is decision theoretic: We introduce a class of auctions for allocating positions that we call compensated position auctions, and we characterize "maxmin perfect" bidding. Finally, we approach the problem from the perspective of non-cooperative game theory, and we characterize the Bayes Nash equilibrium of these auctions when participants are privately informed of their preferences. We show that these three solutions to the allocation problem are related in a precise way: for any compensated position auction, (i) when all bidders follow their maxmin perfect bidding strategies then the Shapley value allocation results, (ii) Bayes Nash equilibrium bidding approaches maxmin perfect bidding as bidders become infinitely risk averse, and thus (iii) the allocation obtained in the Bayes Nash equilibrium approaches the Shapley value allocation as bidders become infinitely risk averse. Hence our results provide both decision theoretic and non-cooperative foundations for the Shapley value in the position allocation problem with incomplete information. These results are useful to a planner who doesn't know the participant's true preferences, but

[1] In our model, as in Varian (2007) and the literature on sponsored search auctions, the ratio of the values of any two positions is the same for all participants.

who wishes to implement an allocation that is ex-post fair in the Shapley sense.

Shapley (1953) introduced the notion of a *value* for cooperative games with transferable utility. The Shapley value is a fundamental solution concept in cooperative game theory, with the Shapley allocation often taken as the benchmark for a fair allocation (see Myerson (1977), Roth (1988), Moulin (1992), and Moulin (2004, Chapter 5)). In the position allocation problem, the Shapley allocation identifies for each participant a position and transfer. We characterize the Shapley value allocation, showing how it depends on the inherent values of the positions and the participants' preferences. We show that the transfers associated with the Shapley allocation can be computed recursively, starting with the transfer received by the participant allocated the worst position, then the transfer of the participant with the second-worst position, and so on. This result is a consequence of the feature of the position allocation problem that a participant exerts externalities only on other participants with lower preference intensities than his own.

Suggested by the recursive nature of the Shapley transfers, next we introduce and study a class of dynamic auctions for allocating positions, with auctions in the class differentiated by their cost sharing rules. A compensated position auction takes place over rounds. At each round, the participants (hereafter "bidders") simultaneously make demands for compensation. The bidder with the smallest demand exits the auction after receiving the worst unallocated position and his demand as compensation. His compensation is paid by the remaining bidders, who will eventually be allocated better positions, according to the auction's cost sharing rule. The auction ends when one position and bidder remain. That bidder receives the most desirable position and pays, but does not receive, compensation. In sum, bidders pay compensation to bidders allocated positions worse than their own and receive compensation from bidders allocated positions better than their own.

The second approach studies bidding behavior in compensated position auctions when players act to maximize their minimum payoff. We say a strategy is "maxmin perfect" if it maximizes a bidder's minimum payoff at every history of play. Maxmin perfection is a natural refinement of maxmin in dynamic games. We characterize the unique maxmin perfect bidding strategy, showing how it depends on the auction's cost sharing rule. The strategy has a natural interpretation similar to the solution to the Contested Garment problem described in the Talmud. Our main result here is that when each bidder follows his maxmin perfect bidding strategy, then each bidder obtains his Shapley value allocation. The allocation that obtains is thus independent of the details of the auction's cost sharing rule.

The last approach studies the Bayes Nash equilibria of compensated position auctions when bidders are privately informed of their preference intensities.[2] We provide general necessary conditions for a bidding strategy to form a symmetric equilibrium in increasing and differentiable strategies. Our main result here is to provide the closed-form solution for the unique such equilibrium when bidders are risk neutral or have constant absolute risk aversion (CARA). We show that bidders demand less compensation as they become more risk averse. Furthermore, the equilibrium bidding strategy of CARA risk averse bidders converges uniformly to the maxmin perfect bidding strategy as bidders become infinitely risk averse. An immediate consequence of this result, and our earlier result that maxmin bidding yields the Shapley allocation, is that the equilibrium allocation of any compensated position auction coincides with the Shapley value allocation as bidders become infinitely risk averse. In other words, when bidders are sufficiently risk averse, the equilibrium allocation would be judged to be fair (in Shapley's sense) if all the bidders' preference intensities were commonly known. To our knowledge, this paper is the first to provide this kind of non-cooperative foundation for the Shapley value.

Related Literature

Our paper connects to several literatures in cooperative and non-cooperative game theory: Sequential Auctions, The Assignment Problem, Non-cooperative Foundations of the Shapley Value, Cake Cutting, Dissolving Partnerships,

[2] We focus on Bayesian equilibrium since Mitra and Sen (2010) show, for the problem of allocating N objects to N agents, that there is generically no dominant strategy (VCG) mechanism that is both efficient and budget balanced. They show, in particular, for the existence of such a mechanism it is necessary that the associated "difference domain" is a segment of a straight line in R^{N-1}. In the context of the position allocation problem we study, when $N = 3$ for example, their condition requires the difference in the inherent values of the first and second position is equal to the difference of the inherent values of the second and third positions. This condition does not hold in our context since the inherent values need only be weakly decreasing.

Queueing, Bidding Rings, Bankruptcy, and Cost/Surplus Sharing.

Sequential Auctions: Our paper contributes to a literature on multi-unit sequential auctions with single-unit demands and bidder risk aversion, with recent contributions by Mezzetti (2011) and Hu and Zou (2015) who provide conditions for the sequence of prices received by a seller to be increasing or decreasing. In our setting we study the allocation of positions, items, or rights when the bidders have equal claims. Although our results apply when items are homogeneous, our focus is on the heterogeneous case. Unlike in most auction theory, in our context there is no seller.

The Assignment Problem: The problem of allocating positions is the assignment problem for the special case where all the players rank objects in the same way, as is natural for example when assignments correspond to priorities, e.g., first priority, second priority, etc. Both cooperative and non-cooperative solutions to the general assignment problem have been studied. Moulin (1992) shows that the Shapley value has several desirable properties in cooperative models of assignment games.[3]

Early examples of non-cooperative approaches to the assignment problem include Leonard (1983) and Demange, Gale, Sotomayor (1986). Leonard (1983) provides a mechanism for which it is a dominant strategy for each player to report his preferences over assignments truthfully and which implements the efficient assignment; he shows it generates Vickrey-Clark-Groves prices. Demange, Gale, Sotomayor (1986) provide a dynamic auction which implements the efficient assignment. In the context of internet advertising, important papers by Edelman, Ostrovsky, and Schwarz (2007) and Varian (2007) study the use of the generalized second-price sealed-bid auction to allocate positions under complete information. Edelman, Ostrovsky, and Schwarz (2007) study, in addition, a generalized English auction with incomplete information and show that payoffs (both to bidders and to the seller) are the same as in the Vickrey-Clarke-Groves mechanism.

In all these papers, the seller collects the auction revenue. We study, in contrast, a setting where there is no seller and the only payments are transfers between the bidders. Budget balancedness is a fundamental requirement

[3] The Shapley value is not the only notion of fairness for assignment games. Alkan, Demange, and Gale (1991), for example, study existence of efficient and envy-free allocations in the assignment problem.

since the positions are the common property of the bidders.

Non-cooperative Foundations of the Shapley Value: Pérez-Castrillo and Wettstein (2001) provide a bidding mechanism whose subgame perfect equilibrium outcomes coincide with the Shapley value payoffs. In bargaining games with complete information, non-cooperative foundations of the Shapley value have been provided by Gul (1989) and Hart and Mas-Colell (1996). Gul (1989) provides a game with bilateral bargaining and the random selection of the proposer and shows that, in the efficient equilibrium of the game, players receive their Shapley value payoffs in the limit as they become perfectly patient. Hart and Mas-Colell (1996) studies a multilateral bargaining game and shows that players receive their Shapley value payoffs in the limit as each player's probability of exogenously exiting from bargaining vanishes. By contrast, we obtain the Shapley-value allocation (i.e., the positions and transfers associated with the Shapley value of the TU cooperative game), as bidders become infinitely risk averse in an environment with incomplete information.

Cake Cutting and Dissolving Partnerships: Although we are concerned with the allocation of indivisible heterogenous positions, the class of auctions we study is inspired by the Dubins and Spanier (1961) moving knife algorithm for the fair division of a divisible cake. In the fair division problem there are N participants, each of whom wants cake. To divide the cake, a third party moves a knife across the cake until some participant cries "stop." The participant crying stop receives the cake to the left of the knife and exits, surrendering his claim to any additional cake. The process then continues with the remaining participants and cake, repeating until the whole cake is divided. In our auction, a participant whose demand for compensation is smallest receives the worst remaining position and compensation equal to his demand, while surrendering his claim to better positions.

Dividing a cake is analogous to dissolving a partnership. McAfee (1992) examines the Texas Shootout, a version of divide and choose, for dissolving two-person partnerships. Van Essen and Wooders (2016) studies a dynamic compensation auction for dissolving N-person partnerships. Van Essen and Wooders (2018) studies dual auctions for the dual problems of allocating homogeneous goods or chores, and relates the two. The present paper studies the problem of allocating heterogenous positions. None of these papers concern the Shapley allocation, which is the focus of the present paper.

Queueing: The problem we study is more general than a queuing problem, but reduces to a queuing problem when the inherent value of position i is $-(i-1)$ for each $i \in \{1, \ldots, N\}$. In queuing problems there exist dominant strategy, efficient, and budget balanced mechanisms. Suijs (1996) identifies one such mechanism, but shows there is no such mechanism which is also individually rational.[4] Maniquet (2003)'s Lemma 1 provides an expression for Shapley values in queuing problems; it is special case of our Proposition 1. More significantly, this paper shows that the Shapley allocation satisfies a collection of fairness axioms that he proposes. In a complete information setting, with commonly known waiting costs, Ju, Chun, and van den Brink (2014) provides a bargaining game which has the Shapley allocation of the queueing problem as a subgame perfect equilibrium outcome. Chun (2016) provides a comprehensive survey of the queuing literature.

Bidding Rings, Bankruptcy, and Cost/Surplus Sharing: The Shapley value also appears in the literature on collusion in auctions. Graham, Marshall, and Richard (1990) studies bidding rings, for a single item, from both a cooperative and non-cooperative perspective. In the complete information cooperative setting, they show that a particular knockout auction gives bidders their Shapley value payoffs.[5] They also study equilibrium bidding when information is incomplete and bidders are risk neutral, and contrast the outcome in that setting to the complete information cooperative outcome. Their knockout auction belongs to the class of compensation auctions we study (for the special case of only one valuable position and the demand of an exiting bidder is shared equally among bidders obtaining better positions). Our results therefore shed additional light on behavior in bidding rings since the equilibrium outcome in *any* compensated position auction (the knockout auction as well) converges to the Shapley value outcome as bidders become infinitely risk averse.

Aumann and Maschler (1985) shows that the solutions provided in the

[4] In fact, Suijs's result applies to a more general class of "scheduling" problems. Chun, Mitra, and Mutuswami (2019) provide a charaterization of the allocation rule in Suijs (1996) as the only rule satisfying efficiency, budget balancedness, equal treatment of equals, Pareto indifference, and upward-invariance.

[5] Littlechild and Owen (1973) obtains the same payoffs when allocating costs to the users of an airport runway.

Talmud to three different bankruptcy problems coincide with the nucleoli of the corresponding cooperative games. These solutions are generalizations of the solution to the contested garment problem: "Two hold a garment; one claims it all, the other claims half. Then the one is awarded three-fourths, the other one-fourth." In this solution, the lesser claimant concedes the uncontested half of the garment to the greater one, and the remainder is split equally. In compensated position auctions, at each round all but the worst remaining position are contested. We show that the maxmin perfect bid at each round can be interpreted as a demand for equal shares of the incremental benefits of the contested positions, and in this respect resembles the solution to the contested garment problem.

Compensated position auctions are reminiscent of serial cost sharing. Moulin and Shenker (1992) studies the problem of allocating costs when agents face a production technology with decreasing returns to scale. It proposes a cost sharing rule in which participants pay equal shares of incremental costs (defined in a particular way) and shows that, given this rule, the game in which the participants announce quantities is dominance solvable and the solution has several nice properties. The cost sharing rule is a primitive, part of the description of the game, whereas here the surplus shares are endogenously determined. In our setting, equilibrium demands for compensation can be interpreted as (inflated) demands for equal shares of the incremental benefits of contested positions. When a seller has multiple heterogeneous units of several goods, Lindsay (2018) proposes a mechanism for allocating surplus that is based on a modification of the Shapley value. It features allocations that are ex-post individually rational (and so bidders do not suffer from the exposure problem) and losing bidders make no payments.

The rest of the paper proceeds as follows: We provide in Section 2 a description of the position allocation problem and we identify the Shapley allocation of the associated cooperative game. Section 3 introduces compensated position auctions. For any given cost sharing rule, Section 4 identifies the maxmin perfect bidding strategy and shows that bidders obtain their Shapley value allocations when every bidder follows the maxmin perfect bidding strategy. Our equilibrium results for the Bayesian game where preference intensities are private information are in Section 5. Section 6 relates the Shapley value, maxmin, and equilibrium allocations. We conclude with a discussion in Section 7. All proofs are in the Appendix.

2 Shapley Value

While our focus is on auctions, we begin with some results on the Shapley value, which is the normative benchmark for our analysis.

$N \geq 2$ positions are to be allocated to N players who have equal claims, with one position to be assigned to each player.[6] Equal claims motivates two features of our analysis. First, it motivates treating the participants anonymously, both in the definition of the characteristic function and in the rules of compensated position auctions described in the next section. Second, it motivates the equiprobable random allocation of positions as the benchmark in Section 7 for the payoff to not participating in the auction.

Following Varian (2007), the positions have inherent values, denoted by $\alpha_1, \ldots, \alpha_N$, which are common to all the players. We order the positions so that $\alpha_1 \geq \ldots \geq \alpha_N$. Positions may be desirable or undesirable, i.e., we allow a mixture of positive and negative α's.[7] The payoff to a player whose preference intensity is x, and who receives position i, is $\alpha_i x$ plus any net transfer he receives. Hereafter, we will refer to a player's preference intensity as his *value*. The assumption that $\alpha_1, \ldots, \alpha_N$ are common across players is a symmetry assumption, which is important for the non-cooperative model of Section 5 to admit a symmetric equilibrium. A consequence of this assumption is that the ratio of $\alpha_i x$ to $\alpha_j x$ doesn't depend on a player's value x.

In this section it is convenient to order the players so that $x_1 \geq \ldots \geq x_N$. For any coalition $S \in 2^N$, let $y_1^{(S)}, \ldots, y_{|S|}^{(S)}$ be a rearrangement of the values $\{x_i | i \in S\}$ of the members of S such that $y_1^{(S)} \geq \ldots \geq y_{|S|}^{(S)}$. The problem is to efficiently and fairly allocate positions to players, while respecting budget balance.

Moulin (1992, Section 5) proposes one way of mapping an assignment

[6] This is without loss of generality since, if there are more players than positions, one can create dummy positions, with α's equal to zero, until the number of positions equals the number of players.

[7] Bogomolnaia, Moulin, Sandomirskiy, and Yanovskaya (2017) study a fair division problem when the goods to be divided are a mixture of both goods and bads.

problem into a cooperative game in characteristic function form. Moulin's construction seems appropriate if the purpose of the analysis is normative, as it is in our case. In our setting, a special case of the general assignment problem, the cooperative game is defined by the characteristic function

$$v(S) = \sum_{j=1}^{|S|} \alpha_j y_j^{(S)},$$

which is obtained by efficiently allocating to each coalition S the best $|S|$ positions. Moulin calls $v(S)$ the "stand alone" utility of coalition S.[8] It is the maximal utility the coalition can achieve by helping itself to the best positions. This characteristic function is appropriate for the normative analysis of fairness that we describe next.

Given a TU game $(\{1, \ldots, N\}, v)$, a payoff vector $(\pi_1, \ldots, \pi_N)$ is in the *anti-core* if (i) $\sum_{i=1}^{N} \pi_i = v(\{1, \ldots, N\})$ and (ii) for every coalition $S \in 2^N$ we have that $\sum_{i \in S} \pi_i \leq v(S)$. In other words, a payoff vector in the anti-core divides the surplus and no coalition S of players receives more than its stand alone utility $v(S)$. Such a payoff vector satisfies the minimal requirement for fairness that no coalition S receives more than what it would obtain were it allocated the $|S|$ best positions: such an allocation would require a subsidy from the complementary coalition, who would object on fairness grounds.[9] Hereafter, we say an allocation is in the anti-core if the payoff vector generated by the assignment of positions and transfers is in the anti-core.

Another sense in which allocations in the anti-core are fair is that transfers are "top down," i.e., for any k, the players allocated the k best positions pay compensation (in aggregate) to the players receiving worse positions. To see this, note that for an allocation in the anti-core we can write $\pi_i = \alpha_i x_i + \tau_i$,

[8] Moulin (1992, Theorem 2) establishes that the general assignment game is concave, and thus the problem of assigning players to positions is also concave. A cooperative game is concave if for any coalitions $S, T \subset N$ we have that

$$v(S \cup T) + v(S \cap T) \leq v(S) + v(T).$$

[9] The anti-core of a cooperative game is motivated by fairness considerations, in contrast to the more-familiar notion of the core which is motivated by strategic considerations. See Moulin (1995, Chapter 7).

where τ_i is the transfer of player i. For the players with the k highest values we have

$$\sum_{i=1}^{k} \pi_i = \sum_{i=1}^{k} (\alpha_i x_i + \tau_i) \leq v(\{1, \ldots, k\}) = \sum_{i=1}^{k} \alpha_i x_i,$$

where the inequality holds by the definition of the anti-core. Thus $\sum_{i=1}^{k} \tau_i \leq 0$, i.e., the aggregate transfer to the players allocated the k best positions must be non-positive.

Cooperative game theory provides a solution to the position allocation problem: allocate positions to maximize surplus and make transfers among the players so that each player obtains his Shapley value. We call such an allocation the Shapley allocation. The Shapley solution is appealing since it the only solution satisfying (i) efficiency, (ii) additivity, (iii) symmetry, and (iv) no surplus to dummy players. For a general characteristic function v, the Shapley value ϕ_i of player i is

$$\phi_i = \sum_{S \subseteq \{1, \ldots, N\}} \frac{(|S| - 1)!(N - |S|)!}{N!} \left[v(S) - v(S \backslash \{i\}) \right].$$

By Shapley (1971, Theorem 7), in a concave (convex) game the Shapley value is in the anti-core (core).

Player i's Shapley value can be interpreted as his expected marginal contribution when players arrive, one at a time, in a random order. Suppose, in particular, that (i) positions are assigned on a first-come-first-serve basis, and (ii) when a new player arrives then positions are efficiently reallocated and the new arrival extracts the full amount of his marginal contribution. Then each player obtains, in expectation, his Shapley value when all arrival orders of players are equally likely.

The cooperative solution takes the players' values as a primitive, which presents a practical obstacle to its implementation when values are privately known. The next sections provide decision theoretic and non-cooperative foundations for generating the Shapley allocation in this environment.

Proposition 1 characterizes Shapley values for the position allocation problem.

Proposition 1: *The Shapley value ϕ_i of player i in the position allocation problem is*

$$\phi_i = \frac{1}{i}\left(\sum_{m=1}^{i} \alpha_m\right) x_i - \sum_{m=1}^{N-i} \frac{1}{i+m-1}\left[\sum_{r=1}^{i+m-1} \frac{r}{i+m}\left(\alpha_r - \alpha_{r+1}\right) x_{i+m}\right].$$

Following the interpretation of a player's Shapley value as being his expected marginal contribution, the first term in the expression for ϕ_i is the expected gross contribution of player i, while the second term is the expected negative externality that i imposes on players who, as a result of i joining the coalition, receive worse positions. It is straightforward to show that the Shapley value of player i is at least his value for an "average" position, i.e.,

$$\phi_i \geq \frac{\alpha_1 + \cdots + \alpha_N}{N} x_i,$$

and players with higher values have higher Shapley values, i.e., $\phi_1 \geq \cdots \geq \phi_N$.

Example 1 provides Shapley values for the $N = 3$ problem.

Example 1: Suppose $N = 3$ and $x_1 > x_2 > x_3$. The players' Shapley values are:

$$\begin{aligned}
\phi_1 &= \alpha_1 x_1 - \frac{1}{2}\left(\alpha_1 - \alpha_2\right) x_2 - \frac{1}{6}\left(\alpha_1 - \alpha_2\right) x_3 - \frac{1}{3}\left(\alpha_2 - \alpha_3\right) x_{3,} \\
\phi_2 &= \frac{1}{2}\left(\alpha_1 + \alpha_2\right) x_2 - \frac{1}{6}\left(\alpha_1 - \alpha_2\right) x_3 - \frac{1}{3}\left(\alpha_2 - \alpha_3\right) x_3, \\
\phi_3 &= \frac{1}{3}\left(\alpha_1 + \alpha_2 + \alpha_3\right) x_3.
\end{aligned}$$

If $\alpha_1 = 6$, $\alpha_2 = 4$, and $\alpha_3 = 2$, and $x_1 = 3/4$, $x_2 = 1/2$, and $x_3 = 1/4$, then $\phi_1 = 15/4$, $\phi_2 = 9/4$, and $\phi_3 = 1$. In the Shapley allocation, player i receives position i. Players 1, 2, and 3, receive transfers of $-3/4$, $1/4$, and $1/2$, respectively.

As shown in the following Corollary, a feature of the problem we study is that Shapley values can be computed recursively, starting with player N and working backwards.

Corollary 1: *The players' Shapley values can be written as* $\phi_i = \alpha_i x_i + \tau_i$ *for* $i = 1, \ldots, N$*, where the transfers* $\tau_1, \ldots, \tau_N$ *are defined recursively as*

$$\tau_N = \frac{1}{N}(\alpha_1 + \cdots + \alpha_N) x_N - \alpha_N x_N$$

$$\vdots$$

$$\tau_k = \frac{1}{k}\left[(\alpha_1 + \cdots + \alpha_k) x_k - \sum_{i=k+1}^{N} \tau_i\right] - \alpha_k x_k$$

$$\vdots$$

$$\tau_1 = -\tau_2 - \tau_3 - \cdots - \tau_N.$$

When written as in Corollary 1, the Shapley value has a natural dynamic interpretation. Player N's Shapley value is an equal share of the surplus as he values it, i.e.,

$$\phi_N = \frac{1}{N}(\alpha_1 + \cdots + \alpha_N) x_N.$$

Player N's Shapley allocation is position N and transfer $\tau_N = \phi_N - \alpha_N x_N$. Following the transfer τ_N to Player N, the residual surplus is

$$(\alpha_1 + \cdots + \alpha_{N-1}) x_{N-1} - \tau_N$$

as Player $N-1$ values it, and Player $N-1$'s Shapley value is an equal share of this among the $N-1$ remaining players, i.e.,

$$\phi_{N-1} = \frac{1}{N-1}\left[(\alpha_1 + \cdots + \alpha_{N-1}) x_{N-1} - \tau_N\right].$$

In general, given transfers τ_N, ..., τ_{k+1}, the Shapley value of the player with the k-th lowest value,

$$\phi_k = \alpha_k x_k + \tau_k = \frac{1}{k}\left[(\alpha_1 + \cdots + \alpha_k) x_k - \sum_{i=k+1}^{N} \tau_i\right],$$

is an equal share, among the k remaining players, of the residual surplus as he values it.

The players' Shapley values can also be interpreted as equal shares of "worst case" residual surpluses. Since player N has the lowest value, ϕ_N is Player N's evaluation of the worst-case surplus in the sense that surplus is minimized were all the other players to also have the same value x_N. After Player N has exited, the term ϕ_{N-1} can likewise be viewed as Player $N-1$'s evaluation of an equal share of the worst-case residual surplus, and so on.

The interpretation of Shapley values as equal shares of worst-case residual surplus suggests a connection to maxmin play. In particular, dynamic mechanisms with the property that *the maxmin payoff of the active player with the lowest value is an equal share of the residual surplus* are natural candidates to implement Shapley allocations under maxmin play. The next section studies a class of auctions with this property.

3 Compensated Position Auctions

We now describe a class of auctions for solving the position allocation problem. These auctions takes place over $N-1$ rounds, where at each round the worst unallocated position is auctioned. At each round t, the active bidders simultaneously submit (possibly negative) demands for compensation. The bidder with the smallest demand receives position $N-t+1$ and his demand d_t as compensation, and he exits the auction. The compensation d_t is paid by the $N-t$ bidders that have not yet received positions according to a linear cost sharing rule

$$\lambda^t = (\lambda^t_1, \lambda^t_2, \ldots, \lambda^t_{N-t}),$$

where λ^t_i is the proportion of d_t paid by the bidder who ultimately receives position $i \leq N-t$. We require that for each $t = 1, \ldots, N-1$ that $\sum_{i=1}^{N-t} \lambda^t_i = 1$.[10]

Bidders have a common utility function u, where $u' > 0$ and $u'' \leq 0$. A bidder with value x who has the smallest demand d_t at round t, is allocated position $N-t+1$ and obtains a payoff of

$$u\left(\alpha_{N-t+1}x + d_t - \sum_{i=1}^{t-1} \lambda^i_{N-t+1} d_i\right),$$

where $\alpha_{N-t+1}x$ is the payoff from his position, d_t is the compensation he

[10] The λ^t_i's need not be positive. It is, however, important that only bidders who have not received positions are liable for d_t, i.e., $\lambda^t_{N-t+1} = \ldots = \lambda^t_N = 0$.

receives, and

$$\sum_{i=1}^{t-1} \lambda_{N-t+1}^{i} d_i$$

is the compensation he pays to bidders who exited at prior rounds. In particular, $\lambda_{N-t+1}^{i} d_i$ is the compensation he pays to the bidder who exited in round i. At round $N-1$, the final round, the bidder with the largest demand receives the best position, pays compensation to every other bidder, and obtains a payoff of

$$u\left(\alpha_1 x - \sum_{i=1}^{N-1} \lambda_1^i d_i\right).$$

In sum, a bidder who submits the smallest demand surrenders his claim to more desirable positions and receives compensation from the bidders who maintain their claims to these positions, while he pays compensation to bidders who have accepted less desirable positions.

Each bidder knows his own value, but not the values of the other bidders, and observes only the smallest demand at each round.[11] A *strategy* is a list of $N-1$ functions $\beta = (\beta_1, \ldots, \beta_{N-1})$, where $\beta_t(x; d_1, \ldots, d_{t-1})$ gives the demand in the t-th round of a bidder whose value is x, when $d_1, \ldots, d_{t-1}$ are the smallest demands in previous rounds. We write $\mathbf{d}_{t-1}$ for $(d_1, \ldots, d_{t-1})$.

Different cost sharing rules define different auctions. If the compensation to a bidder who exits the auction at round t is paid entirely by the next bidder to exit (at round $t+1$ with position $N-t$), then $\lambda_{N-t}^t = 1$ and $\lambda_1^t = \ldots = \lambda_{N-t-1}^t = 0$. If the compensation is paid in equal shares by the bidders obtaining better positions then $\lambda_1^t = \ldots = \lambda_{N-t}^t = 1/(N-t)$.

4 Maxmin

We first take a decision theoretic approach to bidding in compensated position auctions and ask what payoff a bidder can guarantee himself, i.e., what is his maxmin payoff. While a compensated position auction will have many

[11] The assumption that bidders observe only the smallest demand at each round is important in Section 5, as it is necessary for the existence of equilibrium in increasing strategies.

maxmin strategies, since the auction is dynamic we focus on "maxmin perfect" strategies which maximize a bidder's minimum payoff at each point in the auction. In this section we define the notion of a maxmin perfect bidding strategy and show that any compensated position auction has a unique such strategy. Our main result in this section is that every bidder obtains his Shapley allocation when each bidder follows the maxmin bidding strategy.

For a bidder who remains in the auction at round t, let $v_t(x_i, x_{-i}, \beta^i, \beta^{-i}; \mathbf{d}_{t-1})$ be the bidder's payoff when his value is x_i and he follows the strategy β^i, when x_{-i} and β^{-i} are the values and strategies of the remaining bidders, and $\mathbf{d}_{t-1}$ is the sequence of smallest demands at the prior rounds.

Definition: A strategy β^i *guarantees bidder i with value x_i a payoff of $\bar{v}_t$ at round t, given* $\mathbf{d}_{t-1}$, if $v_t(x_i, x_{-i}, \beta^i, \beta^{-i}; \mathbf{d}_{t-1}) \geq \bar{v}_t \ \forall x_{-i}, \beta^{-i}$.

Let $\bar{v}_t(x_i; \mathbf{d}_{t-1})$ be the largest payoff that bidder i with value x_i can guarantee at round t given $\mathbf{d}_{t-1}$.

Definition: A strategy β^i is a *maxmin perfect strategy for bidder i* if β^i guarantees bidder i with value x_i a payoff $\bar{v}_t(x_i; \mathbf{d}_{t-1})$ for each t, $x_i \in [0, \bar{x}]$, and $\mathbf{d}_{t-1}$.

Proposition 2 identifies the unique maxmin perfect strategy and the associated value for the compensated position auction with cost sharing rule $\lambda = (\lambda^1, \ldots, \lambda^{N-1})$.

Proposition 2: *Let $\lambda = (\lambda^1, \ldots, \lambda^{N-1})$ be a cost sharing rule. The strategy profile $\underline{\beta} = (\underline{\beta}_1, \ldots, \underline{\beta}_{N-1})$ given by*

$$\underline{\beta}_t(x; \mathbf{d}_{t-1}) = \sum_{m=1}^{N-t} \frac{m}{N-t+1}(\alpha_m - \alpha_{m+1})x - \sum_{i=1}^{t-1} \left[\sum_{m=1}^{N-t} \frac{m}{N-t+1}(\lambda^i_m - \lambda^i_{m+1}) d_i \right]$$

for each round t is the unique maxmin perfect strategy of the compensated position auction with cost sharing rule λ. In particular, $\underline{\beta}$ guarantees a bidder with value x, who is active at round t, a payoff of

$$\bar{v}_t(x; \mathbf{d}_{t-1}) = \left(\frac{\alpha_1 + \cdots + \alpha_{N-t+1}}{N-t+1} \right) x - \sum_{i=1}^{t-1} \left[\frac{\lambda^i_1 + \cdots + \lambda^i_{N-t+1}}{N-t+1} \right] d_i,$$

when $\mathbf{d}_{t-1}$ is the sequence of smallest demands from prior rounds.

Example 2: If demands are paid equally by players obtaining better positions, then $\lambda^i_m - \lambda^i_{m+1} = 0$ for $m = 1, \ldots, N-t$ and the maxmin perfect bid function in Proposition 2 simplifies to

$$\underline{\beta}_t(x; \mathbf{d}_{t-1}) = \sum_{m=1}^{N-t} \frac{m}{N-t+1}(\alpha_m - \alpha_{m+1})x,$$

and it guarantees

$$\bar{v}_t(x; \mathbf{d}_{t-1}) = \left(\frac{\alpha_1 + \cdots + \alpha_{N-t+1}}{N-t+1} \right) x - \sum_{i=1}^{t-1} \frac{1}{N-i} d_i.$$

The maxmin perfect strategy $\underline{\beta}$ has a natural fairness interpretation akin to the solution to the famous contested garment problem. Recall that the solution to the contest garment problem calls for giving each participant the uncontested portion of his demand and splitting the contested portion equally.

The maxmin bid function likewise calls for a bidder to demand an equal share of the incremental benefits of contested positions: Consider the bid function in Example 2. At round 1, positions 1 though $N-1$ are contested and position N, the worst position, is uncontested. There are $N-1$ bidders who will be allocated position $N-1$ or better, each of whom will enjoy an incremental benefit of $\alpha_{N-1} - \alpha_N$ times their value. A bidder i with value x demands an equal share, $1/N$-th, of this total benefit as he himself values it, i.e., he demands $\frac{N-1}{N}(\alpha_{N-1} - \alpha_N)\, x$. There are $N-2$ bidders who will obtain position $N-2$ or better, each of whom will each enjoy an incremental benefit of $\alpha_{N-2} - \alpha_{N-1}$ times their value. Bidder i demands an equal share of this total benefit too, i.e., he demands $\frac{N-2}{N}(\alpha_{N-2} - \alpha_{N-1})\, x$. Continuing in this fashion, one bidder will obtain position 1, and enjoy an incremental benefit of $\alpha_1 - \alpha_2$ times his value. Bidder i demands an equal share. Adding up these shares of incremental benefits for the contested positions yields $\underline{\beta}_1(x)$, bidder i's demand for compensation at round 1, as

$$\frac{1}{N}(\alpha_1 - \alpha_2)\, x + \cdots + \frac{N-2}{N}(\alpha_{N-2} - \alpha_{N-1})\, x + \frac{N-1}{N}(\alpha_{N-1} - \alpha_N)\, x.$$

The round t maxmin bid function $\underline{\beta}_t$ has an interpretation analogous to $\underline{\beta}_1$, where equal shares are relative to the $N-t+1$ bidders and unallocated positions remaining in the auction. For general cost sharing rules, bidders demand in round t an equal share of the incremental "net" benefits of contested positions, where positions differ in both their inherent values and in their liabilities for compensation.

Proposition 3, which follows, provides the decision theoretic foundation of the Shapley value in compensated position auctions. If all bidders follow the maxmin perfect bidding strategy, then each bidder obtains his Shapley value allocation. Surprisingly, this result is independent of the cost sharing rule.

Proposition 3: *If each bidder follows the maxmin perfect bidding strategy, then each bidder obtains his Shapley value allocation: ordering the players so that $x_1 \geq \ldots \geq x_N$, then player i obtains position i and the transfer τ_i given in Corollary 1.*

The following example illustrates Proposition 3 and the irrelevance of the cost sharing rule when there are three bidders.

Example 3: Suppose, as in Example 1, that $x_1 > x_2 > x_3$. By Proposition 2 the maxmin perfect bidding strategy is

$$\begin{aligned} \underline{\beta}_1(x) &= \frac{1}{3}(\alpha_1 - \alpha_2)\,x + \frac{2}{3}(\alpha_2 - \alpha_3)\,x \\ \underline{\beta}_2(x; d_1) &= \frac{1}{2}(\alpha_1 - \alpha_2)\,x - \frac{1}{2}(\lambda_1^1 - \lambda_2^1)d_1. \end{aligned}$$

Since $\lambda_1^1 + \lambda_2^1 = 1$ we can write

$$\underline{\beta}_2(x; d_1) = \frac{1}{2}(\alpha_1 - \alpha_2)\,x + \left[\lambda_2^1 - \frac{1}{2}\right] d_1.$$

In the auction, Bidder 3's round 1 demand $d_1 = \underline{\beta}_1(x_3)$ is smallest, he wins position 3, and he exits after receiving compensation $\underline{\beta}_1(x_3)$. His transfer is

$$\underline{\beta}_1(x_3) = \frac{1}{3}(\alpha_1 + \alpha_2 + \alpha_3)\,x_3 - \alpha_3 x_3 = \tau_3.$$

Bidder 2's demand $\underline{\beta}_2(x_2; d_1)$ is smallest at round 2, he wins position 2, and he exits after receiving compensation $\underline{\beta}_2(x_2; d_1)$ and paying compensation

$\lambda_2^1 d_1 = \lambda_2^1 \underline{\beta}_1(x_3)$. His transfer is

$$\begin{aligned} & \underline{\beta}_2(x_2; \underline{\beta}_1(x_3)) - \lambda_2^1 \underline{\beta}_1(x_3) \\ = & \frac{1}{2}((\alpha_1 + \alpha_2)x_2 - \tau_3) - \alpha_2 x_2 \\ = & \tau_2. \end{aligned}$$

Bidder 1 wins position 1 and pays total compensation of $d_2 + (1 - \lambda_2^1)d_1 = \underline{\beta}_2(x_2; \underline{\beta}_1(x_3)) + (1 - \lambda_2^1)\underline{\beta}_1(x_3)$. His transfer is

$$\begin{aligned} & -\underline{\beta}_2(x_2; \underline{\beta}_1(x_3)) - (1 - \lambda_2^1)\underline{\beta}_1(x_3) \\ = & -(\underline{\beta}_2(x_2; \underline{\beta}_1(x_3)) - \lambda_2^1 \underline{\beta}_1(x_3)) - \underline{\beta}_1(x_3) \\ = & -\tau_2 - \tau_3. \end{aligned}$$

Thus, each bidder receives his Shapley allocation regardless of the cost sharing rule.

Intuitively, whatever the cost sharing rule, maxmin bids adjust so that bidders allocated better positions ultimately pay equal shares of the surplus received by bidder allocated worse positions. In Example 3, this is evident from the term $\left[\lambda_2^1 - \frac{1}{2}\right] d_1$ that appears in the bid function $\underline{\beta}_2(x; d_1)$. If λ_2^1 exceeds 1/2, then round 2 demands increase by an amount that exactly offsets the compensation that the winner of position 2 pays in excess of an equal share of Bidder 3's demand d_1.

5 Equilibrium

In this section we study compensated position auctions in a standard independent private values setting. To reduce notation we focus on the compensated position auction in which demands are paid equally by bidders obtaining better positions, i.e., we assume at each round t that $\lambda_j^t = \frac{1}{N-t}$ for $j = 1, \ldots, N-t$.[12] Propositions 4 through 7 are stated for this cost sharing rule. Importantly, Proposition 8, which shows that the equilibrium bid

[12] When $\lambda_j^t = \frac{1}{N-t}$ for $j = 1, \ldots, N-t$, one can show that equilibrium demands are always positive. The auction can then equivalently be framed as one in which at each round the bid ascends from zero. The first bidder to drop receives the worst unallocated position and receives compensation equal to the price at which he drops.

function converges uniformly to the maximin perfect bid function as bidders become infinitely risk averse, holds for any cost sharing rule.

Independent Private Values

The bidders' preference intensities (hereafter "values") are independently and identically distributed according to cumulative distribution function F with support $[0, \bar{x}]$, where $\bar{x} < \infty$ and $f \equiv F'$ is continuous and positive on $[0, \bar{x}]$. Let $X_1, \ldots, X_N$ be N independent draws from F. Let $Z_1^{(N)}, \ldots, Z_N^{(N)}$ be a rearrangement of the X_i's such that $Z_1^{(N)} \leq Z_2^{(N)} \leq \ldots \leq Z_N^{(N)}$. The joint density of $Z_1^{(N)}, \ldots, Z_N^{(N)}$ is

$$g_{1,\ldots,N}^{(N)}(z_1, \ldots, z_N) = N! \prod_{i=1}^{N} f(z_i)$$

for $z_1 \leq z_2 \leq \ldots \leq z_N$ and $g_{1,\ldots,N}^{(N)}(z_1, \ldots, z_N) = 0$ otherwise. Let $G_t^{(N)}$ denote the *c.d.f.* of $Z_t^{(N)}$, i.e., $G_t^{(N)}$ is the distribution of the t-th lowest of N draws. The conditional density of $Z_{t+1}^{(N)}$ given $Z_1^{(N)} = z_1, \ldots, Z_t^{(N)} = z_t$ is

$$g_{t+1}^{(N)}(z_{t+1}|z_t) = (N-t) f(z_{t+1}) \frac{[1 - F(z_{t+1})]^{N-(t+1)}}{[1 - F(z_t)]^{N-t}}$$

if $0 \leq z_1 \leq \ldots \leq z_{t+1}$ and is zero otherwise. Define

$$\Gamma_t^N(z) \equiv g_{t+1}^{(N)}(z|z) = (N-t) \frac{f(z)}{1 - F(z)}$$

to be the hazard function.

Necessary Conditions for Equilibrium

Proposition 4 provides necessary conditions for β to be a symmetric Bayes Nash equilibrium (hereafter "equilibrium") in strictly increasing and differentiable bidding strategies.[13] These conditions are also sufficient for risk neutral and CARA bidders, as we establish in Propositions 5 and 6.

Proposition 4: *Any symmetric equilibrium β in increasing and differentiable*

[13] Bayes Nash equilibrium, rather than Bayes perfect equilibrium, is appropriate as there is no need to specify off-the-equilibrium-path beliefs and strategies since the only way a bidder can observably move play off the equilibrium path is by making a bid where he exits the auction.

bidding strategies satisfies the following system of $N-1$ differential equations:

$$u'\left(\alpha_{N-t+1}x+\beta_t(x;\mathbf{d}_{t-1})-\sum_{j=1}^{t-1}\frac{d_j}{N-j}\right)\beta_t'(x;\mathbf{d}_{t-1})$$

$$= -\left[\begin{array}{c} u\left(\alpha_{N-t}x+\beta_{t+1}(x;\mathbf{d}_{t-1},\beta_t(x;\mathbf{d}_{t-1}))-\frac{1}{N-t}\beta_t(x;\mathbf{d}_{t-1})-\sum_{j=1}^{t-1}\frac{d_j}{N-j}\right) \\ -u\left(\alpha_{N-t+1}x+\beta_t(x;\mathbf{d}_{t-1})-\sum_{j=1}^{t-1}\frac{d_j}{N-j}\right)\end{array}\right]\Gamma_t^N(x),$$

for each $t\in\{1,\ldots,N-1\}$ *where* $\beta_N(x;d_{N-1})\equiv 0$.

Risk Neutral Bidders

Proposition 5 identifies the equilibrium when bidders are risk neutral. We write β_t^0 for the equilibrium bid function.

Proposition 5: *Suppose that bidders are risk neutral. The unique symmetric equilibrium in increasing and differentiable strategies is given, for* $t=1,\ldots,N-1$, *by*

$$\beta_t^0(x)=\frac{N-t}{N-t+1}E\left[\left(\alpha_{N-t}-\alpha_{N-t+1}\right)Z_t^{(N)}+\beta_{t+1}^0(Z_t^{(N)})|Z_t^{(N)}>x>Z_{t-1}^{(N)}\right]$$

where $\beta_N^0\equiv 0$. *Equivalently, it is given by*

$$\beta_t^0(x)=\sum_{m=1}^{N-t}\frac{m}{N-t+1}E\left[\left(\alpha_m-\alpha_{m+1}\right)Z_{N-m}^{(N)}|Z_t^{(N)}>x>Z_{t-1}^{(N)}\right].$$

Equilibrium demands at each round are independent of the smallest demands at prior rounds.

Observe from the second expression for β_t^0 that if at some round t all the remaining positions have the same α's, i.e., $\alpha_1=\ldots=\alpha_{N-t+1}$, then bids are zero at round t and every subsequent round. This is intuitive since when the remaining positions are identical and the number of positions is equal to the number of remaining bidders, then no position is contested.

The risk neutral bid function β_t^0, given in Proposition 5, has a similar form and interpretation to the maxmin perfect bid function $\underline{\beta}_t$ when demands are paid equally by players obtaining better positions. At round $t>1$, positions are $1,\ldots,N-t$ are contested. The m-th term of $\underline{\beta}_t$,

$$\frac{m}{N-t+1}\left(\alpha_m-\alpha_{m+1}\right)x,$$

is an equal share (among the $N-t+1$ active bidders at round t) of the total benefit obtained by the m bidders allocated position m or better, as a bidder with value x values it. The m-th term of β_t^0,

$$\frac{m}{N-t+1}\left(\alpha_m-\alpha_{m+1}\right)E\left[Z_{N-m}^{(N)}|Z_t^{(N)}>x>Z_{t-1}^{(N)}\right],$$

is the same except that it is based on a value that is inflated relative to the bidder's true value. Risk neutral bidders demand more compensation, increasing their expected payoff at the expense of reducing their worst-case payoff.

Example 4: If $N=3$ and values are distributed $U[0,1]$, then the equilibrium bid functions for risk neutral bidders are

$$\beta_1^0(x)=(\alpha_1-\alpha_2)\left(\frac{1}{6}x+\frac{1}{6}\right)+(\alpha_2-\alpha_3)\left(\frac{1}{2}x+\frac{1}{6}\right)$$

and

$$\beta_2^0(x)=(\alpha_1-\alpha_2)\left(\frac{1}{3}x+\frac{1}{6}\right).$$

One can show that when bidders are risk neutral, then all symmetric, efficient, and budget-balanced mechanisms for the position allocation problem are payoff equivalent.[14] Thus, in a risk neutral world, any two position auctions are payoff equivalent to each other as well as to other mechanisms such as the expected externality mechanism of Arrow, d'Aspremont, and Gérard-Varet (AGV).

The payoff equivalance result implies that there is no symmetric, efficient, and budget-balanced Bayesian incentive compatible mechanism that ex-post implements allocations in the anti-core. To see this, suppose $\alpha_1=1$ and $\alpha_2=\ldots=a_N=0$, i.e., only the first position has value. Consider a bidder with value zero. The bidder obtains a payoff of zero in the anti-core. (The stand alone utility of a bidder with value zero is zero.) By contrast, the expected payoff of a bidder with value zero in *any* symmetric, efficient, and budget-balanced Bayesian incentive compatible mechanism is

$$\frac{1}{N}E[Z_{N-1}^{(N)}]>0.$$

[14] A proof is provided in the Online Appendix.

Hence in any such mechanism a bidder with value zero obtains, ex-post, a positive payoff with positive probability, i.e., an allocation not in the anti-core. Since the Shapley allocation of a concave game is in the anti-core, there is thus no such mechanism that implements the Shapley value.

Proposition 3 established that maximin bidding in a compensated position auction generates Shapley allocations. We have just argued that when bidders are risk neutral there is no symmetric, efficient, and budget-balanced Bayesian incentive compatible that implements Shapley allocations. The remainder of this section works towards establishing that the compensated position auction generates the Shapley allocation as bidders become infinitely risk averse.

CARA Bidders

The next proposition characterizes equilibrium when bidders have constant absolute risk aversion (CARA), i.e., utility is given by

$$u^{\theta}(x) = \frac{1 - e^{-\theta x}}{\theta},$$

where $\theta > 0$ is the common index of risk aversion. Note that $\lim_{\theta \to 0} u^{\theta}(x) = x$, i.e., bidders are risk neutral in the limit as θ approaches zero. Denote by β_t^{θ} the equilibrium bid function in round t when bidders have CARA index of risk aversion θ. The assumption that bidders have CARA preferences allows for a tractable analysis of the effect of risk aversion on bidding behavior, with risk-neutral bidding and maxmin bidding as extreme cases. With CARA preferences, equilibrium bids depend only on a bidder's value and the round, and are independent of the history of lowest demands.

Proposition 6: *Suppose that bidders are CARA risk averse with index of risk aversion $\theta > 0$. The unique symmetric equilibrium in increasing and differentiable strategies is given recursively, for $t = 1, \ldots, N-1$, by*

$$\beta_t^{\theta}(x) = -\frac{N-t}{(N-t+1)\,\theta} \ln \left\{ E\left[e^{-\theta[(\alpha_{N-t} - \alpha_{N-t+1}) Z_t^{(N)} + \beta_{t+1}^{\theta}(Z_t^{(N)})]} | Z_t^{(N)} > x > Z_{t-1}^{(N)} \right] \right\}$$

where $\beta_N^{\theta} \equiv 0$. Equilibrium bids at each round are independent of prior smallest demands.

Example 5: If $N = 3$ and values are distributed $U[0,1]$, then the equilibrium bid functions for CARA risk averse bidders are

$$\beta_1^\theta(x) = -\frac{2}{3\theta}\ln\left\{\frac{\int_x^1 e^{-\theta\left[(\alpha_2-\alpha_3)z+\beta_2^\theta(z)\right]}3(1-z)^2dz}{(1-x)^3}\right\}$$

and

$$\beta_2^\theta(x) = -\frac{1}{2\theta}\ln\left\{\frac{\int_x^1 e^{-\theta(\alpha_1-\alpha_2)z}2(1-z)dz}{(1-x)^2}\right\}.$$

Bounds and Comparative Statics

Proposition 7 provides upper and lower bounds for the CARA equilibrium bid functions. The risk neutral bid function β_t^0 is an upper bound for the equilibrium bid function of a CARA risk averse bidder while $\underline{\beta}_t$ is a lower bound: a risk averse bidder submits a smaller demand, and thus accepts less compensation, than were he risk neutral, but submits a larger demand than his maxmin perfect demand.

Proposition 7: *Suppose that bidders are CARA risk averse with index of risk aversion $\theta > 0$ and $\alpha_1 > \alpha_2$. Then for each $t = 1,\ldots,N-1$ we have that*

$$\underline{\beta}_t(x) < \beta_t^\theta(x) < \beta_t^0(x) \text{ for } x < \bar{x},$$

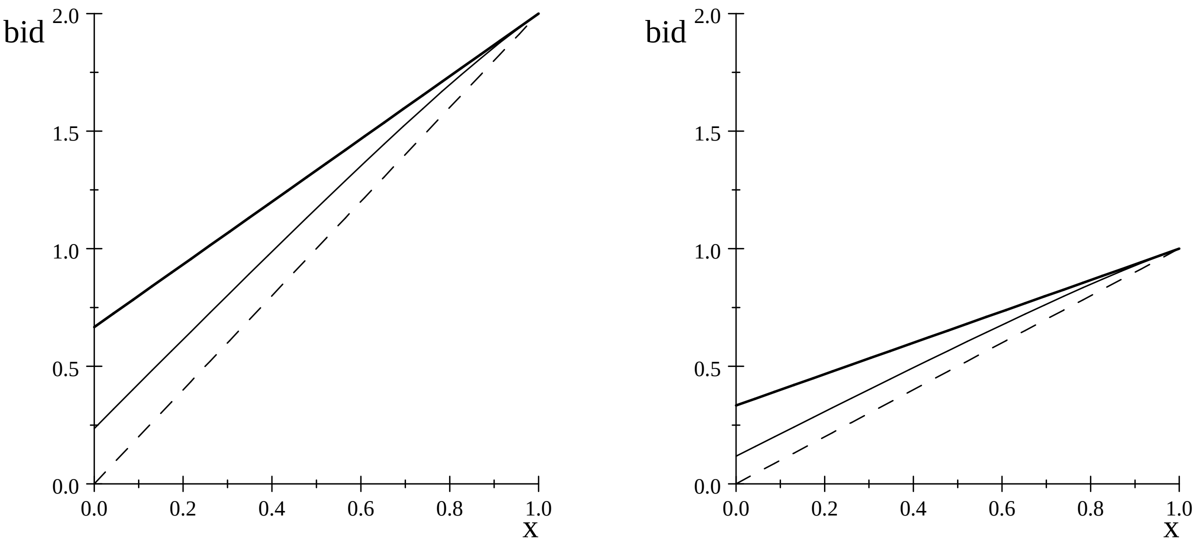
where

$$\underline{\beta}_t(x) = \sum_{m=1}^{N-t}\frac{m}{N-t+1}\left(\alpha_m - \alpha_{m+1}\right)x.$$

Proposition 8 shows that demands decrease as bidders become more risk averse, and that demands converge uniformly to the maxmin demands as bidders become infinitely risk averse.

Proposition 8: *Suppose that bidders are CARA risk averse with index of risk aversion $\theta > 0$. Then for each $t = 1,\ldots,N-1$ we have that $\beta_t^\theta(x)$ is decreasing in θ, and β_t^θ converges uniformly to $\underline{\beta}_t$ on $[0,\bar{x}]$ as $\theta \to \infty$.*

Figure 1 illustrates propositions 7 and 8 when $N = 3$, values are distributed $U[0,1]$, and $\alpha_1 = 6$, $\alpha_2 = 4$, and $\alpha_3 = 2$. In the figure, the bold solid lines are the risk neutral bid functions (i.e., $\theta = 0$) for rounds 1 and 2, which bound above the bids of CARA risk averse bidders. The dashed lines give $\underline{\beta}_1$ and $\underline{\beta}_2$, the maximin bid functions. The thin solid lines are the bid functions when bidders have CARA index of risk aversion of $\theta = 10$.



Round 1: $\theta = 0$, 10, and ∞.

Round 2: $\theta = 0$, 10, and ∞.

Figure 1: CARA Bounds

6 Equilibrium, Maxmin, and the Shapley Value

By Proposition 8, as bidders become infinitely risk averse, the equilibrium bid function converges to the maxmin perfect bid function $\underline{\beta}$. By Proposition 3, when each bidder follows the maxmin perfect strategy, then each obtains his Shapley value allocation. Corollary 2 follows immediately.

Corollary 2: *Suppose that bidders are CARA risk averse with index of risk aversion $\theta > 0$. Then the equilibrium allocation converges to the Shapley-value allocation as $\theta \to \infty$.*

Corollary 2 holds for any class of utility functions such that the equilibrium bid functions converge to the maxmin perfect bid function.

Since the Shapley allocation is in the anti-core, these results imply that the compensated position auction produces allocations in the anti-core of the TU cooperative game when bidders are sufficiently risk averse or when each bidder follows the maxmin perfect strategy. The next example and the associated figure illustrate Corollary 2, showing that the bidders' equilibrium transfers converge to their Shapley value transfers as bidders become infinitely risk averse.

Example 6: Suppose demands are paid equally by players obtaining better positions. Figure 2 shows the equilibrium transfer of each bidder as a function of θ, when $\alpha_1 = 6$, $\alpha_2 = 4$, $\alpha_3 = 2$ and the bidders' values are $x_1 = 3/4$, $x_2 = 1/2$, and $x_3 = 1/4$. The transfer of Bidder 3 is

$$t_3(\theta) = \beta_1^\theta(x_3),$$

of Bidder 2 is

$$t_2(\theta) = \beta_2^\theta(x_2; \beta_1^\theta(x_3)) - \frac{1}{2}\beta_1^\theta(x_3),$$

of Bidder 1 is

$$t_1(\theta) = -\beta_2^\theta(x_2; \beta_1^\theta(x_3)) - \frac{1}{2}\beta_1^\theta(x_3).$$

The dashed lines at 1/2, 1/4, and −3/4 are the bidders' Shapley transfers.

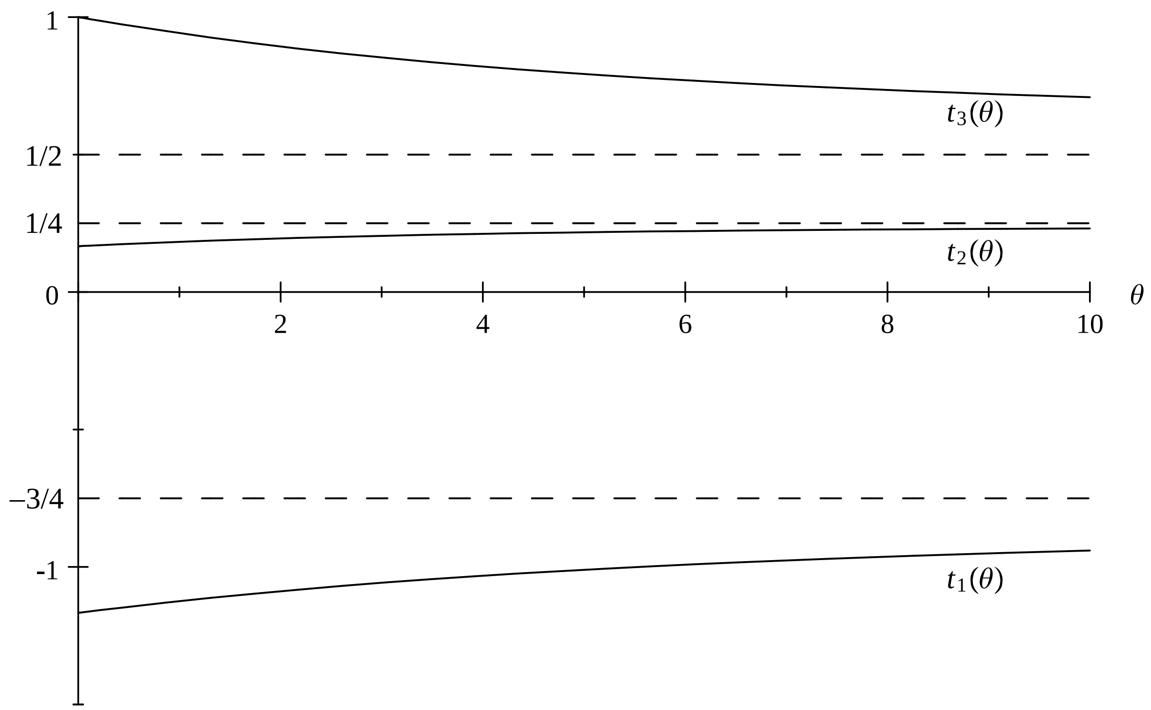


Figure 2: Equilibrium Transfers as a Function of θ.

Since the auction is efficient, each bidder is allocated the same position he would receive in the Shapley allocation. As θ approaches infinity, each bidder also receives the same transfer that he would receive in the Shapley allocation: Bidder 3 receives compensation of $\underline{\beta}_1(1/4) = 1/2$. Bidder 2 receives compensation of $\underline{\beta}_2(1/2) = 1/2$ from Bidder 1, but pays compensation of $\frac{1}{2}\underline{\beta}_1(1/4)$ to Bidder 3, for a net transfer of 1/4. Bidder 1 pays compensation of $\frac{1}{2}\underline{\beta}_1(1/4)$ to Bidder 3 and $\underline{\beta}_2(1/2)$ to Bidder 2, for a net transfer of $-3/4$. These are exactly the transfers identified in Example 1.

7 Discussion

This paper proposes compensated position auctions as a solution to the problem of fairly allocating heterogeneous items, priorities, positions, or rights among participants who have equal claims. Compensated position auctions are efficient and budget balanced. From a purely theoretical perspective these auctions are of interest since they provide decision theoretic and non-cooperative foundations for the Shapley value in an environment with incomplete information. Since the equilibrium allocation approaches the Shapley allocation as bidders become risk averse, then for sufficiently risk averse bidders the equilibrium allocation is in the anti-core and thus transfers have the "top-down" property discussed in Section 2.

Participation in the auction is individually rational for a bidder when the alternative is the random assignment of positions: Following the maxmin perfect bidding strategy, by Proposition 2 a bidder with value x guarantees himself a payoff of at least $\bar{v}_1(x) = \frac{1}{N}\sum_{m=1}^{N}\alpha_m x$ and a utility of at least $u(\frac{1}{N}\sum_{m=1}^{N}\alpha_m x)$. His equilibrium expected utility is therefore at least $u(\frac{1}{N}\sum_{m=1}^{N}\alpha_m x)$ since it must exceed his maxmin utility. Concavity of u implies that

$$u(\frac{1}{N}\sum_{m=1}^{N}\alpha_m x) \geq \frac{1}{N}\sum_{m=1}^{N}u(\alpha_m x),$$

and thus bidders prefer the compensated position auction to the random allocation of positions.[15]

[15] It is well known that the AGV mechanism need not be individually rational. For the position allocation problem, since participation in a position auction is individually rational, then payoff equivalence implies participation is individually rational in the AGV

For the position allocation problem there may be other mechanisms whose equilibrium allocation converges to the Shapley allocation as bidders become infinitely risk averse, and which generates the Shapley allocation under maxmin play. It is easy, however, to construct auctions that do not have these properties. Consider, for example, the auction in which all bidders simultaneously make sealed bids, the highest bidder gets the best position, the second highest bidder gets the second best position, and so on. Suppose further that only the highest bidder pays his bid and his bid is divided equally among all the bidders. If the auction has a symmetric equilibrium in increasing strategies, then the auction will be efficient and budget balanced. It cannot, however, generate the Shapley allocation as all the bidders (except the highest) receive the same net transfer, namely $1/N$-th of the highest bid. As Example 1 illustrates, the Shapley allocation requires different bidders receive different net transfers. As another example, it is easy to show that the AGV mechanism does not produce the Shapley allocation under maxmin bidding.

Auctions in which there is a seller need not generate allocations in the anti-core. Consider, for example, a simple second-price sealed-bid auction with two bidders, whose values are x_1 and x_2, and assume that $x_1 > x_2 > 0$. Label the seller as player 0, and assume the seller's value for the good is zero. The stand alone utility of any individual player is zero, i.e., $v(\{0\}) = v(\{1\}) = v(\{2\}) = 0$. Furthermore, $v(\{0,1\}) = x_1$, $v(\{0,2\}) = x_2$, and $v(\{0,1,2\}) = x_1$. The anti-core is empty since, for a payoff profile (π_0, π_1, π_2) to be in the anti-core, requires that $\pi_i \leq v(\{i\}) = 0$ for $i \in \{0,1,2\}$ and $\pi_0 + \pi_1 + \pi_2 = x_1$, and these conditions can not be simultaneously satisfied. It is straightforward to compute the players' Shapley value payoffs as $\phi_0 = (3x_1 + x_2)/6$, $\phi_1 = (3x_1 - 2x_2)/6$, and $\phi_2 = x_2/6$. These payoffs do not coincide with the Bayes Nash equilibrium payoffs of x_2 for the seller, $x_1 - x_2$ for the high-value bidder, and 0 for the low-value bidder. Since value bidding is a weakly dominant strategy regardless of risk attitudes, the Bayes Nash equilibrium allocation does not approach the Shapley allocation even as bidders become infinitely risk averse.

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as well.

8 Appendix

The proof of Proposition 1 involves combinatorial arguments that play no role in the remaining proofs. It is included for completeness, but the reader is invited to skip it.

Proof of Proposition 1: We compute the Shapley value directly using that

$$\phi_i = \sum_{s=1}^{N} \frac{(N-s)!(s-1)!}{N!} \left[\sum_{B_i(s)} (v(S) - v(S \backslash \{i\})) \right]$$

where

$$B_i(s) = \{S | i \in S \text{ and } |S| = s\}.$$

We first compute the marginal contribution of player i to coalition S. If $i \in S$ has the j-th highest value in coalition S (i.e., $x_i = y_j^{(S)}$) then

$$\begin{aligned} v(S) - v(S \backslash \{i\}) &= \alpha_j y_j^{(S)} - \sum_{m=1}^{|S|-j} (\alpha_{j+m-1} - \alpha_{j+m}) \, y_{j+m}^{(S)} \\ &= \alpha_j x_i - \sum_{m=1}^{|S|-j} (\alpha_{j+m-1} - \alpha_{j+m}) \, y_{j+m}^{(S)}. \end{aligned}$$

This follows since in coalition S player i is assigned the j-th position, players in S with a smaller index than i stay in the same position they occupied in $S \backslash \{i\}$, and players with a higher index than i move down one position.

Player i's Shapley value can be written as

$$\phi_i = c^i x_i - \sum_{m=1}^{N-i} \delta^{im} x_{i+m},$$

where c^i is of the form

$$c^i = c_1^i \alpha_1 + \cdots + c_i^i \alpha_i,$$

and δ^{im} is of the form

$$\delta^{im} = \delta_1^{im} (\alpha_1 - \alpha_2) + \cdots + \delta_{i+m-1}^{im} (\alpha_{i+m-1} - \alpha_{i+m}).$$

The term c^i is the expected contribution of player i and $\delta^{im} x_{i+m}$ is the expected externality that i imposes on player $i+m$.

We now compute c_r^i for $1 \leq r \leq i$, which is the contribution of player i when allocated position r. For each coalition size s, we count the number of coalitions of size s where i is in position r and multiply this number by the appropriate Shapley weight. The coefficient c_r^i is the sum of these terms over all s.

The smallest coalitions where i is in position r are coalitions of size r, and consist of player i and $r-1$ players with a smaller index. The largest coalitions where i is in position r are coalitions of size $N-i+r$, and consist of player i, $r-1$ players with a smaller index, and $N-i$ players with a larger index. The number of coalitions of size s where i is placed in position r is

$$\binom{i-1}{r-1}\binom{N-i}{s-r},$$

where $\binom{i-1}{r-1}$ is the number of ways of choosing $r-1$ players with index smaller than i from $i-1$ players, and $\binom{N-i}{s-r}$ is the number of ways of choosing $s-r$ players with index larger than i from $N-i$ players. The Shapley weight for coalitions of size s is

$$\frac{(s-1)!(N-s)!}{N!},$$

and therefore

$$c_r^i = \sum_{s=r}^{N-i+r} \frac{(s-1)!(N-s)!}{N!}\binom{i-1}{r-1}\binom{N-i}{s-r}.$$

Summing across positions where player i can be placed yields

$$\begin{aligned} c^i &= \sum_{r=1}^{i}\left[\sum_{s=r}^{N-i+r} \frac{(s-1)!(N-s)!}{N!}\binom{i-1}{r-1}\binom{N-i}{s-r}\right]\alpha_r \\ &= \sum_{r=1}^{i}\left[\frac{1}{N}\sum_{s=r}^{N-i+r} \frac{\binom{i-1}{r-1}\binom{N-i}{s-r}}{\binom{N-1}{s-1}}\right]\alpha_r \\ &= \frac{1}{i}\sum_{r=1}^{i}\alpha_r, \end{aligned}$$

where the last equality holds by Claim 4 in the Supplemental Appendix.[16]

Next, we compute δ_r^{im} for $0 < m \leq N-i$ and $1 \leq r < i+m$. The term $\delta_r^{im}(\alpha_r - \alpha_{r+1})x_{i+m}$ will be the expected externality player i imposes on

[16] See http://www.johnwooders.com/papers/PositionAuctionsSupplementalAppendix.pdf

player $i+m$ by pushing player $i+m$ from r to $r+1$. For each player $i+m$, position r, and coalition size s, we count the number of coalitions of size s where player i pushes player $i+m$ from position r to position $r+1$ and we multiply this number by the appropriate Shapley weight. The coefficient δ_r^{im} is the sum of these terms over all s.

The smallest coalitions where i pushes $i+m$ from position r to position $r+1$ are coalitions of size $r+1$, and consist of player i, player $i+m$, and $r-1$ other players with smaller index than $i+m$. The largest coalitions where i pushes $i+m$ from position r to position $r+1$ are coalitions of size $r+1+N-(i+m)$, and consist of player i, player $i+m$, $r-1$ other players with index smaller than $i+m$, and the $N-(i+m)$ players with an index larger than $i+m$. The number of coalitions of size s where i pushes $i+m$ from position r to position $r+1$ is

$$\binom{i+m-2}{r-1}\binom{N-(i+m)}{s-(r+1)},$$

where $\binom{i+m-2}{r-1}$ is the number of ways of choosing $r-1$ players (excluding player i) with index smaller than $i+m$, and $\binom{N-(i+m)}{s-(r+1)}$ is the number of ways of choosing $s-(r+1)$ players with index larger than $i+m$ from $N-(i+m)$ players. The Shapley weight for coalitions of size s is

$$\frac{(s-1)!(N-s)!}{N!},$$

and therefore,

$$\delta_r^{im} = \sum_{s=r+1}^{r+1+N-(i+m)} \frac{(s-1)!(N-s)!}{N!}\binom{i+m-2}{r-1}\binom{N-(i+m)}{s-(r+1)}.$$

Summing across positions where player $i+m$ can be placed yields

$$\begin{aligned}
\delta^{im} &= \sum_{r=1}^{i+m-1}\left[\sum_{s=r+1}^{r+1+N-(i+m)} \frac{(s-1)!(N-s)!}{N!}\binom{i+m-2}{r-1}\binom{N-(i+m)}{s-(r+1)}\right](\alpha_r-\alpha_{r+1}) \\
&= \sum_{r=1}^{i+m-1}\left[\frac{1}{N}\sum_{s=r+1}^{r+1+N-(i+m)} \frac{\binom{i+m-2}{r-1}\binom{N-(i+m)}{s-(r+1)}}{\binom{N-1}{S-1}}\right](\alpha_r-\alpha_{r+1}).
\end{aligned}$$

The identity in Claim 4 holds for all $i \leq N$. Replacing i with $i+m$ and r with $r+1$ in this identity, and noting that $i+m \leq N$ also, we obtain the following new identity

$$\frac{1}{N}\sum_{s=(r+1)}^{N+(r+1)-(i+m)}\binom{(i+m)-1}{(r+1)-1}\frac{\binom{N-(i+m)}{s-(r+1)}}{\binom{N-1}{s-1}}=\frac{1}{i+m}.$$

Applying this new identity to δ^{im} yields

$$\begin{aligned}\delta^{im} &= \sum_{r=1}^{i+m-1}\frac{\binom{i+m-2}{r-1}}{\binom{(i+m)-1}{(r+1)-1}}\left[\frac{1}{N}\sum_{s=r+1}^{r+1+N-(i+m)}\frac{\binom{(i+m)-1}{(r+1)-1}\binom{N-(i+m)}{s-(r+1)}}{\binom{N-1}{S-1}}\right](\alpha_r-\alpha_{r+1})\\ &= \frac{1}{i+m}\sum_{r=1}^{i+m-1}\frac{\binom{i+m-2}{r-1}}{\binom{i+m-1}{r}}(\alpha_r-\alpha_{r+1}).\end{aligned}$$

The total expected externality that player i imposes on the other players is

$$\begin{aligned}\sum_{m=1}^{N-i}\delta^{im}x_{i+m} &= \sum_{m=1}^{N-i}\left[\frac{1}{i+m}\sum_{r=1}^{i+m-1}\frac{\binom{i+m-2}{r-1}}{\binom{i+m-1}{r}}(\alpha_r-\alpha_{r+1})\right]x_{i+m}\\ &= \sum_{m=1}^{N-i}\frac{1}{i+m-1}\left[\frac{i+m-1}{i+m}\sum_{r=1}^{i+m-1}\frac{\binom{i+m-2}{r-1}}{\binom{i+m-1}{r}}(\alpha_r-\alpha_{r+1})\right]x_{i+m}.\end{aligned}$$

Noting that

$$\begin{aligned}(i+m-1)\frac{\binom{i+m-2}{r-1}}{\binom{i+m-1}{r}} &= (i+m-1)\frac{\frac{(i+m-2)!}{(i+m-2-(r-1))!(r-1)!}}{\frac{(i+m-1)!}{(i+m-1-r))!r!}}\\ &= (i+m-1)\frac{\frac{(i+m-2)!}{(i+m-1-r)!(r-1)!}}{\frac{(i+m-1)!}{(i+m-1-r))!r!}}\\ &= r,\end{aligned}$$

we can write

$$\sum_{m=1}^{N-i}\delta^{im}x_{i+m}=\sum_{m=1}^{N-i}\frac{1}{i+m-1}\left[\sum_{r=1}^{i+m-1}\frac{r}{i+m}(\alpha_r-\alpha_{r+1})\,x_{i+m}\right].$$

Collecting terms, the Shapley value of player i is

$$\phi_i=\frac{1}{i}\left(\sum_{m=1}^{i}\alpha_m\right)x_i-\sum_{m=1}^{N-i}\frac{1}{i+m-1}\left[\sum_{r=1}^{i+m-1}\frac{r}{i+m}(\alpha_r-\alpha_{r+1})\,x_{i+m}\right].$$

□

Proof of Corollary 1: We first show that

$$\tau_k = \left[\frac{\alpha_1 + \cdots + \alpha_{k-1}}{k} - \frac{k-1}{k}\alpha_k\right] x_k - \sum_{j=k+1}^{N} \frac{1}{j-1}\left[\frac{\alpha_1 + \cdots + \alpha_{j-1}}{j} - \frac{j-1}{j}\alpha_j\right] x_j.$$

Clearly true for $k = N$. Assume that it is true for $k = k' + 1$. We show it is true for k'.

Subclaim: We first establish the following: If for $j = k' + 1, ..., N$ we have

$$\tau_j = s_j - \sum_{m=j+1}^{N} \frac{1}{m-1} s_m,$$

then

$$\tau_N + \cdots + \tau_{k'+1} = s_{k'+1} + \frac{k'}{k'+1} s_{k'+2} + \cdots + \frac{k'}{N-2} s_{N-1} + \frac{k'}{N-1} s_N.$$

We have $\tau_N = s_N$. Assume that the claim is true for $\tau_N + \cdots + \tau_{k'+2}$. We show it is true for $\tau_N + \cdots + \tau_{k'+1}$. We have

$$\tau_N + \cdots + \tau_{k'+2} = s_{k'+2} + \cdots + \frac{k'+1}{N-2} s_{N-1} + \frac{k'+1}{N-1} s_N$$

and

$$\tau_{k'+1} = s_{k'+1} - \frac{1}{k'+1} s_{k'+2} - \cdots - \frac{1}{N-2} s_{N-1} - \frac{1}{N-1} s_N.$$

Adding these equations gives us the result.

Define

$$s_j = \left[\frac{\alpha_1 + \cdots + \alpha_{j-1}}{j} - \frac{j-1}{j}\alpha_j\right] x_j.$$

Simple algebra shows that

$$s_j = \sum_{m=1}^{j-1} \frac{m}{j}(\alpha_m - \alpha_{m+1}) x_j$$

We show that

$$\tau_{k'} = \left[\frac{\alpha_1 + \cdots + \alpha_{k'-1}}{k'} - \frac{k'-1}{k'}\alpha_{k'}\right] x_{k'} - \sum_{j=k'+1}^{N} \frac{1}{j-1}\left[\frac{\alpha_1 + \cdots + \alpha_{j-1}}{j} - \frac{j-1}{j}\alpha_j\right] x_j,$$

which we can write as

$$\tau_{k'} = s_{k'} - \sum_{j=k'+1}^{N} \frac{1}{j-1} s_j.$$

We have

$$\begin{aligned}
\tau_{k'} &= \frac{1}{k'}\left[(\alpha_1 + \cdots + \alpha_{k'})\, x_{k'} - \sum_{i=k'+1}^{N} \tau_i\right] - \alpha_{k'} x_{k'} \\
&= \left[\frac{\alpha_1 + \cdots + \alpha_{k'-1}}{k'} - \frac{k'-1}{k'}\alpha_{k'}\right] x_{k'} - \frac{1}{k'} \sum_{i=k'+1}^{N} \tau_i
\end{aligned}$$

By the subclaim

$$\frac{1}{k'} \sum_{i=k'+1}^{N} \tau_i = \frac{1}{k'} s_{k'+1} + \frac{1}{k'+1} s_{k'+2} + \cdots + \frac{1}{N-2} s_{N-1} + \frac{1}{N-1} s_N.$$

Hence,

$$\tau_{k'} = s_{k'} - \left(\frac{1}{k'} s_{k'+1} + \frac{1}{k'+1} s_{k'+2} + \cdots + \frac{1}{N-2} s_{N-1} + \frac{1}{N-1} s_N\right),$$

which establishes the claim.

Next we show that $\phi_{k'} = \alpha_{k'} x_{k'} + \tau_{k'}$, which establishes the Corollary.

$$\begin{aligned}
\alpha_{k'} x_{k'} + \tau_{k'} &= \frac{\alpha_1 + \cdots + \alpha_{k'}}{k'} x_{k'} - \left(\frac{1}{k'} s_{k'+1} + \frac{1}{k'+1} s_{k'+2} + \cdots + \frac{1}{N-2} s_{N-1} + \frac{1}{N-1} s_N\right) \\
&= \frac{\alpha_1 + \cdots + \alpha_{k'}}{k'} x_{k'} - \sum_{m=1}^{N-k'} \frac{1}{k'+m-1} \left[\sum_{r=1}^{k'+m-1} \frac{r}{k'+m} (\alpha_r - \alpha_{r+1}) x_{k'+m}\right],
\end{aligned}$$

where we use

$$s_{k'+m} = \sum_{r=1}^{k'+m-1} \frac{r}{k'+m} (\alpha_r - \alpha_{r+1}) x_{k'+m}.$$

Hence $\alpha_{k'} x_{k'} + \tau_{k'} = \phi_{k'}$ as given in Proposition 1. This establishes the Corollary. □

Proof of Proposition 2: We first show that following $\underline{\beta}$ guarantees a bidder with value x a payoff at round t of at least

$$\bar{v}_t(x;\mathbf{d}_{t-1}) = \left(\frac{\alpha_1 + \cdots + \alpha_{N-t+1}}{N-t+1}\right)x - \sum_{i=1}^{t-1}\left[\frac{\lambda_1^i + \cdots + \lambda_{N-t+1}^i}{N-t+1}\right]d_i,$$

when d_{t-1} is the sequence of smallest demands at prior rounds.

The proof is by induction. Consider round $N-1$. A bidder with value x whose demand is d_{N-1} either (i) has the smallest demand and obtains a payoff of

$$\alpha_2 x + d_{N-1} - \sum_{i=1}^{N-2}\lambda_2^i d_i$$

or (ii) his rival has the smallest demand $b \leq d_{N-1}$ and he obtains a payoff of $\alpha_1 x - b - \sum_{i=1}^{N-2}\lambda_1^i d_i$. In the second case, his payoff is at least

$$\alpha_1 x - d_{N-1} - \sum_{i=1}^{N-2}\lambda_1^i d_i.$$

The bidder maximizes his minimum payoff when d_{N-1} satisfies

$$\alpha_2 x + d_{N-1} - \sum_{i=1}^{N-2}\lambda_2^i d_i = \alpha_1 x - d_{N-1} - \sum_{i=1}^{N-2}\lambda_1^i d_i,$$

i.e.,

$$d_{N-1} = \frac{\alpha_1 - \alpha_2}{2}x - \sum_{i=1}^{N-2}\frac{\lambda_1^i - \lambda_2^i}{2}d_i.$$

Hence at round $N-1$ the bidder guarantees himself a payoff of at least

$$\bar{v}_{N-1}(x;\mathbf{d}_{N-2}) = \frac{\alpha_1 + \alpha_2}{2}x - \sum_{i=1}^{N-2}\frac{\lambda_1^i + \lambda_2^i}{2}d_i$$

by following

$$\underline{\beta}_{N-1}(x;\mathbf{d}_{N-2}) = \frac{\alpha_1 - \alpha_2}{2}x - \sum_{i=1}^{N-2}\frac{\lambda_1^i - \lambda_2^i}{2}d_i.$$

Suppose that at round $t+1$, given smallest demands d_t, a bidder with value x can guarantee himself at least

$$\bar{v}_{t+1}(x;\mathbf{d}_t) = \sum_{m=1}^{N-t}\frac{\alpha_m}{N-t}x - \sum_{i=1}^{t}\frac{\lambda_1^i + \ldots + \lambda_{N-t}^i}{N-t}d_i,$$

by following

$$\underline{\beta}_s(x;\mathbf{d}_{s-1}) = \sum_{m=1}^{N-s} \frac{m}{N-s+1}(\alpha_m - \alpha_{m+1})x - \sum_{i=1}^{s-1}\left[\sum_{m=1}^{N-s} \frac{m}{N-s+1}(\lambda_m^i - \lambda_{m+1}^i)d_i\right]$$

for $s = t+1, \ldots, N-1$. We show that at round t he can guarantee himself at least

$$\bar{v}_t(x;\mathbf{d}_{t-1}) = \left(\frac{\alpha_1 + \cdots + \alpha_{N-t+1}}{N-t+1}\right)x - \sum_{i=1}^{t-1}\left[\frac{\lambda_1^i + \cdots + \lambda_{N-t+1}^i}{N-t+1}\right]d_i,$$

by following

$$\underline{\beta}_s(x;\mathbf{d}_{s-1}) = \sum_{m=1}^{N-s} \frac{m}{N-s+1}(\alpha_m - \alpha_{m+1})x - \sum_{i=1}^{s-1}\left[\sum_{m=1}^{N-s} \frac{m}{N-s+1}(\lambda_m^i - \lambda_{m+1}^i)d_i\right]$$

for $s = t, \ldots, N-1$.

A bidder with value x whose demand is d_t at round t either (i) has the smallest demand and obtains a payoff of

$$\alpha_{N-t+1}x + d_t - \sum_{i=1}^{t-1}\lambda_{N-t+1}^i d_i,$$

or (ii) a rival has the smallest demand $b \leq d_t$ and he obtains a payoff of at least

$$\begin{aligned}
\bar{v}_{t+1}(x;(\mathbf{d}_{t-1},b)) &= \left(\frac{\alpha_1 + \cdots + \alpha_{N-t}}{N-t}\right)x - \frac{1}{N-t}b - \sum_{i=1}^{t-1}\left[\frac{\lambda_1^i + \cdots + \lambda_{N-t}^i}{N-t}\right]d_i \\
&\geq \left(\frac{\alpha_1 + \cdots + \alpha_{N-t}}{N-t}\right)x - \frac{1}{N-t}d_t - \sum_{i=1}^{t-1}\left[\frac{\lambda_1^i + \cdots + \lambda_{N-t}^i}{N-t}\right]d_i,
\end{aligned}$$

where the first equality holds since $\lambda_1^t + \cdots + \lambda_{N-t}^t = 1$.

The bidder maximizes his minimum payoff when d_t satisfies

$$\alpha_{N-t+1}x + d_t - \sum_{i=1}^{t-1}\lambda_{N-t+1}^i d_i = \left(\frac{\alpha_1 + \cdots + \alpha_{N-t}}{N-t}\right)x - \frac{1}{N-t}d_t - \sum_{i=1}^{t-1}\left[\frac{\lambda_1^i + \cdots + \lambda_{N-t}^i}{N-t}\right]d_i$$

i.e.,

$$\begin{aligned}
d_t &= \left(\frac{\alpha_1 + \cdots + \alpha_{N-t}}{N-t+1} - \frac{N-t}{N-t+1}\alpha_{N-t+1}\right)x \\
&\quad - \sum_{i=1}^{t-1}\left[\frac{\lambda_1^i + \cdots + \lambda_{N-t}^i}{N-t+1} - \frac{N-t}{N-t+1}\lambda_{N-t+1}^i\right]d_i.
\end{aligned}$$

Hence at round t the bidder guarantees himself a payoff of at least

$$\begin{aligned}\bar{v}_t(x;\mathbf{d}_{t-1}) &= \alpha_{N-t+1}x + d_t - \sum_{i=1}^{t-1}\lambda^i_{N-t+1}d_i \\ &= \left(\frac{\alpha_1+\cdots+\alpha_{N-t+1}}{N-t+1}\right)x - \sum_{i=1}^{t-1}\left[\frac{\lambda^i_1+\cdots+\lambda^i_{N-t+1}}{N-t+1}\right]d_i,\end{aligned}$$

by following

$$\begin{aligned}\underline{\beta}_t(x;\mathbf{d}_{t-1}) &= \left(\frac{\alpha_1+\cdots+\alpha_{N-t}}{N-t+1} - \frac{N-t}{N-t+1}\alpha_{N-t+1}\right)x \\ &\quad -\sum_{i=1}^{t-1}\left[\frac{\lambda^i_1+\cdots+\lambda^i_{N-t}}{N-t+1} - \frac{N-t}{N-t+1}\lambda^i_{N-t+1}\right]d_i \\ &= \sum_{m=1}^{N-t}\frac{m}{N-t+1}(\alpha_m-\alpha_{m+1})x - \sum_{i=1}^{t-1}\left[\sum_{m=1}^{N-t}\frac{m}{N-t+1}(\lambda^i_m-\lambda^i_{m+1})d_i\right].\end{aligned}$$

Next, we show that $\bar{v}_t(x;d_{t-1})$ is the largest payoff a bidder with value x can guarantee at round t given smallest demands d_{t-1}. Suppose to the contrary he can guarantee himself $v'_t > \bar{v}_t(x;d_{t-1})$. If all active bidder have the same value x then, since the game is symmetric, each such bidder can guarantee himself at least v'_t and hence the total guaranteed payoffs of the active bidders is at least

$$\begin{aligned}(N-t+1)v'_t &> (N-t+1)\left[\left(\frac{\alpha_1+\cdots+\alpha_{N-t+1}}{N-t+1}\right)x - \sum_{i=1}^{t-1}\left[\frac{\lambda^i_1+\cdots+\lambda^i_{N-t+1}}{N-t+1}\right]d_i\right] \\ &= \sum_{m=1}^{N-t+1}\alpha_m x - \sum_{i=1}^{t-1}[\lambda^i_1+\cdots+\lambda^i_{N-t+1}]d_i,\end{aligned}$$

which is a contraction since the RHS is the total surplus that can be obtained by the active bidders at round t. The first term is the surplus realized from allocating positions 1 through $N-t+1$ to the active bidders, and the second term is the compensation they owe.

We have established that $\underline{\beta}$ is a maxmin perfect strategy. Next we show that $\underline{\beta}$ is the unique maxmin perfect strategy. As a first step, we establish at each round t that a bidder with value x can be held to a payoff $\bar{v}_t(x;d_{t-1})$ given smallest demands d_{t-1}.

Consider a bidder with value x at round $N-1$ given smallest demands d_{N-2}. Suppose his rival bids $\underline{\beta}_{N-1}(x; \mathbf{d}_{N-2})$. If the bidder demands $d_{N-1} < \underline{\beta}_{N-1}(x; \mathbf{d}_{N-2})$, then his payoff is

$$\alpha_2 x + d_{N-1} - \sum_{i=1}^{N-2} \lambda_2^i d_i < \alpha_2 x + \underline{\beta}_{N-1}(x; \mathbf{d}_{N-2}) - \sum_{i=1}^{N-2} \lambda_2^i d_i = \bar{v}_{N-1}(x; \mathbf{d}_{N-2}).$$

If he demands $d_{N-1} > \underline{\beta}_{N-1}(x; \mathbf{d}_{N-2})$ then his payoff is

$$\alpha_1 x - \underline{\beta}_{N-1}(x; \mathbf{d}_{N-2}) - \sum_{i=1}^{N-2} \lambda_2^i d_i = \bar{v}_{N-1}(x; \mathbf{d}_{N-2}).$$

In both cases, his payoff is at most $\bar{v}_{N-1}(x;d_{N-2})$, which establishes he is held to $\bar{v}_{N-1}(x;d_{N-2})$.

Suppose the claim is true for rounds $t+1, \ldots, N-1$. We show it holds for round t. Consider a bidder with value x at round t with smallest demands d_{t-1}. Suppose at each round $s = t, \ldots, N-1$ that each of his rivals demands $\underline{\beta}_s(x; \mathbf{d}_{s-1})$ at round s given smallest demands d_{s-1}. If at round t the bidder demands $d_t < \underline{\beta}_t(x; \mathbf{d}_{t-1})$ his payoff is

$$\alpha_{N-t+1} x + d_t - \sum_{i=1}^{t-1} \lambda_{N-t+1}^i d_i < \alpha_{N-t+1} x + \underline{\beta}_t(x; \mathbf{d}_{t-1}) - \sum_{i=1}^{t-1} \lambda_{N-t+1}^i d_i = \bar{v}_t(x; \mathbf{d}_{t-1}).$$

If he demands $d_t > \underline{\beta}_t(x; \mathbf{d}_{t-1})$, then he continues to round $t+1$ and by the induction hypothesis his rivals hold him to $\bar{v}_{t+1}(x;d_{t-1}, \underline{\beta}_t(x; \mathbf{d}_{t-1}))$. Straight forward algebra establishes that

$$\bar{v}_{t+1}(x; \mathbf{d}_{t-1}, \underline{\beta}_t(x; \mathbf{d}_{t-1})) = \bar{v}_t(x; \mathbf{d}_{t-1}).$$

This establishes the claim holds for all rounds.

Finally, we show that $\underline{\beta}$ is the unique maxmin perfect strategy. Suppose that there is another maxmin perfect strategy $\hat{\beta} \neq \underline{\beta}$. Then for some x, t, and d_{t-1} we have that $\hat{\beta}_t(x;d_{t-1}) \neq \underline{\beta}_t(x; \mathbf{d}_{t-1})$. Consider a bidder with value x at round t, given smallest demands d_{t-1}, who follows $\hat{\beta}$. Suppose that at each round $s = t, \ldots, N-1$ that his rivals bid $\underline{\beta}_s(x; \mathbf{d}_{s-1})$. If $\hat{\beta}_t(x;d_{t-1}) < \underline{\beta}_t(x; \mathbf{d}_{t-1})$ then bidder i drops out at round t and obtains the payoff

$$\alpha_{N-t+1} x + \hat{\beta}_t(x; \mathbf{d}_{t-1}) - \sum_{i=1}^{t-1} \lambda_{N-t+1}^i d_i < \alpha_{N-t+1} x + \underline{\beta}_t(x; \mathbf{d}_{t-1}) - \sum_{i=1}^{t-1} \lambda_{N-t+1}^i d_i = \bar{v}_t(x; \mathbf{d}_{t-1}).$$

If $\hat{\beta}_t(x;d_{t-1}) > \underline{\beta}_t(x;\mathbf{d}_{t-1})$ and his rivals bid $(\hat{\beta}_t(x;d_{t-1}) + \underline{\beta}_t(x;\mathbf{d}_{t-1}))/2$ at round t and bids $\underline{\beta}_s(x;\mathbf{d}_{s-1})$ at each round $s = t+1, \ldots, N-1$ then the bidder's payoff at round t is at most

$$\bar{v}_{t+1}(x;\mathbf{d}_{t-1}, \frac{1}{2}(\hat{\beta}_t(x;\mathbf{d}_{t-1}) + \underline{\beta}_t(x;\mathbf{d}_{t-1})))$$

by the immediately prior claim. Since

$$\begin{aligned}\bar{v}_{t+1}(x;\mathbf{d}_{t-1}, d_t) &= \left(\frac{\alpha_1 + \cdots + \alpha_{N-t}}{N-t}\right)x - \left[\frac{\lambda_1^t + \cdots + \lambda_{N-t}^t}{N-t}\right]d_t - \sum_{i=1}^{t-1}\left[\frac{\lambda_1^i + \cdots + \lambda_{N-t}^i}{N-t}\right]d_i \\ &= \left(\frac{\alpha_1 + \cdots + \alpha_{N-t}}{N-t}\right)x - \frac{1}{N-t}d_t - \sum_{i=1}^{t-1}\left[\frac{\lambda_1^i + \cdots + \lambda_{N-t}^i}{N-t}\right]d_i\end{aligned}$$

is decreasing in d_t, we have

$$\bar{v}_{t+1}(x;\mathbf{d}_{t-1}, \frac{1}{2}(\hat{\beta}_t(x;\mathbf{d}_{t-1}) + \underline{\beta}_t(x;\mathbf{d}_{t-1}))) < \bar{v}_{t+1}(x;\mathbf{d}_{t-1}, \underline{\beta}_t(x;\mathbf{d}_{t-1})) = \bar{v}_t(x;\mathbf{d}_{t-1}),$$

which contradicts that $\hat{\beta}$ is a maxmin perfect strategy. $\square$

Proof of Proposition 3: It is convenient to order the players so that $x_1 \geq \ldots \geq x_N$ and to define

$$s_t(x) = \sum_{m=1}^{N-t} \frac{m}{N-t+1}(\alpha_m - \alpha_{m+1})x,$$

which can also be written as

$$s_t(x) = \left[\frac{\alpha_1 + \cdots + \alpha_{N-t+1}}{N-t+1} - \alpha_{N-t+1}\right]x.$$

When positions $1, \ldots, N-t+1$ remain to be allocated, then $s_t(x)$ can be interpreted as an equal share of incremental benefits of the contested positions to a bidder with value x. It is straightforward to show that the Shapley transfers given in Corollary 1 satisfy

$$\tau_{N-t+1} = s_t(x_{N-t+1}) - \sum_{i=1}^{t-1} \frac{1}{N-i} s_i(x_{N-i+1}).$$

Let $\mathbf{d}_{t-1}$ be the sequence of dropout prices at round t. When all bidders follow the maxmin bidding strategy, then at round t the active bidders have values $x_1, \dots, x_{N-t+1}$. The bidder with value x_{N-t+1} submits the smallest demand, he receives position $N-t+1$ and, by the construction of the maxmin bid function (see the proof of Proposition 2), his payoff is equal to his value, i.e., $\bar{v}_t(x_{N-t+1}; \mathbf{d}_{t-1})$. In particular, his payoff is

$$\bar{v}_t(x_{N-t+1}; \mathbf{d}_{t-1}) = \left(\frac{\alpha_1 + \cdots + \alpha_{N-t+1}}{N-t+1} \right) x_{N-t+1} - \sum_{i=1}^{t-1} \left[\frac{\lambda_1^i + \cdots + \lambda_{N-t+1}^i}{N-t+1} \right] d_i.$$

We show that $\bar{v}_t(x_{N-t+1}; \mathbf{d}_{t-1}) = \phi_{N-t+1}$.

We first show that

$$\sum_{i=1}^{t-1} \left[\frac{\lambda_1^i + \cdots + \lambda_{N-t+1}^i}{N-t+1} \right] d_i = \sum_{i=1}^{t-1} \frac{1}{N-i} s_i(x_{N-i+1}).$$

Since $\lambda_1^{t-1} + \cdots + \lambda_{N-t+1}^{t-1} = 1$, we have

$$\sum_{i=1}^{t-1} \left[\frac{\lambda_1^i + \cdots + \lambda_{N-t+1}^i}{N-t+1} \right] d_i = \frac{1}{N-t+1} d_{t-1} + \sum_{i=1}^{t-2} \left[\frac{\lambda_1^i + \cdots + \lambda_{N-t+1}^i}{N-t+1} \right] d_i.$$

Since the maxmin perfect bid at round $t-1$ was

$$d_{t-1} = s_{t-1}(x_{N-t+2}) - \sum_{i=1}^{t-2} \left[\frac{\lambda_1^i + \cdots + \lambda_{N-t+1}^i}{N-t+2} - \frac{N-t+1}{N-t+2} \lambda_{N-t+2}^i \right] d_i,$$

we have $\sum_{i=1}^{t-1} \left[\frac{\lambda_1^i + \cdots + \lambda_{N-t+1}^i}{N-t+1} \right] d_i$

$$\begin{aligned}
&= \frac{s_{t-1}(x_{N-t+2})}{N-t+1} - \frac{1}{N-t+1} \sum_{i=1}^{t-2} \left[\frac{\lambda_1^i + \cdots + \lambda_{N-t+1}^i}{N-t+2} - \frac{N-t+1}{N-t+2} \lambda_{N-t+2}^i \right] d_i \\
&\quad + \sum_{i=1}^{t-2} \left[\frac{\lambda_1^i + \cdots + \lambda_{N-t+1}^i}{N-t+1} \right] d_i \\
&= \frac{s_{t-1}(x_{N-t+2})}{N-t+1} - \frac{1}{N-t+1} \sum_{i=1}^{t-2} \left[\frac{\lambda_1^i + \cdots + \lambda_{N-t+1}^i + \lambda_{N-t+2}^i}{N-t+2} - \lambda_{N-t+2}^i \right] d_i \\
&\quad + \sum_{i=1}^{t-2} \left[\frac{\lambda_1^i + \cdots + \lambda_{N-t+1}^i}{N-t+1} \right] d_i \\
&= \frac{s_{t-1}(x_{N-t+2})}{N-t+1} - \frac{1}{N-t+1} \sum_{i=1}^{t-2} \left[\frac{\lambda_1^i + \cdots + \lambda_{N-t+1}^i + \lambda_{N-t+2}^i}{N-t+2} - (\lambda_1^i + \cdots + \lambda_{N-t+2}^i) \right] d_i \\
&= \frac{s_{t-1}(x_{N-t+2})}{N-t+1} + \frac{1}{N-t+1} \sum_{i=1}^{t-2} \left[\frac{\lambda_1^i + \cdots + \lambda_{N-t+1}^i + \lambda_{N-t+2}^i}{N-t+2} \right] d_i.
\end{aligned}$$

Hence we have

$$\sum_{i=1}^{t-1}\left[\frac{\lambda_1^i+\cdots+\lambda_{N-t+1}^i}{N-t+1}\right]d_i=\frac{s_{t-1}(x_{N-t+2})}{N-t+1}+\cdots+\frac{s_1(x_N)}{N-1}=\sum_{i=1}^{t-1}\frac{1}{N-i}s_i(x_{N-i+1}).$$

Substituting into the expression for $\bar{v}_t(x_{N-t+1};\mathbf{d}_{t-1})$ above, we obtain

$$\bar{v}_t(x_{N-t+1};\mathbf{d}_{t-1})=\left(\frac{\alpha_1+\cdots+\alpha_{N-t+1}}{N-t+1}\right)x_{N-t+1}-\sum_{i=1}^{t-1}\frac{1}{N-i}s_i(x_{N-i+1}).$$

By rearranging $s_t(x)$, as it is defined above, we have

$$\frac{\alpha_1+\cdots+\alpha_{N-t+1}}{N-t+1}x_{N-t+1}=\alpha_{N-t+1}x_{N-t+1}+s_t(x_{N-t+1}).$$

As noted earlier, the Shapley transfers satisfy

$$-\sum_{i=1}^{t-1}\frac{1}{N-i}s_i(x_{N-i+1})=\tau_{N-t+1}-s_t(x_{N-t+1}).$$

Thus the bidder with value x_{N-t+1} exits at round t obtains

$$\bar{v}_t(x_{N-t+1};\mathbf{d}_{t-1})=\alpha_{N-t+1}x_{N-t+1}+\tau_{N-t+1},$$

his Shapley value payoff: he obtains position $N-t+1$ and a net transfer τ_{N-t+1}, i.e., his Shapley allocation. □

Proof of Proposition 4: Let $\beta=(\beta_1,\ldots,\beta_{N-1})$ be a symmetric equilibrium in increasing and differentiable strategies. Since equilibrium is in increasing strategies, the sequence of smallest demands $(d_1,\ldots,d_{t-1})$ at round t reveals the $t-1$ lowest values $(z_1,\ldots,z_{t-1})$. In the proof it is convenient to write the round t equilibrium bid as a function of the prior dropout values rather than as a function of the prior smallest demands. In particular, we write $\beta_t(x|\mathbf{z}_{t-1})$ rather than $\beta_t(x;d_{t-1})$.

For each $t<N$, let $\pi_t(y,x|\mathbf{z}_{t-1})$ be the expected payoff to a bidder with value x who in round t deviates from equilibrium and bids as though his value is y (i.e., he bids $\beta_t(y|\mathbf{z}_{t-1})$), when $\mathbf{z}_{t-1}$ is the profile of values of the $t-1$ bidders to drop so far. In this case we will sometimes say the bidder "bids y". Let

$$\Pi_t(x|\mathbf{z}_{t-1})=\pi_t(x,x|\mathbf{z}_{t-1})$$

be the equilibrium payoff of a bidder in round t when his value is x and $\mathbf{z}_{t-1}$ is the profile of values of the $t-1$ bidders to drop in prior rounds.

We now derive the necessary conditions in Proposition 4. Let $\mathbf{z}_{t-1}$ be arbitrary. Consider a bid y. If $z_t \in [z_{t-1}, y]$ the bidder continues to round $t+1$ and has an expected payoff of $\Pi_{t+1}(x|\mathbf{z}_{t-1}, z_t)$. If $z_t \geq y$ he obtains a payoff of $\alpha_{N-t+1}x + \beta_t(y|\mathbf{z}_{t-1}) - \Sigma_{j=1}^{t-1}\frac{1}{N-j}d_j$ in round t. Hence his expected payoff is

$$\begin{aligned}\pi_t(y, x|\mathbf{z}_{t-1}) = \quad & \int_{z_{t-1}}^{y} \Pi_{t+1}(x|\mathbf{z}_{t-1}, z_t) g_t^{(N-1)}(z_t|z_{t-1}) dz_t \\ & + \int_y^{\bar{x}} u\left(\alpha_{N-t+1}x + \beta_t(y|\mathbf{z}_{t-1}) - \Sigma_{j=1}^{t-1}\frac{1}{N-j}d_j\right) g_t^{(N-1)}(z_t|z_{t-1}) dz_t.\end{aligned}$$

Differentiating with respect to y yields

$$\begin{aligned}\frac{\partial \pi_t(y, x|\mathbf{z}_{t-1})}{\partial y} = \quad & [\Pi_{t+1}(x|\mathbf{z}_{t-1}, y) - u\left(\alpha_{N-t+1}x + \beta_t(y|\mathbf{z}_{t-1}) - \Sigma_{j=1}^{t-1}\frac{1}{N-j}d_j\right)] g_t^{(N-1)}(y|z_{t-1}) \\ & + u'\left(\alpha_{N-t+1}x + \beta_t(y|\mathbf{z}_{t-1}) - \Sigma_{j=1}^{t-1}\frac{1}{N-j}d_j\right) \beta_t'(y|\mathbf{z}_{t-1})(1 - G_t^{(N-1)}(y|z_{t-1})).\end{aligned}$$

A necessary condition for equilibrium is that $\partial \pi_t(y, x|\mathbf{z}_{t-1})/\partial y|_{y=x} = 0$, i.e.,

$$\begin{aligned}& [\Pi_{t+1}(x|\mathbf{z}_{t-1}, x) - u\left(\alpha_{N-t+1}x + \beta_t(x|\mathbf{z}_{t-1}) - \Sigma_{j=1}^{t-1}\frac{1}{N-j}d_j\right)] g_t^{(N-1)}(x|z_{t-1}) \\ & + u'\left(\alpha_{N-t+1}x + \beta_t(x|\mathbf{z}_{t-1}) - \Sigma_{j=1}^{t-1}\frac{1}{N-j}d_j\right) \beta_t'(x|\mathbf{z}_{t-1})(1 - G_t^{(N-1)}(x|z_{t-1})) = 0.\end{aligned}$$

Since

$$\begin{aligned}\Pi_{t+1}(x|\mathbf{z}_{t-1}, x) &= \pi_{t+1}(x, x|\mathbf{z}_{t-1}, x) \\ &= u\left(\alpha_{N-t}x + \beta_{t+1}(x|\mathbf{z}_{t-1}, x) - \frac{1}{N-t}\beta_t(x|\mathbf{z}_{t-1}) - \Sigma_{j=1}^{t-1}\frac{1}{N-j}d_j\right)\end{aligned}$$

the necessary condition can be written as

$$\begin{aligned}& u'\left(\alpha_{N-t+1}x + \beta_t(x|\mathbf{z}_{t-1}) - \Sigma_{j=1}^{t-1}\frac{1}{N-j}d_j\right) \beta_t'(x|\mathbf{z}_{t-1}) \\ = \; & - \left[\begin{array}{c} u\left(\alpha_{N-t}x + \beta_{t+1}(x|\mathbf{z}_{t-1}, x) - \frac{1}{N-t}\beta_t(x|\mathbf{z}_{t-1}) - \Sigma_{j=1}^{t-1}\frac{1}{N-j}d_j\right) \\ -u\left(\alpha_{N-t+1}x + \beta_t(x|\mathbf{z}_{t-1}) - \Sigma_{j=1}^{t-1}\frac{1}{N-j}d_j\right) \end{array}\right] \Gamma_t^N(x),\end{aligned}$$

where $\beta_N(x; \mathbf{z}_{N-1}) \equiv 0$. Replacing $\mathbf{z}_{t-1}$ with d_{t-1} and the x in $\beta_{t+1}(x|\mathbf{z}_{t-1}, x)$ with $\beta_t(x|d_{t-1})$ yields the differential equation given in the Proposition for round t. □

Proof of Proposition 5: We first show that the bidding functions in Proposition 5 satisfies the system of differential equations in Proposition 4. The proof is by induction. Consider round $N-1$. The differential equation for round $N-1$ is

$$\beta_{N-1}^{0\prime}(x|\mathbf{z}_{N-2}) = -[(\alpha_1 - \alpha_2)x - 2\beta_{N-1}^{0}(x|\mathbf{z}_{N-2})]\Gamma_{N-1}^{N}(x). \qquad (1)$$

The unique solution is

$$\begin{aligned}\beta_{N-1}^{0}(x) &= \frac{1}{2}\frac{\int_x^{\bar{x}}(\alpha_1-\alpha_2)\,z2f(z)(1-F(z))dz}{(1-F(x))^2}\\ &= \frac{1}{2}E\left[(\alpha_1-\alpha_2)Z_{N-1}^{(N)}|Z_{N-1}^{(N)} > x > Z_{N-2}^{(N)}\right],\end{aligned}$$

which is $\beta_{N-1}^{0}(x)$, as given in Proposition 5.

Suppose $\beta_{t+1}^{0}, \ldots, \beta_{N-1}^{0}$ are as given in Proposition 5 for round $t+1, \ldots, N-1$. Consider round t. The differential equation in Proposition 4 for round t, using the notation from the proof of Proposition 4, is

$$\beta_t^{0\prime}(x|\mathbf{z}_{t-1}) = -\left[(\alpha_{N-t}-\alpha_{N-t+1})x + \beta_{t+1}^{0}(x|\mathbf{z}_{t-1},x) - \frac{N-t+1}{N-t}\beta_t^0(x|\mathbf{z}_{t-1})\right]\Gamma_t^N(x).$$

Since $\beta_{t+1}^{0}(x|\mathbf{z}_{t-1},x)$ is independent of $(\mathbf{z}_{t-1},x)$, we can write

$$\beta_t^{0\prime}(x) = -\left[(\alpha_{N-t}-\alpha_{N-t+1})x + \beta_{t+1}^{0}(x) - \frac{N-t+1}{N-t}\beta_t^0(x)\right]\Gamma_t^N(x).$$

The unique solution is

$$\begin{aligned}\beta_t^0(x) &= \frac{N-t}{N-t+1}\int_x^{\bar{x}}\frac{\left((\alpha_{N-t}-\alpha_{N-t+1})z+\beta_{t+1}^0(z)\right)(N-t+1)f(z)(1-F(z))^{N-t}}{(1-F(x))^{N-t+1}}dz\\ &= \frac{N-t}{N-t+1}E\left[(\alpha_{N-t}-\alpha_{N-t-1})\,Z_t^{(N)} + \beta_{t+1}^0(Z_t^{(N)})|Z_t^{(N)} > x > Z_{t-1}^{(N)}\right]\\ &= \sum_{m=1}^{N-t}\frac{m}{N-t+1}E\left[(\alpha_m-\alpha_{m+1})\,Z_{N-m}^{(N)}|Z_t^{(N)} > x > Z_{t-1}^{(N)}\right],\end{aligned}$$

where the second equality restates the first equality as an expected value. The third equality is established as Claim 5 in the Supplemental Appendix. This establishes the result for round t and hence, by induction, the result for all t.

Next we establish that the bidding strategies are an equilibrium. It is sufficient to show that the following three-part claim holds for every t:

1. If $x \geq z_{t-1}$ then $x \in \arg\max_y \pi_t(y, x|\mathbf{z}_{t-1})$, i.e., it is optimal for a bidder with value x to bid $\beta_t^0(x)$ in round t.

2. If $x < z_{t-1}$ then $z_{t-1} \in \arg\max_y \pi_t(y, x|\mathbf{z}_{t-1})$.

3. $\frac{d\Pi_t(x|\mathbf{z}_{t-1})}{dx} \geq \alpha_{N-t+1}$.

Parts 2 and 3 are ancillary results needed to establish Part 1 for rounds prior to the last round. Part 2 is necessary to evaluate the consequence at round t of a bid y greater than the equilibrium bid x. In this case, a rival bidder with value $z_t > x$ may drop out before the bidder, and we need to evaluate the consequence for his optimal bid in round $t+1$. Part 2 shows that in this event it is optimal for the bidder to bid z_t (rather than x) in round $t+1$.

The proof is by induction. Consider round $N-1$. Any bid below z_{N-2} is strictly dominated by a bid of z_{N-2} since both bids result in the same position while a bid of z_{N-2} yields higher compensation. Suppose $y \geq z_{N-2}$. When bidders are risk neutral we have

$$\begin{aligned}\pi_{N-1}(y, x|\mathbf{z}_{N-2}) = & \int_{z_{N-2}}^{y} \left(\alpha_1 x - \beta_{N-1}^0(z_{N-1}) - \Sigma_{j=1}^{N-2} \tfrac{1}{N-j} d_j\right) g_{N-1}^{(N-1)}(z_{N-1}|z_{N-2}) dz_{N-1} \\ & + \int_{y}^{\bar{x}} \left(\alpha_2 x + \beta_{N-1}^0(y) - \Sigma_{j=1}^{N-2} \tfrac{1}{N-j} d_j\right) g_{N-1}^{(N-1)}(z_{N-1}|z_{N-2}) dz_{N-1}.\end{aligned}$$

Differentiating with respect to y yields $\partial\pi_{N-1}(y, x|\mathbf{z}_{N-2})/\partial y =$

$$\begin{aligned} & (\alpha_1 x - \beta_{N-1}^0(y) - \Sigma_{j=1}^{N-2} \tfrac{1}{N-j} d_j) g_{N-1}^{(N-1)}(y|z_{N-2}) \\ - & (\alpha_2 x + \beta_{N-1}^0(y) - \Sigma_{j=1}^{N-2} \tfrac{1}{N-j} d_j) g_{N-1}^{(N-1)}(y|z_{N-2}) \\ + & \beta_{N-1}^{0\prime}(y)(1 - G_{N-1}^{(N-1)}(y|z_{N-2})).\end{aligned}$$

Substituting the differential equation (1)

$$\beta_{N-1}^{0\prime}(y) = -[(\alpha_1 - \alpha_2)y - 2\beta_{N-1}^0(y)]\Gamma_{N-1}^N(y) \tag{2}$$

into the expression for $\partial\pi_{N-1}(y, x|\mathbf{z}_{N-2})/\partial y$ yields

$$\partial\pi_{N-1}(y, x|\mathbf{z}_{N-2})/\partial y = (\alpha_1 - \alpha_2)(x - y) g_{N-1}^{(N-1)}(y|z_{N-2}).$$

If $y < x$ then $\partial\pi_{N-1}(y, x|\mathbf{z}_{N-2})/\partial y > 0$, and if $y > x$ then $\partial\pi_{N-1}(y, x|\mathbf{z}_{N-2})/\partial y < 0$. Thus $x \geq z_{N-2}$ implies $x \in \arg\max_y \pi_{N-1}(y, x|\mathbf{z}_{N-2})$, which establishes Part 1.

If $x < z_{N-2}$, then any bid below z_{N-2} is strictly dominated. For any bid $y \geq z_{N-2}$ then $y > x$ and the above argument establishes $\partial \pi_{N-1}(y, x|\mathbf{z}_{N-2})/\partial y < 0$ for all $y \geq z_{N-2}$, i.e., $z_{N-2} \in \arg\max_y \pi_{N-1}(y, x|\mathbf{z}_{N-2})$. This establishes Part 2.

By the Envelope Theorem

$$\begin{aligned} \frac{d\Pi_{N-1}(x|\mathbf{z}_{N-2})}{dx} &= \left.\frac{\partial \pi_{N-1}(y, x|\mathbf{z}_{N-2})}{\partial x}\right|_{y=x} \\ &= \alpha_1 G_{N-1}^{(N-1)}(x|z_{N-2}) + \alpha_2(1 - G_{N-1}^{(N-1)}(x|z_{N-2})) \\ &\geq \alpha_2, \end{aligned}$$

which establishes Part 3. This completes the claim for round $N-1$.

Assume the three-part claim is true for rounds $t+1$ through $N-1$. We show it is true for round t. Let $\mathbf{z}_{t-1}$ be a sequence of dropout values. Suppose $x \geq z_{t-1}$. Consider an active bidder in the t-th round whose value is x and who bids y. A bid below z_{t-1} is dominated. Since his payoff function differs in each case, we need to distinguish (i) $y \in [z_{t-1}, x]$ and (ii) $y > x$. In what follows, we denote the payoff to a bid of y as $\pi_t^L(y, x|\mathbf{z}_{t-1})$ if $y \in [z_{t-1}, x]$ and as $\pi_t^H(y, x|\mathbf{z}_{t-1})$ if $y \geq x$.

Case (i): Consider a bid $y \in [z_{t-1}, x]$. If the next highest value of a rival bidder is $z_t \in [z_{t-1}, y]$, then the bidder continues to round $t+1$ where, by the induction hypothesis, he optimally bids x and he has an expected payoff of $\Pi_{t+1}(x|\mathbf{z}_{t-1}, z_t)$. If $z_t \geq y$ he obtains a payoff of $\alpha_{N-t+1}x + \beta_t^0(y) - \Sigma_{j=1}^{t-1} \frac{1}{N-j} d_j$ in round t. Hence his payoff is

$$\begin{aligned} \pi_t^L(y, x|\mathbf{z}_{t-1}) = \; & \textstyle\int_{z_{t-1}}^{y} \Pi_{t+1}(x|\mathbf{z}_{t-1}, z_t) g_t^{(N-1)}(z_t|z_{t-1}) dz_t \\ & + \textstyle\int_y^{\bar{x}} \left(\alpha_{N-t+1}x + \beta_t^0(y) - \Sigma_{j=1}^{t-1} \frac{1}{N-j} d_j\right) g_t^{(N-1)}(z_t|z_{t-1}) dz_t. \end{aligned}$$

Differentiating with respect to y yields

$$\begin{aligned} \frac{\partial \pi_t^L(y, x|\mathbf{z}_{t-1})}{\partial y} = \; & [\Pi_{t+1}(x|\mathbf{z}_{t-1}, y) - \left(\alpha_{N-t+1}x + \beta_t^0(y) - \Sigma_{j=1}^{t-1} \tfrac{1}{N-j} d_j\right)] g_t^{(N-1)}(y|z_{t-1}) \\ & + \beta_t^{0\prime}(y)(1 - G_t^{(N-1)}(y|z_{t-1})). \end{aligned}$$

By the induction hypothesis we have

$$\frac{\partial^2 \pi_t^L(y, x|\mathbf{z}_{t-1})}{\partial x \partial y} = \left(\frac{d\Pi_{t+1}(x|\mathbf{z}_{t-1}, y)}{dx} - \alpha_{N-t+1}\right) g_t^{(N-1)}(y|z_{t-1}) \geq 0.$$

Case (ii): Consider a bid $y \geq x$. If the next highest value of a rival bidder is $z_t \in [z_{t-1}, x]$, then the bidder continues to round $t+1$ and, by the induction hypothesis, he bids x and obtains $\Pi_{t+1}(x|\mathbf{z}_{t-1}, z_t)$. If $z_t \in [x, y]$, then he continues to round $t+1$ and, by the part 2 of the induction hypothesis, he optimally bids z_t, he wins position $N-t$, and obtains compensation $\beta^0_{t+1}(z_t)$. His payoff is $\alpha_{N-t}x + \beta^0_{t+1}(z_t) - \frac{1}{N-t}\beta^0_t(z_t) - \Sigma_{j=1}^{t-1}\frac{1}{N-j}d_j$. If $z_t > y$, then in round t his payoff is $\alpha_{N-t+1}x + \beta^0_t(y|\mathbf{z}_{t-1}) - \Sigma_{j=1}^{t-1}\frac{1}{N-j}d_j$. Thus his expected payoff at round t is

$$\begin{aligned}\pi^H_t(y, x|\mathbf{z}_{t-1}) = & \int_{z_{t-1}}^{x} \Pi_{t+1}(x|\mathbf{z}_{t-1}, z_t) g_t^{(N-1)}(z_t|z_{t-1})dz_t \\ & + \int_x^y \left(\alpha_{N-t}x + \beta^0_{t+1}(z_t) - \tfrac{1}{N-t}\beta^0_t(z_t) - \Sigma_{j=1}^{t-1}\tfrac{1}{N-j}d_j\right) g_t^{(N-1)}(z_t|z_{t-1})dz_t \\ & + \int_y^{\bar{x}} \left(\alpha_{N-t+1}x + \beta^0_t(y) - \Sigma_{j=1}^{t-1}\tfrac{1}{N-j}d_j\right) g_t^{(N-1)}(z_t|z_{t-1})dz_t.\end{aligned}$$

Differentiating with respect to y yields

$$\begin{aligned}\frac{\partial \pi^H_t(y, x|\mathbf{z}_{t-1})}{\partial y} = & \left[\left((\alpha_{N-t} - \alpha_{N-t+1})x + \beta^0_{t+1}(y) - \frac{N-t+1}{N-t}\beta^0_t(y)\right)\right] g_t^{(N-1)}(y|z_{t-1}) \\ & + \beta^{0\prime}_t(y)(1 - G_t^{(N-1)}(y|z_{t-1})).\end{aligned}$$

Since $\alpha_{N-t} - \alpha_{N-t+1} \geq 0$ then

$$\frac{\partial^2 \pi^H_t(y, x|\mathbf{z}_{t-1})}{\partial x \partial y} = (\alpha_{N-t} - \alpha_{N-t+1}) g_t^{(N-1)}(y|z_{t-1}) \geq 0.$$

We have shown that

$$\left.\frac{\partial \pi^H_t(y, x|\mathbf{z}_{t-1})}{\partial y}\right|_{y=x} = \left.\frac{\partial \pi^L_t(y, x|\mathbf{z}_{t-1})}{\partial y}\right|_{y=x} = 0$$

and

$$\frac{\partial^2 \pi^L_t(y, x|\mathbf{z}_{t-1})}{\partial x \partial y} \geq 0 \text{ for } y \leq x \text{ and } \frac{\partial^2 \pi^H_t(y, x|\mathbf{z}_{t-1})}{\partial x \partial y} \geq 0 \text{ for } y \leq x.$$

hence by Lemma 0 in Van Essen and Wooders (2016) we have $x \in \arg\max_{y \in [z_{t-1}, \bar{x}]} \pi_t(y, x|\mathbf{z}_{t-1})$. This establishes Part 1 for round t.

Suppose $x < z_{t-1}$. Any $y < z_{t-1}$ is strictly dominated by a bid of z_{t-1}. For $y \geq z_{t-1}$ we can write

$$\begin{aligned}\pi_t(y, x|\mathbf{z}_{t-1}) = & \int_{z_{t-1}}^{y} \left(\alpha_{N-t}x + \beta^0_{t+1}(z_t) - \tfrac{1}{N-t}\beta^0_t(z_t) - \Sigma_{j=1}^{t-1}\tfrac{1}{N-j}d_j\right) g_t^{(N-1)}(z_t|z_{t-1})dz_t \\ & + \int_y^{\bar{x}} \left(\alpha_{N-t+1}x + \beta^0_t(y) - \Sigma_{j=1}^{t-1}\tfrac{1}{N-j}d_j\right) g_t^{(N-1)}(z_t|z_{t-1})dz_t.\end{aligned}$$

Differentiating with respect to y and replacing $\beta_t^{0'}(y)$ with the equilibrium differential equation yields

$$\frac{\partial \pi_t(y, x|\mathbf{z}_{t-1})}{\partial y} = (\alpha_{N-t} - \alpha_{N-t+1})(x-y)\, g_t^{(N-1)}(y|z_{t-1}) \le 0$$

since $y > x$ and $\alpha_{N-t} - \alpha_{N-t+1} \ge 0$. Hence, if $x < z_{t-1}$ then $z_{t-1} \in \arg\max_y \pi_t(y, x|\mathbf{z}_{t-1})$. This establishes Part 2 for round t.

Finally, by the Envelope Theorem, we have

$$\begin{aligned}
\frac{d\Pi_t(x|\mathbf{z}_{t-1})}{dx} &= \left.\frac{\partial \pi_t^L(y, x|\mathbf{z}_{t-1})}{\partial x}\right|_{y=x} = \left.\frac{\partial \pi_t^H(y, x|\mathbf{z}_{t-1})}{\partial x}\right|_{y=x} \\
&= \int_{z_{t-1}}^{x} \frac{d\Pi_{t+1}(x|\mathbf{z}_{t-1}, z_t)}{dx} g_t^{(N-1)}(z_t|z_{t-1})dz_t + \alpha_{N-t+1}(1 - G_t^{(N-1)}(x|z_{t-1}) \\
&\ge \alpha_{N-t}G_t^{(N-1)}(x|z_{t-1}) + \alpha_{N-t+1}(1 - G_t^{(N-1)}(x|z_{t-1}) \\
&\ge \alpha_{N-t+1}
\end{aligned}$$

where the first inequality follows from the induction hypothesis and the second inequality follows since $\alpha_{N-t} \ge \alpha_{N-t+1}$. This establishes Part 3 for round t, and completes the proof by induction. □

Proof of Proposition 6: We first show that the bidding functions given in Proposition 6 are the unique solution to the system of differential equations in Proposition 4 when bidders have CARA utility. The proof is by induction. Consider round $N-1$. The differential equation for round $N-1$ is

$$\begin{gathered}
-\theta e^{-\theta[\alpha_2 x + \beta_{N-1}^{\theta}(x|\mathbf{z}_{N-2}) - \Sigma_{j=1}^{N-2}\frac{1}{N-j}d_j]}\beta_{N-1}^{\theta\prime}(x|\mathbf{z}_{N-2}) = \\
[e^{-\theta[\alpha_2 x + \beta_{N-1}^{\theta}(x|\mathbf{z}_{N-2}) - \Sigma_{j=1}^{N-2}\frac{1}{N-j}d_j]} - e^{-\theta[\alpha_1 x - \beta_{N-1}^{\theta}(x|\mathbf{z}_{N-2}) - \Sigma_{j=1}^{N-2}\frac{1}{N-j}d_j]}]\Gamma_{N-1}^{N}(x).
\end{gathered}$$

Dividing both sides by $e^{-\theta[\alpha_2 x - \beta_{N-1}^{\theta}(x|\mathbf{z}_{N-2}) - \Sigma_{j=1}^{N-2}\frac{1}{N-j}d_j]}$ yields

$$-\theta e^{-2\theta\beta_{N-1}^{\theta}(x|\mathbf{z}_{N-2})}\beta_{N-1}^{\theta\prime}(x|\mathbf{z}_{N-2}) = [e^{-2\theta\beta_{N-1}^{\theta}(x|\mathbf{z}_{N-2})} - e^{-\theta(\alpha_1-\alpha_2)x}]\Gamma_{N-1}^{N}(x).$$

Multiplying both sides by $2(1-F(x))^2$ yields

$$\begin{aligned}
&-2(1-F(x))^2\theta e^{-2\theta\beta_{N-1}^{\theta}(x|\mathbf{z}_{N-2})}\beta_{N-1}^{\theta\prime}(x|\mathbf{z}_{N-2}) \\
= \;& 2f(x)(1-F(x))[e^{-2\theta\beta_{N-1}^{\theta}(x|\mathbf{z}_{N-2})} - e^{-\theta(\alpha_1-\alpha_2)x}].
\end{aligned}$$

Rearranging

$$\begin{aligned} & -2\theta(1-F(x))^2 e^{-2\theta\beta^\theta_{N-1}(x|\mathbf{z}_{N-2})}\beta^{\theta\prime}_{N-1}(x|\mathbf{z}_{N-2}) - 2f(x)(1-F(x))e^{-2\theta\beta^\theta_{N-1}(x|\mathbf{z}_{N-2})} \\ = & -e^{-\theta(\alpha_1-\alpha_2)x}2f(x)(1-F(x)). \end{aligned}$$

or

$$\frac{d}{dx}\left(e^{-2\theta\beta^\theta_{N-1}(x|\mathbf{z}_{N-2})}(1-F(x))^2\right) = -e^{-\theta(\alpha_1-\alpha_2)x}2f(x)(1-F(x)).$$

By the Fundamental Theorem of Calculus

$$e^{-2\theta\beta^\theta_{N-1}(x|\mathbf{z}_{N-2})}(1-F(x))^2 = \int_0^x -e^{-\theta(\alpha_1-\alpha_2)s}2f(s)(1-F(s))ds + C.$$

Since the LHS is zero at $x=\bar{x}$ then

$$C = \int_0^{\bar{x}} e^{-\theta(\alpha_1-\alpha_2)s}2f(s)(1-F(s))ds.$$

The unique solution $\beta^\theta_{N-1}(x|\mathbf{z}_{N-2})$ therefore satisfies

$$e^{-2\theta\beta^\theta_{N-1}(x|\mathbf{z}_{N-2})}(1-F(x))^2 = \int_x^{\bar{x}} e^{-\theta(\alpha_1-\alpha_2)s}2f(s)(1-F(s))ds.$$

Rearranging yields

$$\begin{aligned} \beta^\theta_{N-1}(x) &= -\frac{1}{2\theta}\ln\left(\frac{\int_x^{\bar{x}} e^{-\theta[(\alpha_1-\alpha_2)s]}2f(s)(1-F(s))ds}{(1-F(x))^2}\right) \\ &= -\frac{1}{2\theta}\ln\left(E\left[e^{-\theta(\alpha_1-\alpha_2)Z^{(N)}_{N-1}}|Z^{(N)}_{N-1} > x > Z^{(N)}_{N-2}\right]\right), \end{aligned}$$

which is $\beta^\theta_{N-1}(x)$, as given in Proposition 6.

Suppose $\beta^\theta_{t+1},\ldots,\beta^\theta_{N-1}$ are as given in Proposition 6 for rounds $t+1,\ldots,N-1$. Consider round t. The differential equation in the proof of Proposition 4 for round t is

$$\begin{aligned} & u'\left(\alpha_{N-t+1}x + \beta^\theta_t(x|\mathbf{z}_{t-1}) - \sum_{j=1}^{t-1}\frac{1}{N-j}d_j\right)\beta^{\theta\prime}_t(x|\mathbf{z}_{t-1}) \\ = & -\left[\begin{array}{c} u\left(\alpha_{N-t}x + \beta^\theta_{t+1}(x) - \frac{1}{N-t}\beta^\theta_t(x|\mathbf{z}_{t-1}) - \sum_{j=1}^{t-1}\frac{1}{N-j}d_j\right) \\ -u\left(\alpha_{N-t+1}x + \beta^\theta_t(x|\mathbf{z}_{t-1}) - \sum_{j=1}^{t-1}\frac{1}{N-j}d_j\right) \end{array}\right]\Gamma^N_t(x), \end{aligned}$$

where we have used that $\beta^\theta_{t+1}(x)$ is independent of $\mathbf{z}_t$ by the induction hypothesis. We have

$$\begin{aligned} & \theta e^{-\theta[\alpha_{N-t+1}x+\beta^\theta_t(x|\mathbf{z}_{t-1})-\sum_{j=1}^{t-1}\frac{1}{N-j}d_j]}\beta^{\theta\prime}_t(x|\mathbf{z}_{t-1}) \\ = & -\left[e^{-\theta[\alpha_{N-t+1}x+\beta^\theta_t(x|\mathbf{z}_{t-1})-\sum_{j=1}^{t-1}\frac{1}{N-j}d_j]} - e^{-\theta[\alpha_{N-t}x+\beta^\theta_{t+1}(x)-\frac{1}{N-t}\beta^\theta_t(x|\mathbf{z}_{t-1})-\sum_{j=1}^{t-1}\frac{1}{N-j}d_j]}\right]\Gamma^N_t(x). \end{aligned}$$

Dividing both sides by $e^{-\theta[\alpha_{N-t+1}x-\frac{1}{N-t}\beta^\theta_t(x|\mathbf{z}_{t-1})-\sum_{j=1}^{t-1}\frac{1}{N-j}d_j]}$ yields

$$\begin{aligned} & -\theta e^{-\theta\frac{N-t+1}{N-t}\beta^\theta_t(x|\mathbf{z}_{t-1})}\beta^{\theta\prime}_t(x|\mathbf{z}_{t-1}) \\ = & [e^{-\theta\frac{N-t+1}{N-t}\beta^\theta_t(x|\mathbf{z}_{t-1})} - e^{-\theta[(\alpha_{N-t}-\alpha_{N-t+1})x+\beta^\theta_{t+1}(x)]}](N-t)\frac{f(x)}{1-F(x)}. \end{aligned}$$

Multiplying both sides by $\frac{N-t+1}{N-t}(1-F(x))^{N-t+1}$ yields

$$\begin{aligned} & -\theta\frac{N-t+1}{N-t}(1-F(x))^{N-t+1}e^{-\theta\frac{N-t+1}{N-t}\beta^\theta_t(x|\mathbf{z}_{t-1})}\beta^{\theta\prime}_t(x|\mathbf{z}_{t-1}) \\ = & [e^{-\theta\frac{N-t+1}{N-t}\beta^\theta_t(x|\mathbf{z}_{t-1})} - e^{-\theta[(\alpha_{N-t}-\alpha_{N-t+1})x+\beta^\theta_{t+1}(x)]}](N-t+1)(1-F(x))^{N-t}f(x). \end{aligned}$$

This equation can be rewritten as

$$\begin{aligned} & -\theta\frac{N-t+1}{N-t}(1-F(x))^{N-t+1}e^{-\theta\frac{N-t+1}{N-t}\beta^\theta_t(x|\mathbf{z}_{t-1})}\beta^{\theta\prime}_t(x|\mathbf{z}_{t-1}) \\ & -e^{-\theta\frac{N-t+1}{N-t}\beta^\theta_t(x|\mathbf{z}_{t-1})}(N-t+1)(1-F(x))^{N-t}f(x) \\ = & -e^{-\theta[(\alpha_{N-t}-\alpha_{N-t+1})x+\beta^\theta_{t+1}(x)]}(N-t+1)(1-F(x))^{N-t}f(x), \end{aligned}$$

i.e.,

$$\frac{d}{dx}((1-F(x))^{N-t+1}e^{-\theta\frac{N-t+1}{N-t}\beta^\theta_t(x|\mathbf{z}_{t-1})}) = -e^{-\theta[(\alpha_{N-t}-\alpha_{N-t+1})x+\beta^\theta_{t+1}(x)]}(N-t+1)(1-F(x))^{N-t}f(x).$$

By the Fundamental Theorem of Calculus

$$\begin{aligned} & (1-F(x))^{N-t+1}e^{-\theta\frac{N-t+1}{N-t}\beta^\theta_t(x|\mathbf{z}_{t-1})} \\ = & \int_0^x -e^{-\theta[(\alpha_{N-t}-\alpha_{N-t+1})s+\beta^\theta_{t+1}(s)]}(N-t+1)(1-F(s))^{N-t}f(s)ds + C. \end{aligned}$$

Since the LHS is zero at $x=\bar{x}$ then

$$C = \int_0^{\bar{x}} e^{-\theta[(\alpha_{N-t}-\alpha_{N-t+1})s+\beta^\theta_{t+1}(s)]}(N-t+1)(1-F(s))^{N-t}f(s)ds.$$

Hence the unique solution $\beta_t^\theta(x|\mathbf{z}_{t-1})$ satisfies

$$(1-F(x))^{N-t+1}e^{-\theta\frac{N-t+1}{N-t}\beta_t^\theta(x|\mathbf{z}_{t-1})} = \int_x^{\bar{x}} e^{-\theta[(\alpha_{N-t}-\alpha_{N-t+1})s+\beta_{t+1}^\theta(s)]}(N-t+1)(1-F(s))^{N-t}f(s)ds.$$

Thus

$$\begin{aligned}
\beta_t^\theta(x) &= -\frac{N-t}{(N-t+1)\theta}\ln\left(\int_x^{\bar{x}} e^{-\theta[(\alpha_{N-t}-\alpha_{N-t+1})s+\beta_{t+1}^\theta(s)]}(N-t+1)\frac{(1-F(s))^{N-t}}{(1-F(x))^{N-t+1}}f(s)ds\right) \\
&= -\frac{N-t}{(N-t+1)\theta}\ln\left\{E\left[e^{-\theta\left((\alpha_{N-t}-\alpha_{N-t+1})Z_t^{(N)}+\beta_{t+1}^\theta(Z_t^{(N)})\right)}|Z_t^{(N)} > x > Z_{t-1}^{(N)}\right]\right\},
\end{aligned}$$

which establishes the result for round t and hence, by induction, the result for all t.

Next we establish that the bidding strategies are an equilibrium. It is sufficient to show that the following two-part claim holds for every t:

1. If $x \geq z_{t-1}$ then $x \in \arg\max_y \pi_t(y, x|\mathbf{z}_{t-1})$, i.e., it is optimal for a bidder with value x to bid $\beta_t^\theta(x)$ in round t.

2. If $x < z_{t-1}$ then $z_{t-1} \in \arg\max_y \pi_t(y, x|\mathbf{z}_{t-1})$.

The proof is by induction. Consider round $N-1$. Suppose that $x \geq z_{N-2}$. Consider an active bidder whose value is x and who bids y. Any bid below z_{N-2} is strictly dominated by a bid of z_{N-2} since both bids result in the same position while a bid of z_{N-2} yields higher compensation. Hence consider bids $y \geq z_{N-2}$.

With a bid of y the bidder wins Position 1 and obtains $\alpha_1 x-\beta_{N-1}^\theta(z_{N-1})-\Sigma_{j=1}^{N-2}\frac{1}{N-j}d_j$ if $y > z_{N-1}$, and he obtains $\alpha_2 x + \beta_{N-1}^\theta(y) - \Sigma_{j=1}^{N-2}\frac{1}{N-j}d_j$ if $y < z_{N-1}$. Hence

$$\begin{aligned}
\pi_{N-1}(y, x|\mathbf{z}_{N-2}) = & \int_{z_{N-2}}^{y} u\left(\alpha_1 x - \beta_{N-1}^\theta(z_{N-1}) - \Sigma_{j=1}^{N-2}\tfrac{1}{N-j}d_j\right) g_{N-1}^{(N-1)}(z_{N-1}|z_{N-2})dz_{N-1} \\
&+\int_{y}^{\bar{x}} u\left(\alpha_2 x + \beta_{N-1}^\theta(y) - \Sigma_{j=1}^{N-2}\tfrac{1}{N-j}d_j\right) g_{N-1}^{(N-1)}(z_{N-1}|z_{N-2})dz_{N-1}.
\end{aligned}$$

Differentiating with respect to y yields $\partial\pi_{N-1}(y, x|\mathbf{z}_{N-2})/\partial y =$

$$\begin{aligned}
& u(\alpha_1 x - \beta_{N-1}^\theta(y) - \Sigma_{j=1}^{N-2}\tfrac{1}{N-j}d_j)g_{N-1}^{(N-1)}(y|z_{N-2}) \\
- & u(\alpha_2 x + \beta_{N-1}^\theta(y) - \Sigma_{j=1}^{N-2}\tfrac{1}{N-j}d_j)g_{N-1}^{(N-1)}(y|z_{N-2}) \\
+ & u'(\alpha_2 x + \beta_{N-1}^\theta(y) - \Sigma_{j=1}^{N-2}\tfrac{1}{N-j}d_j)\beta_{N-1}^{\theta\prime}(y)(1 - G_{N-1}^{(N-1)}(y|z_{N-2})).
\end{aligned} \quad (3)$$

The necessary condition given in Proposition 4 for the general utility function u is

$$u'\left(\alpha_2 y+\beta^{\theta}_{N-1}(y)-\sum_{j=1}^{N-2}\frac{1}{N-j}d_j\right)\beta^{\theta\prime}_{N-1}(y)$$
$$= -\left[\begin{array}{c} u\left(\alpha_1 y-\beta^{\theta}_{N-1}(y)-\sum_{j=1}^{N-2}\frac{1}{N-j}d_j\right) \\ -u\left(\alpha_2 y+\beta^{\theta}_{N-1}(y)-\sum_{j=1}^{N-2}\frac{1}{N-j}d_j\right)\end{array}\right]\Gamma^N_{N-1}(y).$$

Substituting this expression into $\partial\pi_{N-1}(y,x|\mathbf{z}_{N-2})/\partial y$ yields

$$\begin{array}{ll} & u(\alpha_1 x-\beta^{\theta}_{N-1}(y)-\Sigma_{j=1}^{N-2}\frac{1}{N-j}d_j)g^{(N-1)}_{N-1}(y|z_{N-2}) \\ - & u(\alpha_2 x+\beta^{\theta}_{N-1}(y)-\Sigma_{j=1}^{N-2}\frac{1}{N-j}d_j)g^{(N-1)}_{N-1}(y|z_{N-2}) \\ - & \frac{u'(\alpha_2 x+\beta^{\theta}_{N-1}(y)-\Sigma_{j=1}^{N-2}\frac{1}{N-j}d_j)}{u'(\alpha_2 y+\beta^{\theta}_{N-1}(y)-\Sigma_{j=1}^{N-2}\frac{1}{N-j}d_j)}\left[\begin{array}{c} u\left(\alpha_1 y-\beta^{\theta}_{N-1}(y)-\sum_{j=1}^{N-2}\frac{1}{N-j}d_j\right) \\ -u\left(\alpha_2 y+\beta^{\theta}_{N-1}(y)-\sum_{j=1}^{N-2}\frac{1}{N-j}d_j\right)\end{array}\right]g^{(N-1)}_{N-1}(y|z_{N-2}).\end{array}$$

This derivative has the same sign as

$$\begin{array}{ll} & u(\alpha_1 x-\beta^{\theta}_{N-1}(y)-\Sigma_{j=1}^{N-2}\frac{1}{N-j}d_j)-u(\alpha_2 x+\beta^{\theta}_{N-1}(y)-\Sigma_{j=1}^{N-2}\frac{1}{N-j}d_j) \\ - & \frac{u'(\alpha_2 x+\beta^{\theta}_{N-1}(y)-\Sigma_{j=1}^{N-2}\frac{1}{N-j}d_j)}{u'(\alpha_2 y+\beta^{\theta}_{N-1}(y)-\Sigma_{j=1}^{N-2}\frac{1}{N-j}d_j)}\left[\begin{array}{c} u\left(\alpha_1 y-\beta^{\theta}_{N-1}(y)-\sum_{j=1}^{N-2}\frac{1}{N-j}d_j\right) \\ -u\left(\alpha_2 y+\beta^{\theta}_{N-1}(y)-\sum_{j=1}^{N-2}\frac{1}{N-j}d_j\right)\end{array}\right].\end{array}$$

Using that $u(x)$ has CARA we can write

$$\frac{u'(\alpha_2 x+\beta^{\theta}_{N-1}(y)-\Sigma_{j=1}^{N-2}\frac{1}{N-j}d_j)}{u'(\alpha_2 y+\beta^{\theta}_{N-1}(y)-\Sigma_{j=1}^{N-2}\frac{1}{N-j}d_j)}=e^{-\theta\alpha_2(x-y)}.$$

We can write

$$u(\alpha_1 x-\beta^{\theta}_{N-1}(y)-\Sigma_{j=1}^{N-2}\frac{1}{N-j}d_j)-u(\alpha_2 x+\beta^{\theta}_{N-1}(y)-\Sigma_{j=1}^{N-2}\frac{1}{N-j}d_j)$$

as

$$\frac{e^{-\theta[\alpha_2 x+\beta^{\theta}_{N-1}(y)-\Sigma_{j=1}^{N-2}\frac{1}{N-j}d_j]}-e^{-\theta[\alpha_1 x-\beta^{\theta}_{N-1}(y)-\Sigma_{j=1}^{N-2}\frac{1}{N-j}d_j]}}{\theta}.$$

Hence the sign of the derivative is the same as the sign of

$$\begin{array}{l} e^{-\theta[\alpha_2 x+\beta^{\theta}_{N-1}(y)-\Sigma_{j=1}^{N-2}\frac{1}{N-j}d_j]}-e^{-\theta[\alpha_1 x-\beta^{\theta}_{N-1}(y)-\Sigma_{j=1}^{N-2}\frac{1}{N-j}d_j]} \\ -e^{-\theta\alpha_2(x-y)}\left(e^{-\theta[\alpha_2 y+\beta^{\theta}_{N-1}(y)-\Sigma_{j=1}^{N-2}\frac{1}{N-j}d_j]}-e^{-\theta[\alpha_1 y-\beta^{\theta}_{N-1}(y)-\Sigma_{j=1}^{N-2}\frac{1}{N-j}d_j]}\right).\end{array}$$

We can rewrite this as

$$-e^{-\theta[\alpha_1 x-\beta^{\theta}_{N-1}(y)-\Sigma_{j=1}^{N-2}\frac{1}{N-j}d_j]}+e^{-\theta\alpha_2(x-y)}e^{-\theta[\alpha_1 y-\beta^{\theta}_{N-1}(y)-\Sigma_{j=1}^{N-2}\frac{1}{N-j}d_j]}$$

which has the same sign as

$$-e^{-\theta\alpha_1 x}+e^{-\theta\alpha_2(x-y)}e^{-\theta\alpha_1 y}$$

which has the same sign as

$$-e^{-\theta\alpha_1(x-y)}+e^{-\theta\alpha_2(x-y)}.$$

Since $\alpha_1>\alpha_2$, this expression is positive if $y<x$ and is negative if $y>x$. Thus $\partial\pi_{N-1}(y,x|\mathbf{z}_{N-2})/\partial y>0$ if $y<x$ and $\partial\pi_{N-1}(y,x|\mathbf{z}_{N-2})/\partial y<0$ if $y>x$.

We have shown if $x\geq z_{N-2}$ then $x\in\arg\max_y\pi_{N-1}(y,x|\mathbf{z}_{N-2})$, which establishes part 1 of the two-part claim. If $x<z_{N-2}$, then $y\geq z_{N-2}$ (since any bid below z_{N-2} is strictly dominated) implies $y\geq z_{N-2}>x$ and the above argument establishes bidding z_{N-2} is optimal since $\partial\pi_{N-1}(y,x|\mathbf{z}_{N-2})/\partial y<0$ for all $y\geq z_{N-2}$, i.e., $z_{N-2}\in\arg\max_y\pi_{N-1}(y,x|\mathbf{z}_{N-2})$. This establishes part 2 of the two-part claim for round $N-1$.

Assume the two-part claim is true for rounds $t+1$ through $N-1$. We show it is true for round t. Let $\mathbf{z}_{t-1}$ be arbitrary. Suppose $x\geq z_{t-1}$. Consider an active bidder in the t-th round whose value is x and who bids as though his value is $y\geq z_{t-1}$. A bid below z_{t-1} is not optimal. We need to distinguish between two cases: (i) $y\in[z_{t-1},x]$ and (ii) $y>x$, since his payoff function differs in each case. In what follows, we denote the payoff to a bid of y as $\pi^L_t(y,x|\mathbf{z}_{t-1})$ if $y\in[z_{t-1},x]$ and as $\pi^H_t(y,x|\mathbf{z}_{t-1})$ if $y\geq x$.

Case (i): Consider a bid $y\in[z_{t-1},x]$. If $z_t\in[z_{t-1},y]$ the bidder continues to round $t+1$ where, by the induction hypothesis, he optimally bids x and he has an expected payoff of $\Pi_{t+1}(x|\mathbf{z}_{t-1},z_t)$. If $z_t\geq y$ he obtains a payoff of $\alpha_{N-t+1}x+\beta^{\theta}_t(y)-\Sigma_{j=1}^{t-1}\frac{1}{N-j}d_j$ in round t. Hence his payoff is

$$\begin{aligned}\pi^L_t(y,x|\mathbf{z}_{t-1})= &\int_{z_{t-1}}^{y}\Pi_{t+1}(x|\mathbf{z}_{t-1},z_t)g_t^{(N-1)}(z_t|z_{t-1})dz_t\\ &+\int_y^{\bar{x}}u\left(\alpha_{N-t+1}x+\beta^{\theta}_t(y)-\Sigma_{j=1}^{t-1}\frac{1}{N-j}d_j\right)g_t^{(N-1)}(z_t|z_{t-1})dz_t.\end{aligned}$$

Differentiating with respect to y yields

$$\begin{aligned}\frac{\partial\pi^L_t(y,x|\mathbf{z}_{t-1})}{\partial y}= &[\Pi_{t+1}(x|\mathbf{z}_{t-1},y)-u\left(\alpha_{N-t+1}x+\beta^{\theta}_t(y)-\Sigma_{j=1}^{t-1}\frac{1}{N-j}d_j\right)]g_t^{(N-1)}(y|z_{t-1})\\ &+u'\left(\alpha_{N-t+1}x+\beta^{\theta}_t(y)-\Sigma_{j=1}^{t-1}\frac{1}{N-j}d_j\right)\beta^{\theta\prime}_t(y)(1-G_t^{(N-1)}(y|z_{t-1})).\end{aligned}$$

Rewriting

$$\begin{aligned}\frac{\partial \pi_t^L(y,x|\mathbf{z}_{t-1})}{\partial y} &= [\Pi_{t+1}(x|\mathbf{z}_{t-1},y) - \tfrac{1-e^{-\theta[\alpha_{N-t+1}x+\beta_t^\theta(y)-\Sigma_{j=1}^{t-1}\frac{1}{N-j}d_j]}}{\theta}]g_t^{(N-1)}(y|z_{t-1}) \\ &\quad + e^{-\theta[\alpha_{N-t+1}x+\beta_t^\theta(y)-\Sigma_{j=1}^{t-1}\frac{1}{N-j}d_j]}\beta_t^{\theta\prime}(y)(1-G_t^{(N-1)}(y|z_{t-1})).\end{aligned}$$

Using the expression for $\beta_t^{\theta\prime}(y)$ from the necessary condition for equilibrium from Proposition 4 for round t and substituting yields

$$\begin{aligned}\frac{\partial \pi_t^L(y,x|\mathbf{z}_{t-1})}{\partial y} &= [\Pi_{t+1}(x|\mathbf{z}_{t-1},y) - \tfrac{1-e^{-\theta[\alpha_{N-t+1}x+\beta_t^\theta(y)-\Sigma_{j=1}^{t-1}\frac{1}{N-j}d_j]}}{\theta}]g_t^{(N-1)}(y|z_{t-1}) \\ &\quad - e^{-\theta\alpha_{N-t+1}(x-y)}\frac{1}{\theta}\left[\begin{array}{c} e^{-\theta[\alpha_{N-t+1}y+\beta_t^\theta(y)-\sum_{j=1}^{t-1}\frac{1}{N-j}d_j]} \\ -e^{-\theta[\alpha_{N-t}y+\beta_{t+1}^\theta(y)-\frac{1}{N-t}\beta_t^\theta(y)-\sum_{j=1}^{t-1}\frac{1}{N-j}d_j]}\end{array}\right] g_t^{(N-1)}(y|z_{t-1}).\end{aligned}$$

Simplifying yields $\partial \pi_t^L(y,x|\mathbf{z}_{t-1})/\partial y$ as

$$\left(\Pi_{t+1}(x|\mathbf{z}_{t-1},y) - \frac{1}{\theta}\left[1-e^{-\theta[\alpha_{N-t}y+\alpha_{N-t+1}(x-y)+\beta_{t+1}^\theta(y)-\frac{1}{N-t}\beta_t^\theta(y)-\sum_{j=1}^{t-1}\frac{1}{N-j}d_j]}\right]\right) g_t^{(N-1)}(y|z_{t-1}).$$

We show that $\partial \pi_t^L(y,x|\mathbf{z}_{t-1})/\partial y > 0$ for $y < x$. If the bid at round t is y, then $\Pi_{t+1}(x|\mathbf{z}_{t-1},y)$ is the equilibrium payoff at round $t+1$ of a bidder with value x. If he were to deviate from equilibrium and bid y at round $t+1$, then he obtains position $N-t$ (since y is the smallest value of a rival bidder) and he receives $\beta_{t+1}^\theta(y)$ at round $t+1$ and pays $\frac{1}{N-t}\beta_t^\theta(y)+\sum_{j=1}^{t-1}\frac{1}{N-j}d_j$. By the induction hypothesis, this payoff is less than his equilibrium payoff, i.e.,

$$\Pi_{t+1}(x|\mathbf{z}_{t-1},y) > \frac{1}{\theta}\left[1-e^{-\theta[\alpha_{N-t}x+\beta_{t+1}^\theta(y)-\frac{1}{N-t}\beta_t^\theta(y)-\sum_{j=1}^{t-1}\frac{1}{N-j}d_j]}\right].$$

Since $\alpha_{N-t} > \alpha_{N-t+1}$ and $x > y$ we have

$$\alpha_{N-t}x > \alpha_{N-t}y + \alpha_{N-t+1}(x-y)$$

and hence

$$\begin{aligned} &\frac{1}{\theta}\left[1-e^{-\theta[\alpha_{N-t}x+\beta_{t+1}^\theta(y)-\frac{1}{N-t}\beta_t^\theta(y)-\sum_{j=1}^{t-1}\frac{d_j}{N-j}]}\right] \\ > &\frac{1}{\theta}\left[1-e^{-\theta[\alpha_{N-t}y+\alpha_{N-t+1}(x-y)+\beta_{t+1}^\theta(y)-\frac{1}{N-t}\beta_t^\theta(y)-\sum_{j=1}^{t-1}\frac{d_j}{N-j}]}\right].\end{aligned}$$

Thus $\Pi_{t+1}(x|\mathbf{z}_{t-1},y)$ is greater than the RHS of this inequality and hence $\partial \pi_t^L(y,x|\mathbf{z}_{t-1})/\partial y > 0$ for $y < x$.

Case (ii): Consider a bid $y \geq x$. If $z_t \in [z_{t-1}, x]$, then the bidder continues to round $t+1$ and, by part 1 of induction hypothesis, he bids x and obtains $\Pi_{t+1}(x|\mathbf{z}_{t-1}, z_t)$. If $z_t \in [x, y]$, then he continues to round $t+1$ and, by part 2 of the induction hypothesis, he bids z_t and obtains a payoff of

$$\alpha_{N-t}x + \beta^{\theta}_{t+1}(z_t) - \frac{1}{N-t}\beta^{\theta}_t(z_t) - \Sigma^{t-1}_{j=1}\frac{1}{N-j}d_j.$$

If $z_t > y$ then in round t he obtains position $N-t+1$ and his payoff is $\alpha_{N-t+1}x+ \beta^{\theta}_t(y) - \Sigma^{t-1}_{j=1}\frac{1}{N-j}d_j$. Thus his expected payoff at round t is

$$\begin{aligned}\pi^H_t(y, x|\mathbf{z}_{t-1}) = \; & \int_{z_{t-1}}^{x} \Pi_{t+1}(x|\mathbf{z}_{t-1}, z_t)g^{(N-1)}_t(z_t|z_{t-1})dz_t \\ & + \int_x^y u(\alpha_{N-t}x + \beta^{\theta}_{t+1}(z_t) - \tfrac{1}{N-t}\beta^{\theta}_t(z_t) - \Sigma^{t-1}_{j=1}\tfrac{1}{N-j}d_j)g^{(N-1)}_t(z_t|z_{t-1})dz_t, \\ & + \int_y^{\bar{x}} u\left(\alpha_{N-t+1}x + \beta^{\theta}_t(y) - \Sigma^{t-1}_{j=1}\tfrac{1}{N-j}d_j\right) g^{(N-1)}_t(z_t|z_{t-1})dz_t.\end{aligned}$$

Differentiating with respect to y yields

$$\begin{aligned}\frac{\partial\pi^H_t(y, x|\mathbf{z}_{t-1})}{\partial y} = & \left[\begin{array}{c} u\left(\alpha_{N-t}x + \beta^{\theta}_{t+1}(y) - \frac{1}{N-t}\beta^{\theta}_t(y) - \Sigma^{t-1}_{j=1}\frac{1}{N-j}d_j\right) \\ -u\left(\alpha_{N-t+1}x + \beta^{\theta}_t(y) - \Sigma^{t-1}_{j=1}\frac{1}{N-j}d_j\right)\end{array}\right] g^{(N-1)}_t(y|z_{t-1}) \\ & +u'\left(\alpha_{N-t+1}x + \beta^{\theta}_t(y) - \Sigma^{t-1}_{j=1}\frac{1}{N-j}d_j\right) \beta^{\theta\prime}_t(y)(1 - G^{(N-1)}_t(y|z_{t-1})).\end{aligned}$$

Using the expression for $\beta^{\theta\prime}_t(y)$ from the necessary condition for equilibrium from Proposition 4 for round t and substituting gives $\partial\pi^H_t(y, x|\mathbf{z}_{t-1})/\partial y$ as

$$\begin{aligned}& \left[\begin{array}{c} u(\alpha_{N-t}x + \beta^{\theta}_{t+1}(y) - \frac{1}{N-t}\beta^{\theta}_t(y) - \Sigma^{t-1}_{j=1}\frac{1}{N-j}d_j) \\ -u(\alpha_{N-t+1}x + \beta^{\theta}_t(y) - \Sigma^{t-1}_{j=1}\frac{1}{N-j}d_j)\end{array}\right] g^{(N-1)}_t(y|z_{t-1}) \\ & -\frac{u'(\alpha_{N-t+1}x + \beta^{\theta}_t(y) - \Sigma^{t-1}_{j=1}\frac{1}{N-j}d_j)}{u'(\alpha_{N-t+1}y + \beta^{\theta}_t(y) - \Sigma^{t-1}_{j=1}\frac{1}{N-j}d_j)} \\ & \times \left[\begin{array}{c} u(\alpha_{N-t}y + \beta^{\theta}_{t+1}(y) - \frac{1}{N-t}\beta^{\theta}_t(y) - \Sigma^{t-1}_{j=1}\frac{1}{N-j}d_j) \\ -u(\alpha_{N-t+1}y + \beta^{\theta}_t(y) - \Sigma^{t-1}_{j=1}\frac{1}{N-j}d_j)\end{array}\right] g^{(N-1)}_t(y|z_{t-1}).\end{aligned}$$

Since bidders have CARA preferences, then $\partial\pi^H_t(y, x|\mathbf{z}_{t-1})/\partial y$ is

$$\begin{aligned}& \frac{1}{\theta}\left[\begin{array}{c} e^{-\theta[\alpha_{N-t+1}x+\beta^{\theta}_t(y)-\Sigma^{t-1}_{j=1}\frac{1}{N-j}d_j]} \\ -e^{-\theta[\alpha_{N-t}x+\beta^{\theta}_{t+1}(y)-\frac{1}{N-t}\beta^{\theta}_t(y)-\Sigma^{t-1}_{j=1}\frac{1}{N-j}d_j]}\end{array}\right] g^{(N-1)}_t(y|z_{t-1}) \\ & -e^{-\theta\alpha_{N-t+1}(x-y)}\frac{1}{\theta}\left[\begin{array}{c} e^{-\theta[\alpha_{N-t+1}y+\beta^{\theta}_t(y)-\Sigma^{t-1}_{j=1}\frac{1}{N-j}d_j]} \\ -e^{-\theta[\alpha_{N-t}y+\beta^{\theta}_{t+1}(y)-\frac{1}{N-t}\beta^{\theta}_t(y)-\Sigma^{t-1}_{j=1}\frac{1}{N-j}d_j]}\end{array}\right] g^{(N-1)}_t(y|z_{t-1}). \\ = \; & \frac{1}{\theta}\left[\begin{array}{c} e^{-\theta[\alpha_{N-t}y+\alpha_{N-t+1}(x-y)+\beta^{\theta}_{t+1}(y)-\frac{1}{N-t}\beta^{\theta}_t(y)-\Sigma^{t-1}_{j=1}\frac{1}{N-j}d_j]} \\ -e^{-\theta[\alpha_{N-t}x+\beta^{\theta}_{t+1}(y)-\frac{1}{N-t}\beta^{\theta}_t(y)-\Sigma^{t-1}_{j=1}\frac{1}{N-j}d_j]}\end{array}\right] g^{(N-1)}_t(y|z_{t-1}).\end{aligned}$$

This has the same sign as

$$e^{-\theta\alpha_{N-t+1}(x-y)} - e^{-\theta\alpha_{N-t}(x-y)},$$

which is negative since $\alpha_{N-t} > \alpha_{N-t+1}$ and $y > x$.

We have shown if $x \geq z_{t-1}$ then $x \in \arg\max_y \pi_t(y, x|\mathbf{z}_{t-1})$. If $x < z_{t-1}$, then $y \geq z_{t-1}$ (since any bid below z_{t-1} is strictly dominated) implies $y \geq z_{t-1} > x$ and the above argument establishes bidding z_{t-1} is optimal since $\partial\pi_t(y, x|\mathbf{z}_{t-1})/\partial y < 0$ for all $y \geq z_{t-1}$, i.e., $z_{t-1} \in \arg\max_y \pi_t(y, x|\mathbf{z}_{t-1})$. This establishes the two-part claim for round t, and completes the proof by induction. □

Proof of Proposition 7: We first show that for each t we have that $\beta_t^0(x) > \beta_t^\theta(x)$ for $\theta > 0$ and $x < \bar{x}$. Consider round $t = N - 1$. Since e^{-x} is convex, Jensen's inequality implies

$$e^{-E\left[\theta(\alpha_1-\alpha_2)Z_{N-1}^{(N)}|Z_{N-1}^{(N)}>x>Z_{N-2}^{(N)}\right]} < E\left[e^{-\theta(\alpha_1-\alpha_2)Z_{N-1}^{(N)}}|Z_{N-1}^{(N)} > x > Z_{N-2}^{(N)}\right].$$

Taking the log of both sides and then dividing both sides by -2θ yields

$$\begin{aligned} \beta_{N-1}^0(x) &= \frac{1}{2}E\left[(\alpha_1 - \alpha_2)\, Z_{N-1}^{(N)}|Z_{N-1}^{(N)} > x > Z_{N-2}^{(N)}\right] \\ &> -\frac{1}{2\theta}\ln\left\{E\left[e^{-\theta(\alpha_1-\alpha_2)Z_{N-1}^{(N)}}|Z_{N-1}^{(N)} > x > Z_{N-2}^{(N)}\right]\right\} \\ &= \beta_{N-1}^\theta(x). \end{aligned}$$

Assume that $\beta_{t+1}^0(x) > \beta_{t+1}^\theta(x)$ for $x < \bar{x}$. We show that $\beta_t^0(x) > \beta_t^\theta(x)$ for $x < \bar{x}$. For $z < \bar{x}$ we have

$$(\alpha_{N-t} - \alpha_{N-t+1})\, z + \beta_{t+1}^0(z) > (\alpha_{N-t} - \alpha_{N-t+1})\, z + \beta_{t+1}^\theta(z).$$

Multiplying through by $-\theta$ and applying the exponential function to both sides gives

$$e^{-\theta\left[(\alpha_{N-t}-\alpha_{N-t+1})z+\beta_{t+1}^0(z)\right]} < e^{-\theta\left[(\alpha_{N-t}-\alpha_{N-t+1})z+\beta_{t+1}^\theta(z)\right]}.$$

Hence

$$\begin{aligned} & E\left[e^{-\theta\left[(\alpha_{N-t}-\alpha_{N-t+1})Z_t^{(N)}+\beta_{t+1}^0(Z_t^{(N)})\right]}|Z_t^{(N)} > x > Z_{t-1}^{(N)}\right] \\ < \; & E\left[e^{-\theta\left[(\alpha_{N-t}-\alpha_{N-t+1})Z_t^{(N)}+\beta_{t+1}^\theta(Z_t^{(N)})\right]}|Z_t^{(N)} > x > Z_{t-1}^{(N)}\right]. \end{aligned}$$

By Jensen's inequality, we have

$$\begin{aligned} & e^{-\theta E\left[(\alpha_{N-t}-\alpha_{N-t+1})Z_t^{(N)}+\beta^0_{t+1}(Z_t^{(N)})|Z_t^{(N)}>x>Z_{t-1}^{(N)}\right]} \\ < \;& E\left[e^{-\theta\left[(\alpha_{N-t}-\alpha_{N-t+1})Z_t^{(N)}+\beta^0_{t+1}(Z_t^{(N)})\right]}|Z_t^{(N)}>x>Z_{t-1}^{(N)}\right], \end{aligned}$$

and thus

$$\begin{aligned} & e^{-\theta E\left[(\alpha_{N-t}-\alpha_{N-t+1})Z_t^{(N)}+\beta^0_{t+1}(Z_t^{(N)})|Z_t^{(N)}>x>Z_{t-1}^{(N)}\right]} \\ < \;& E\left[e^{-\theta\left[(\alpha_{N-t}-\alpha_{N-t+1})Z_t^{(N)}+\beta^\theta_{t+1}(Z_t^{(N)})\right]}|Z_t^{(N)}>x>Z_{t-1}^{(N)}\right]. \end{aligned}$$

Taking the log of both sides and then multiplying both sides by $-(N-t)/((N-t+1)\,\theta)$ yields $\beta^0_t(x) > \beta^\theta_t(x)$. We have shown for each t that $\beta^0_t(x) > \beta^\theta_t(x)$ for $\theta > 0$ and $x < \bar{x}$.

Next we show that for each t we have that $\beta^\theta_t(x) > \underline{\beta}_t(x)$ for $\theta > 0$ and $x < \bar{x}$. Consider $t = N-1$. For $z > x$ we have

$$e^{-\theta(\alpha_1-\alpha_2)z} < e^{-\theta(\alpha_1-\alpha_2)x},$$

and hence

$$E\left[e^{-\theta(\alpha_1-\alpha_2)Z^{(N)}_{N-1}}|Z^{(N)}_{N-1}>x>Z^{(N)}_{N-2}\right] < e^{-\theta(\alpha_1-\alpha_2)x}.$$

Taking the log of both sides and then dividing both sides by -2θ yields

$$\beta^\theta_{N-1}(x) = -\frac{1}{2\theta}\ln\left\{E\left[e^{-\theta(\alpha_1-\alpha_2)Z^{(N)}_{N-1}}|Z^{(N)}_{N-1}>x>Z^{(N)}_{N-2}\right]\right\} > \frac{\alpha_1-\alpha_2}{2}x = \underline{\beta}_{N-1}(x).$$

Assume that $\beta^\theta_{t+1}(x) > \underline{\beta}_{t+1}(x)$ for $x < \bar{x}$. We show that $\beta^\theta_t(x) > \underline{\beta}_t(x)$ for $x < \bar{x}$. For $z > x$ we have

$$e^{-\theta\left[(\alpha_{N-t}-\alpha_{N-t+1})z+\beta^\theta_{t+1}(z)\right]} < e^{-\theta\left[(\alpha_{N-t}-\alpha_{N-t+1})x+\underline{\beta}_{t+1}(x)\right]},$$

and hence

$$E\left[e^{-\theta\left[(\alpha_{N-t}-\alpha_{N-t+1})Z_t^{(N)}+\beta^\theta_{t+1}(Z_t^{(N)})\right]}|Z_t^{(N)}>x>Z_{t-1}^{(N)}\right] < e^{-\theta\left[(\alpha_{N-t}-\alpha_{N-t+1})x+\underline{\beta}_{t+1}(x)\right]}.$$

By the analogous argument as above, we obtain

$$\begin{aligned} \beta^\theta_t(x) &= -\frac{N-t}{(N-t+1)\,\theta}\ln\left\{E\left[e^{-\theta[(\alpha_{N-t}-\alpha_{N-t+1})Z_t^{(N)}+\beta^\theta_{t+1}(Z_t^{(N)})]}|Z_t^{(N)}>x>Z_{t-1}^{(N)}\right]\right\} \\ &> \frac{N-t}{N-t+1}\left[(\alpha_{N-t}-\alpha_{N-t+1})\,x+\underline{\beta}_{t+1}(x)\right] \\ &= \underline{\beta}_t(x), \end{aligned}$$

where the last equality follows since

$$\begin{aligned}
& \frac{N-t}{N-t+1}\left[\left(\alpha_{N-t}-\alpha_{N-t+1}\right)x+\underline{\beta}_{t+1}(x)\right] \\
= & \frac{N-t}{N-t+1}\left(\alpha_{N-t}-\alpha_{N-t+1}\right)x+\frac{N-t}{N-t+1}\left(\sum_{m=1}^{N-t-1}\frac{1}{N-t}\alpha_m-\frac{N-t-1}{N-t}\alpha_{N-t}\right)x \\
= & \left(\sum_{m=1}^{N-t-1}\frac{1}{N-t+1}\alpha_m-\frac{N-t-1}{N-t+1}\alpha_{N-t}+\frac{N-t}{N-t+1}\left(\alpha_{N-t}-\alpha_{N-t+1}\right)\right)x \\
= & \left(\sum_{m=1}^{N-t}\frac{1}{N-t+1}\alpha_m-\frac{N-t}{N-t+1}\alpha_{N-t+1}\right)x \\
= & \underline{\beta}_t(x).
\end{aligned}$$

□

Proof of Proposition 8: We first show that for each t we have $\beta_t^{\theta'}(x) < \beta_t^{\theta}(x)$ for $\theta' > \theta$ and $x < \bar{x}$. Consider round $t = N-1$. Since $f(s) = s^{\frac{\theta}{\theta'}}$ is concave, by Jensen's inequality we have

$$\left(E\left[e^{-\theta'(\alpha_1-\alpha_2)Z_{N-1}^{(N)}}|Z_{N-1}^{(N)} > x > Z_{N-2}^{(N)}\right]\right)^{\frac{\theta}{\theta'}} > E\left[e^{-\theta(\alpha_1-\alpha_2)Z_{N-1}^{(N)}}|Z_{N-1}^{(N)} > x > Z_{N-2}^{(N)}\right].$$

Taking the log of both sides and then dividing both sides by -2θ yields

$$\begin{aligned}
\beta_{N-1}^{\theta}(x) &= -\frac{1}{2\theta}\ln\left(E\left[e^{-\theta(\alpha_1-\alpha_2)Z_{N-1}^{(N)}}|Z_{N-1}^{(N)} > x > Z_{N-2}^{(N)}\right]\right) \\
&> -\frac{1}{2\theta'}\ln\left(E\left[e^{-\theta'(\alpha_1-\alpha_2)Z_{N-1}^{(N)}}|Z_{N-1}^{(N)} > x > Z_{N-2}^{(N)}\right]\right) \\
&= \beta_{N-1}^{\theta'}(x).
\end{aligned}$$

Assume that $\beta_{t+1}^{\theta'}(x) < \beta_{t+1}^{\theta}(x)$ for $x < \bar{x}$. We show that $\beta_t^{\theta'}(x) < \beta_t^{\theta}(x)$ for $x < \bar{x}$. By Jensen's inequality we have

$$\begin{aligned}
& E\left[e^{-\theta[(\alpha_{N-t}-\alpha_{N-t+1})Z_t^{(N)}+\beta_{t+1}^{\theta'}(Z_t^{(N)})]}|Z_t^{(N)} > x > Z_{t-1}^{(N)}\right] \\
< & \left(E\left[e^{-\theta'[(\alpha_{N-t}-\alpha_{N-t+1})Z_t^{(N)}+\beta_{t+1}^{\theta'}(Z_t^{(N)})]}|Z_t^{(N)} > x > Z_{t-1}^{(N)}\right]\right)^{\frac{\theta}{\theta'}}
\end{aligned}$$

and since $\beta_{t+1}^{\theta'}(x) < \beta_{t+1}^{\theta}(x)$ then

$$\begin{aligned}
& E\left[e^{-\theta[(\alpha_{N-t}-\alpha_{N-t+1})Z_t^{(N)}+\beta_{t+1}^{\theta}(Z_t^{(N)})]}|Z_t^{(N)} > x > Z_{t-1}^{(N)}\right] \\
< & E\left[e^{-\theta[(\alpha_{N-t}-\alpha_{N-t+1})Z_t^{(N)}+\beta_{t+1}^{\theta'}(Z_t^{(N)})]}|Z_t^{(N)} > x > Z_{t-1}^{(N)}\right].
\end{aligned}$$

Hence

$$E\left[e^{-\theta[(\alpha_{N-t}-\alpha_{N-t+1})Z_t^{(N)}+\beta_{t+1}^{\theta}(Z_t^{(N)})]}|Z_t^{(N)}>x>Z_{t-1}^{(N)}\right]$$
$$<\quad\left(E\left[e^{-\theta'[(\alpha_{N-t}-\alpha_{N-t+1})Z_t^{(N)}+\beta_{t+1}^{\theta'}(Z_t^{(N)})]}|Z_t^{(N)}>x>Z_{t-1}^{(N)}\right]\right)^{\frac{\theta}{\theta'}}.$$

Taking the log of both sides and multiplying both sides by $-(N-t)/((N-t+1)\,\theta)$ yields

$$\begin{aligned}\beta_t^{\theta'}(x) &= -\frac{N-t}{(N-t+1)\,\theta'}\ln\left\{E\left[e^{-\theta'[(\alpha_{N-t}-\alpha_{N-t+1})Z_t^{(N)}+\beta_{t+1}^{\theta'}(Z_t^{(N)})]}|Z_t^{(N)}>x>Z_{t-1}^{(N)}\right]\right\}\\ &< -\frac{N-t}{(N-t+1)\,\theta}\ln\left\{E\left[e^{-\theta[(\alpha_{N-t}-\alpha_{N-t+1})Z_t^{(N)}+\beta_{t+1}^{\theta}(Z_t^{(N)})]}|Z_t^{(N)}>x>Z_{t-1}^{(N)}\right]\right\}\\ &= \beta_t^{\theta}(x).\end{aligned}$$

Next we show that for each t we have $\lim_{\theta\to\infty}\beta_t^{\theta}(x)=\underline{\beta}_t(x)$ for all x. For $t=N-1$ the limit is obtained directly. Specifically, after applying l'Hopital's rule, we see that

$$\lim_{\theta\to\infty}\beta_{N-1}^{\theta}(x)=\frac{1}{2}\left(\alpha_1-\alpha_2\right)\lim_{\theta\to\infty}\frac{\int_x^{\bar{x}} ze^{-\theta(\alpha_1-\alpha_2)z}g_{N-1}^{(N)}(z|x)dz}{\int_x^{\bar{x}} e^{-\theta(\alpha_1-\alpha_2)z}g_{N-1}^{(N)}(z|x)dz}$$

where $g_{N-1}^{(N)}(z|x)=2f(z)(1-F(z))/(1-F(x))^2$. Van Essen and Wooders (2016, p. 239) established that

$$\lim_{\theta\to\infty}\frac{\int_x^{\bar{x}} ze^{-\theta z}g_{N-1}^{(N)}(z|x)dz}{\int_x^{\bar{x}} e^{-\theta z}g_{N-1}^{(N)}(z|x)dz}=x,$$

which implies that

$$\lim_{\theta\to\infty}\frac{\int_x^{\bar{x}} ze^{-\theta(\alpha_1-\alpha_2)z}g_{N-1}^{(N)}(z|x)dz}{\int_x^{\bar{x}} e^{-\theta(\alpha_1-\alpha_2)z}g_{N-1}^{(N)}(z|x)dz}=x,$$

Hence,

$$\lim_{\theta\to\infty}\beta_{N-1}^{\theta}(x)=\frac{1}{2}\left(\alpha_1-\alpha_2\right)x=\underline{\beta}_{N-1}(x).$$

Observe that $\beta_{N-1}^{\theta}(x)$ is continuous in x on the compact set $[0,\bar{x}]$ for each θ, it converges pointwise to $\underline{\beta}_{N-1}(x)$, which is continuous on $[0,\bar{x}]$, and it is decreasing in θ. Hence β_{N-1}^{θ} converges uniformly to $\underline{\beta}_{N-1}$ on $[0,\bar{x}]$ by Theorem 7.12 of Rudin (1976).

Assume that $\beta_{t+1}^{\theta}(x)$ converges uniformly to $\underline{\beta}_{t+1}(x)$ on $[0, \bar{x}]$. We show that $\beta_t^{\theta}(x)$ converges uniformly to $\underline{\beta}_t(x)$. The CARA bid function in round t is

$$\beta_t^{\theta}(x) = -\frac{N-t}{(N-t+1)\,\theta} \ln \left(\int_x^{\bar{x}} e^{-\theta[(\alpha_{N-t}-\alpha_{N-t+1})z+\beta_{t+1}^{\theta}(z)]} g_t^{(N)}(z|x)dz \right).$$

Let $\Delta > 0$ be arbitrary. Since $\beta_t^{\theta}(x)$ is decreasing in θ and since $\beta_{t+1}^{\theta} \to \underline{\beta}_{t+1}$ uniformly as $\theta \to \infty$, then there is a $\bar{\theta}$ such that for all $\theta \geq \bar{\theta}$ we have

$$\beta_{t+1}^{\theta}(x) \leq \sum_{m=1}^{N-t-1} \frac{m}{N-t} (\alpha_m - \alpha_{m+1})\, x + \Delta$$

for $x \in [0, \bar{x}]$. Define

$$\bar{\beta}_t^{\theta}(x) \equiv -\frac{N-t}{(N-t+1)\,\theta} \ln \left(\int_x^{\bar{x}} e^{-\theta[z(\alpha_{N-t}-\alpha_{N-t+1}+\sum_{m=1}^{N-t-1} \frac{m}{N-t}(\alpha_m-\alpha_{m+1}))+\Delta]} g_t^{(N)}(z|x) \right) dz.$$

Then $\beta_t^{\theta}(x) \leq \bar{\beta}_t^{\theta}(x)$ for $\theta \geq \bar{\theta}$ and $x \in [0, \bar{x}]$. By Proposition 7 we have $\underline{\beta}_t(x) \leq \beta_t^{\theta}(x)$ and thus

$$\underline{\beta}_t(x) \leq \beta_t^{\theta}(x) \leq \bar{\beta}_t^{\theta}(x)$$

for $\theta \geq \bar{\theta}$ and $x \in [0, \bar{x}]$.

We establish that $\beta_t^{\theta}(x)$ converges pointwise to $\underline{\beta}_t(x)$ for each $x \in [0, \bar{x}]$. Define

$$C = \alpha_{N-t} - \alpha_{N-t+1} + \sum_{m=1}^{N-t-1} \frac{m}{N-t} (\alpha_m - \alpha_{m+1}).$$

Applying L'Hopital's rule and using the same argument as for round $N-1$, we have

$$\begin{aligned}
\lim_{\theta\to\infty} \bar{\beta}_t^{\theta}(x) &= \frac{N-t}{N-t+1} \lim_{\theta\to\infty} \frac{\int_x^{\bar{x}} (zC+\Delta) e^{-\theta(zC+\Delta)} g_t^{(N)}(z|x)dz}{\int_x^{\bar{x}} e^{-\theta(zC+\Delta)} g_t^{(N)}(z|x)dz} \\
&= \frac{N-t}{N-t+1} \left(C \lim_{\theta\to\infty} \frac{\int_x^{\bar{x}} z e^{-\theta z C} g_t^{(N)}(z|x)dz}{\int_x^{\bar{x}} e^{-\theta z C} g_t^{(N)}(z|x)dz} + \Delta \right) \\
&= \frac{N-t}{N-t+1} (Cx + \Delta),
\end{aligned}$$

where the last inequality holds by Van Essen and Wooders (2016). Substituting for C and simplifying yields

$$\lim_{\theta\to\infty} \bar{\beta}_t^{\theta}(x) = \underline{\beta}_t(x) + \frac{N-t}{N-t+1}\Delta.$$

Since the inequality

$$\underline{\beta}_t(x) \leq \lim_{\theta \to \infty} \beta_t^\theta(x) \leq \lim_{\theta \to \infty} \bar{\beta}_t^\theta(x) = \underline{\beta}_t(x) + \frac{N-t}{N-t+1}\Delta$$

holds for arbitrary $\Delta > 0$, it follows that $\lim_{\theta \to \infty} \beta_t^\theta(x) = \underline{\beta}_t(x)$. By the same argument as for β_{N-1}^θ, we have that β_t^θ converges uniformly to $\underline{\beta}_t$ on $[0, \bar{x}]$. □

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