Allocating Positions Fairly: Auctions and Shapley Value*

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August 3, 2019

Abstract

We study the problem of fairly allocating heterogenous items, priorities, positions, or property rights to participants with equal claims from three perspectives: cooperative, decision theoretic, and non-cooperative. We characterize the Shapley value of the cooperative game and then introduce a class of auctions for non-cooperatively allocating positions. We show that for any auction in this class, each bidder obtains his Shapley value when every bidder follows the auction's unique maxmin perfect bidding strategy. When information is incomplete we characterize the Bayesian equilibrium of these auctions, and show that equilibrium play converges to maxmin perfect play as bidders become infinitely risk averse. The equilibrium allocations thus converges to the Shapley value allocation as bidders become risk averse. Together these results provide both decision theoretic and non-cooperative equilibrium foundations for the Shapley value in an environment with incomplete information.

*The authors are grateful for comments from seminar participants at Cambridge University and participants at the 2017 NYUAD APET conference, the EWGET 2018 conference, and the 2018 CPMD Workshop on Market Design at UTS. The authors are also grateful for comments from Gabrielle Demange, Audrey Hu, and Herve Moulin.

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1 Introduction

This paper studies the problem of allocating heterogeneous items, priorities, positions, or rights to participants who have equal claims. Examples of this type of problem include allocating the priority of service in a queue, allocating items to heirs in an estate, allocating positions of ads on a webpage, or allocating fishing rights to different geographical areas. In our environment, participants have unit demands and a common ranking of the priorities/items/positions/rights, which we hereafter simply refer to as "positions." In particular, all participants agree that one position is the most desirable, a second position is the next most desirable, and so on. Despite the common ranking of positions, participants vary in the intensity of their preferences and these intensities are private information. The problem is to find an allocation that is efficient, budget balanced, and fair, with those participants receiving more desirable positions compensating the ones receiving less desirable positions.

We study the allocation problem from three different perspectives. The first approach, from cooperative game theory, is to characterize the Shapley value allocation. The second approach is decision theoretic. We introduce a class of auctions for allocating positions that we call "compensated position auctions" and we characterize maxmin perfect bidding. Finally, we approach the problem from the perspective of non-cooperative game theory, and we characterize Bayes Nash equilibrium of these auctions. We show that these three solutions to the allocation problem are related in a precise way: for any compensated position auction, (i) when all bidders follow their maxmin perfect bidding strategies then the Shapley value allocation results, and (ii) the allocation obtained in the Bayes Nash equilibrium approaches the Shapley value allocation as bidders become risk averse. Hence our results provide decision theoretic and non-cooperative foundations for the Shapley value in an environment with incomplete information.

Shapley (1953) introduced the notion of a *value* for a cooperative game, now called the Shapley value. The Shapley value is a fundamental solution concept in cooperative game theory with the Shapley allocation often taken as the benchmark for a fair allocation (see Myerson (1977), Roth (1988), Moulin

(1992), and Moulin (2004, Chapter 5)). In the position allocation problem, the Shapley allocation identifies for each player a position and transfer. We characterize the Shapley value allocation. We show that the transfers associated with the Shapley allocation can be computed recursively, starting with the transfer received by the player allocated the worst position, then the transfer of the player with the second-worst position, and so on. This result is a consequence of the feature of the position allocation problem that a player exerts externalities only on players with lower intensities of preferences than his own.

Next we introduce and study a class of dynamic auctions for allocating positions that is suggested by the recursive nature of the Shapley transfers. The auction takes place over rounds. At each round, the participants (hereafter "bidders") simultaneously make demands for compensation. The bidder with the smallest demand receives the worst unallocated position, he receives his demand as compensation, and he exits. His compensation is paid by the remaining bidders, who will eventually be allocated better positions, according to the auction's cost sharing rule. The auction ends when one position and bidder remain. That bidder receives the most desirable position, but pays compensation to every other bidder. In sum, bidders pay compensation to bidders allocated positions worse than their own and receive compensation from bidders allocated positions better than their own.

Our second approach studies bidding behavior in compensated position auctions when players act to maximize their minimum payoff. We will say a strategy is "maxmin perfect" if it maximizes a bidder's minimum payoff at every history of play. Maxmin perfection is a natural refinement of maxmin in dynamic games. We characterize the unique maxmin perfect bidding strategy, showing how it depends on the cost sharing rule. The strategy has a natural interpretation similar to the solution to the Contested Garment problem described in the Talmud. Our main result here is that when each bidder follows his maxmin perfect bidding strategy, then each bidder obtains his Shapley value allocation. The allocation that obtains is thus independent of the cost sharing rule.

The last approach is to study the Bayes Nash equilibria of compensated position auctions when bidders are privately informed of their preference intensities. We provide general necessary conditions for a bidding strategy to form a symmetric equilibrium in increasing and differentiable strategies. We give closed-form solutions for the unique such equilibrium when bidders are risk neutral and when they are CARA risk averse. We show that bidders demand less compensation as they become more risk averse. Our main result here is that the equilibrium bidding strategy of CARA risk averse bidders converges uniformly to the maxmin perfect bidding strategy as bidders become infinitely risk averse. An immediate consequence of this result, and our earlier result that maxmin bidding yields the Shapley allocation, is that the equilibrium allocation of a compensated position auction coincides with the Shapley value allocation as bidders become infinitely risk averse. To our knowledge, this paper is the first to provide non-cooperative foundations for the Shapley value in a setting with incomplete information.

RELATED LITERATURE

Our paper connects to several literatures in cooperative and non-cooperative game theory.

The Assignment Problem: The problem of allocating positions is the assignment problem for the special case where all the players rank assignments in the same way, as is natural for example when assignments correspond to priorities, e.g., first priority, second priority, etc. Both cooperative and non-cooperative solutions to the general assignment problem have been studied. Moulin (1992) shows that the Shapley value has several desirable properties in cooperative models of assignment games.[1]

Early examples of non-cooperative approaches to the assignment problem include Leonard (1983) and Demange, Gale, Sotomayor (1986). Leonard (1983) provides a mechanism for which it is a dominant strategy for each player to report his preferences over assignments truthfully and which implements the efficient assignment; he shows it generates Vickrey-Clark-Groves prices. Demange, Gale, Sotomayor (1986) provide a dynamic auction which implements the efficient assignment. In the context of internet advertising,

[1]The Shapley value is not the only notion of fairness for assignment games. Alkan, Demange, and Gale (1991), for example, study existence of efficient and envy-free allocations in the assignment problem.

important papers by Edelman, Ostrovsky, and Schwarz (2007) and Varian (2007) study the use of the generalized second-price sealed-bid auction to allocate positions under complete information. Edelman, Ostrovsky, and Schwarz (2007) study, in addition, a generalized English auction with incomplete information and show that payoffs (both to bidders and to the seller) are the same as in the Vickrey-Clarke-Groves mechanism.

In all these papers, the seller collects the auction revenue. We study, in contrast, a setting where there is no seller and the only payments are transfers between the bidders. Budget balancedness is a fundamental requirement since the positions are the common property of the bidders.

Non-cooperative Foundations of the Shapley Value: In bargaining games with complete information, non-cooperative foundations of the Shapley value have been provided by Gul (1989) and Hart and Mas Colell (1996). Gul (1989) provides a game with bilateral bargaining and the random selection of the proposer and shows that, in the efficient equilibrium of the game, players receive their Shapley value payoffs in the limit as they become perfectly patient. Hart and Mas-Colell (1996) studies a multilateral bargaining game and shows that players receive their Shapley value payoffs in the limit as each player's probability of exogenously exiting from bargaining vanishes. By contrast, we obtain Shapley value payoffs as bidders become infinitely risk averse in an environment with incomplete information.

Bidding Rings, Bankruptcy, and Cost Sharing: The Shapley value also appears in the literature on collusion in auctions. Graham, Marshall, and Richard (1990) shows that bidders receive their Shapley value payoffs in a nested knockout auction when bidding rings are perfectly nested. In their setting, the bidders' values for the item are commonly known and bidders are assumed to remain active in the knockout auction until the bid reaches their value (although this is not equilibrium behavior).[2]

Aumann and Maschler (1985) shows that the solutions provided in the Talmud to three different bankruptcy problems coincide with the nucleoli of the corresponding cooperative games. These solutions are generalizations of the solution to the contested garment problem: "Two hold a garment;

[2]Littlechild and Owen (1973) obtains the same payoffs when allocating costs to the users of an airport runway.

one claims it all, the other claims half. Then the one is awarded three-fourths, the other one-fourth." In this solution, the lesser claimant concedes the uncontested half the garment to the greater one, and the remainder is split equally. In compensated position auctions, at each round all but the worst remaining position are contested. We show that the maxmin perfect bid at each round can be interpreted as a demand for equal shares of the incremental benefits of the contested positions, and in this respect resembles the solution to the contested garment problem.

Compensated position auctions are reminiscent of serial cost sharing. Moulin and Shenker (1992) studies the problem of allocating costs when agents face a production technology with decreasing returns to scale. It proposes a cost sharing rule in which participants pay equal shares of incremental costs (defined in a particular way) and show that, given this rule, the game in which the participants announce quantities is dominance solvable and equilibrium has several nice properties. The cost sharing rule is a primitive, part of the description of the game, whereas here the surplus shares are endogenously determined. In our setting, equilibrium demands for compensation can be interpreted as (inflated) demands for equal shares of the incremental benefits of contested positions.

Cake Cutting and Dissolving Partnerships: Although we are concerned with the allocation of indivisible heterogenous positions, the class of auctions we study is inspired by the Dubins and Spanier (1961) moving knife algorithm for the fair division of a divisible cake. In the fair division problem there are N participants, each of whom wants cake. To divide the cake, a third party moves a knife across the cake until some participant cries "stop." The participant crying stop receives the cake to the left of the knife and exits, surrendering his claim to any additional cake. The process then continues with the remaining participants and cake, repeating until the whole cake is divided. In our auction, a participant whose demand for compensation is smallest receives the worst remaining position and compensation equal to his demand, while surrendering his claim to better positions.

Dividing a cake is analogous to dissolving a partnership. McAfee (1992) examines the Texas Shootout, a version of divide and choose, for dissolving two-person partnerships. Van Essen and Wooders (2016) studies a dynamic compensation auction for dissolving N-person partnerships. Van Essen and Wooders (2018) studies dual auctions for the dual problems of allocating homogeneous goods or chores, and relates the two. The present paper studies the problem of allocating heterogenous positions.

The rest of the paper proceeds as follows: We provide in Section 2 a description of the position allocation problem and we identify the Shapley allocation of the associated cooperative game. Section 3 introduces compensated position auctions. For any given cost sharing rule, Section 4 identifies the maxmin perfect bidding strategy and shows that bidders obtain their Shapley value allocations when every bidder follows the maxmin perfect bidding strategy. Our equilibrium results for the Bayesian game when information is incomplete are in Section 5. Section 6 relates the Shapley value, maxmin, and equilibrium allocations. We conclude with a discussion in Section 7. All proofs are in the Appendix.

2 Shapley Value

$N \geq 2$ positions are to be allocated to N players who have equal claims, with one position to be assigned to each player.[3] The positions have inherent values, denoted by $\alpha_1, ..., \alpha_N$, which are commonly known. We order the positions so that $\alpha_1 \geq ... \geq \alpha_N$. Positions may be desirable or undesirable, i.e., we allow a mixture of positive and negative α's.[4] The payoff to a player whose preference intensity is x, and who receives position i, is $\alpha_i x$ plus any net transfer he receives. Hereafter, we will refer to a player's preference intensity as his *value*. In this section it is convenient to order the players so that $x_1 \geq ... \geq x_N$. The problem is to efficiently and fairly allocate positions to players, while respecting budget balance.

Cooperative game theory provides one solution: allocate positions to maximize surplus and make transfers among the players so that each player ob-

[3]This is without loss of generality since, if there are more players than positions, one can create dummy positions, with α's equal to zero, until the number of positions equals the number of players.

[4]Bogomolnaia, Moulin, Sandomirskiy, and Yanovskaya (2017) study a fair division problem when the goods to be divided are a mixture of both goods and bads.

tains his Shapley value payoff. The Shapley solution is appealing since it the only solution satisfying (i) efficiency, (ii) additivity, (iii) symmetry, and (iv) no surplus to dummy players. For a general characteristic function v, the Shapley value ϕ_i of player i is

$$\phi_i = \sum_{S \subseteq \{1,...,N\}} \frac{(|S|-1)!(N-|S|)!}{N!} \left[v(S) - v(S \backslash \{i\}) \right],$$

where $v(S)$ gives the value of coalition S. Player i's Shapley value can be interpreted as his expected marginal contribution when the grand coalition is formed by adding players, one at a time, in a random order.

We now describe the Shapley solution to the position allocation problem. For any coalition $S \in 2^N$, let $y_1^{(S)}, ..., y_{|S|}^{(S)}$ be a rearrangement of the values $\{x_i | i \in S\}$ of the members of S such that $y_1^{(S)} \geq ... \geq y_{|S|}^{(S)}$. Surplus is maximized by assigning players with lower indexes to positions with lower indexes. Following Moulin (1992), the characteristic function

$$v(S) = \sum_{j=1}^{|S|} \alpha_j y_j^{(S)}$$

defines the cooperative game.

Proposition 1 characterizes Shapley values for the position allocation problem.

Proposition 1: *The Shapley value ϕ_i of player i in the position allocation problem is*

$$\phi_i = \frac{1}{i} \left(\sum_{m=1}^{i} \alpha_m \right) x_i - \sum_{m=1}^{N-i} \frac{1}{i+m-1} \left[\sum_{r=1}^{i+m-1} \frac{r}{i+m} \left(\alpha_r - \alpha_{r+1} \right) x_{i+m} \right].$$

Following the interpretation of a player's Shapley value as being his expected marginal contribution, the first term in the expression for ϕ_i is the expected gross contribution of player i, while the second term is the expected negative externality that i imposes on players who, as a result of i joining the coalition, receive worse positions.

Example 1 provides the Shapley values for the $N = 3$ problem.

Example 1: Suppose $N = 3$ and $x_1 > x_2 > x_3$. The players' Shapley values are:

$$\begin{aligned}
\phi_1 &= \alpha_1 x_1 - \frac{1}{2}(\alpha_1 - \alpha_2) x_2 - \frac{1}{6}(\alpha_1 - \alpha_2) x_3 - \frac{1}{3}(\alpha_2 - \alpha_3) x_{3,} \\
\phi_2 &= \frac{1}{2}(\alpha_1 + \alpha_2) x_2 - \frac{1}{6}(\alpha_1 - \alpha_2) x_3 - \frac{1}{3}(\alpha_2 - \alpha_3) x_3, \\
\phi_3 &= \frac{1}{3}(\alpha_1 + \alpha_2 + \alpha_3) x_3.
\end{aligned}$$

If $\alpha_1 = 6$, $\alpha_2 = 4$, and $\alpha_1 = 2$, and $x_1 = 3/4$, $x_2 = 1/2$, and $x_3 = 1/4$, then $\phi_1 = 15/4$, $\phi_2 = 9/4$, and $\phi_3 = 1$. In the Shapley allocation, player i receives position i. Players 1, 2, and 3, receive transfers of $-3/4$, $1/4$, and $1/2$, respectively.

In addition to its axiomatic foundation as fair, the Shapley value solution is fair in the sense that it belongs to the "anti-core." The notion of the anti-core describes the minimally fair payoffs of a cooperative game. Formally, given a characteristic function v a payoff vector $(\pi_1, \ldots, \pi_N)$ is in the *anti-core* if (i) $\sum_{i \in N} \pi_i = v(N)$ and (ii) for every coalition $S \subset N$ we have that $\sum_{i \in S} \pi_i \leq v(S)$. In other words, a payoff vector that divides the surplus is in the anti-core if no coalition S of players receives more than $v(S)$, what it could obtain if the coalition had complete command over the allocation of resources. A payoff to a coalition S that exceeded $v(S)$ would require a subsidy from the members of $N \backslash S$, who would object on fairness grounds.[5]

As shown in the following Corollary, a feature of the problem we study is that Shapley values can be computed recursively, starting with player N and working backwards.

Corollary 1: *The Shapley value payoffs can be written as* $\phi_i = \alpha_i x_i + \tau_i$ *for*

[5]The anti-core of a cooperative game is motivated by normative/fairness considerations, in contrast to the core which is motivated by strategic considerations. See Moulin (1995, Chapter 7).

$i = 1, \ldots, N$, *where the transfers* $\tau_1, \ldots, \tau_N$ *are defined recursively as*

$$\begin{aligned} \tau_N &= \frac{1}{N}(\alpha_1 + \cdots + \alpha_N)\, x_N - \alpha_N x_N \\ &\vdots \\ \tau_k &= \frac{1}{k}\left[(\alpha_1 + \cdots + \alpha_k)\, x_k - \sum_{i=k+1}^{N} \tau_i\right] - \alpha_k x_k \\ &\vdots \\ \tau_1 &= -\tau_2 - \tau_3 - \cdots - \tau_N. \end{aligned}$$

When written as in Corollary 1, the Shapley value has a natural dynamic interpretation. Player N's Shapley value is an equal share of the surplus as he values it, i.e.,

$$\phi_N = \frac{1}{N}(\alpha_1 + \cdots + \alpha_N)\, x_N.$$

Player N's Shapley allocation is position N and transfer $\tau_N = \phi_N - \alpha_N x_N$. Following the transfer τ_N to Player N, the residual surplus is

$$(\alpha_1 + \cdots + \alpha_{N-1})\, x_{N-1} - \tau_N$$

as Player $N-1$ values it, and Player $N-1$'s Shapley value is an equal share of this among the $N-1$ remaining players

$$\phi_{N-1} = \frac{1}{N-1}\left[(\alpha_1 + \cdots + \alpha_{N-1})\, x_{N-1} - \tau_N\right].$$

In general, given transfers τ_N, ..., τ_{k+1}, the Shapley value of the player with the k-th lowest value,

$$\phi_k = \alpha_k x_k + \tau_k = \frac{1}{k}\left[(\alpha_1 + \cdots + \alpha_k)\, x_k - \sum_{i=k+1}^{N} \tau_i\right],$$

is an equal share, among the k remaining players, of the residual surplus as he values it.

The players' Shapley values can also be interpreted as equal shares of "worst case" residual surpluses. Since player N has the lowest value, ϕ_N is Player N's evaluation of the worst-case surplus in the sense that surplus is minimized were all the other players to also have the same value x_N. After Player N has exited, the term ϕ_{N-1} can likewise be viewed as Player $N-1$'s evaluation of an equal share of the worst-case residual surplus, and so on.

The interpretation of Shapley payoffs as equal shares of worst-case residual surplus suggests a connection to maxmin play. In particular, dynamic mechanisms with the property that the maxmin payoff of the active player with the lowest value is an equal share of the residual surplus are natural candidates to implement Shapley allocations under maxmin play. The next section studies a class of auctions with this property.

We conclude this section by noting an attractive feature of the Shapley transfers: they are "top-down" in the sense that, for any k, the players allocated the k best positions pay compensation (in aggregate) to the players receiving worse positions. To see this, note that Moulin (1992, Theorem 2) established that the general assignment game is concave, and thus the problem of assigning players to positions is also concave. It follows, by Shapley (1971, Theorem 7), that the anti-core of the game contains the Shapley value. Hence, for the players with the k highest values we have

$$\sum_{i=1}^{k}(\alpha_i x_i + \tau_i) = \sum_{i=1}^{k}\phi_i \leq v(\{1,\ldots,k\}) = \sum_{i=1}^{k}\alpha_i x_i,$$

and thus $\sum_{i=1}^{k}\tau_i \leq 0$, i.e., the aggregate transfer to the players allocated the k best positions must be non-positive.

3 Compensated Position Auctions

We now describe a class of auctions for solving the position allocation problem. These auctions takes place over $N-1$ rounds, where at each round the worst unallocated position is auctioned. At each round t, the active bidders simultaneously submit (possibly negative) demands for compensation. The bidder with the smallest demand receives position $N-t+1$ and compensation d_t and exits the auction. The compensation d_t is paid by the $N-t$ bidders that have not yet received positions according to a linear cost sharing rule

$$\lambda^t = (\lambda_1^t, \lambda_2^t, ..., \lambda_{N-t}^t),$$

where λ_i^t is the proportion of d_t paid by the bidder who ultimately receives position $i \leq N-t$. We require that for each $t = 1, \ldots, N-1$ that $\sum_{i=1}^{N-t} \lambda_i^t = 1$.[6]

Bidders have a common utility function u, where $u' > 0$ and $u'' \leq 0$. A bidder with value x who has the smallest demand d_t at round t, is allocated position $N-t+1$ and obtains a payoff of

$$u\left(\alpha_{N-t+1}x + d_t - \sum_{i=1}^{t-1} \lambda_{N-t+1}^i d_i\right),$$

where $\alpha_{N-t+1}x$ is the payoff from his position, d_t is the compensation he receives, and

$$\sum_{i=1}^{t-1} \lambda_{N-t+1}^i d_i$$

is the compensation he pays to bidders who exited at prior rounds. In particular, $\lambda_{N-t+1}^i d_i$ is the compensation he pays to the bidder who exited in round i. At round $N-1$, the final round, the bidder with the largest demand receives the best position, pays compensation to every other bidder, and obtains a payoff of

$$u\left(\alpha_{N-t+1}x - \sum_{i=1}^{N-1} \lambda_1^i d_i\right).$$

In sum, a bidder who submits the smallest demand surrenders his claim to more desirable positions and receives compensation from the bidders who maintain their claims to these positions, while he pays compensation to bidders who have accepted less desirable positions.

Each bidder knows his own value, but not the values of the other bidders, and observes the smallest demand at each round. A *strategy* is a list of $N-1$ functions $\beta = (\beta_1, ..., \beta_{N-1})$, where $\beta_t(x; d_1, ..., d_{t-1})$ gives the demand in the t-th round of a bidder whose value is x, where $d_1, \ldots, d_{t-1}$ are the smallest demands in previous rounds. We write $\mathbf{d}_{t-1}$ for $(d_1, \ldots, d_{t-1})$.

Different cost sharing rules define different auctions. If the compensation of a bidder who exits the auction at round t is paid entirely by the bidder

[6] The λ_i^t's need not be positive. It is, however, important that only bidders who have not received positions are liable for d_t, i.e., $\lambda_{N-t+1}^t = \ldots = \lambda_N^t = 0$.

who exits at round $t+1$ then $\lambda^t_{N-t} = 1$ and $\lambda^t_j = 0$ for $j = 1, ..., N-t-1$. If the compensation is paid in equal shares by the bidders obtaining better positions then $\lambda^t_1 = ... = \lambda^t_{N-t} = 1/(N-t)$.

4 Maxmin

We first take a decision theoretic approach to bidding in compensated position auctions and ask what payoff a bidder can guarantee himself, i.e., his maxmin payoff. While a compensated position auction will have many maxmin strategies, since the auction is dynamic we focus on "maxmin perfect" strategies which maximize a bidder's minimum payoff at each point in the auction. In this section we define the notion of a maxmin perfect bidding strategy and show that any compensated position auction has a unique such strategy. Our main result is that every bidder obtains his Shapley allocation when each bidder follows the maxmin bidding strategy.

For a bidder who remains in the auction at round t, let $v_t(x_i, x_{-i}, \beta^i, \beta^{-i}; \mathbf{d}_{t-1})$ be the bidder's payoff when his value is x_i and he follows the strategy β^i, when x_{-i} and β^{-i} are the values and strategies of the remaining bidders, and $\mathbf{d}_{t-1}$ is the sequence of smallest demands at the prior rounds.

Definition: A strategy β^i *guarantees bidder* i *with value* x_i *a payoff of* $\bar{v}_t$ *at round* t*, given* $\mathbf{d}_{t-1}$, if $v_t(x_i, x_{-i}, \beta^i, \beta^{-i}; \mathbf{d}_{t-1}) \geq \bar{v}_t \ \forall x_{-i}, \beta^{-i}$.

Let $\bar{v}_t(x_i; \mathbf{d}_{t-1})$ be the largest payoff that bidder i with value x_i can guarantee at round t given $\mathbf{d}_{t-1}$.

Definition: A strategy β^i is a *maxmin perfect strategy for bidder* i if β^i guarantees $\bar{v}_t(x_i; \mathbf{d}_{t-1})$ for each t, $x_i \in [0, \bar{x}]$, and $\mathbf{d}_{t-1}$.

Proposition 2 identifies the unique maxmin perfect strategy and the associated value for the compensated position auction with cost sharing rule $\lambda = (\lambda^1, \ldots, \lambda^{N-1})$.

Proposition 2: *Let* $\lambda = (\lambda^1, \ldots, \lambda^{N-1})$ *be a cost sharing rule. The strategy* *profile* $\underline{\beta} = (\underline{\beta}_1, \ldots, \underline{\beta}_{N-1})$ *given by*

$$\underline{\beta}_t(x; \mathbf{d}_{t-1}) = \sum_{m=1}^{N-t} \frac{m}{N-t+1}(\alpha_m - \alpha_{m+1})x - \sum_{i=1}^{t-1}\left[\sum_{m=1}^{N-t} \frac{m}{N-t+1}(\lambda_m^i - \lambda_{m+1}^i)d_i\right]$$

for each round t *is the unique maxmin perfect strategy of the compensated position auction with cost sharing rule* λ*. In particular,* $\underline{\beta}$ *guarantees a bidder with value* x*, who is active at round* t*, a payoff of*

$$\bar{v}_t(x; \mathbf{d}_{t-1}) = \left(\frac{\alpha_1 + \cdots + \alpha_{N-t+1}}{N-t+1}\right)x - \sum_{i=1}^{t-1}\left[\frac{\lambda_1^i + \cdots + \lambda_{N-t+1}^i}{N-t+1}\right]d_i,$$

when $\mathbf{d}_{t-1}$ *is the sequence of smallest demands from prior rounds.*

Example 2: If demands are paid equally by players obtaining better positions, then $\lambda_m^i - \lambda_{m+1}^i = 0$ for $m = 1, \ldots, N-t$ and the maxmin perfect bid function in Proposition 2 simplifies to

$$\underline{\beta}_t(x; \mathbf{d}_{t-1}) = \sum_{m=1}^{N-t} \frac{m}{N-t+1}(\alpha_m - \alpha_{m+1})x,$$

and it guarantees

$$\bar{v}_t(x; \mathbf{d}_{t-1}) = \left(\frac{\alpha_1 + \cdots + \alpha_{N-t+1}}{N-t+1}\right)x - \sum_{i=1}^{t-1} \frac{1}{N-i}d_i.$$

The maxmin perfect strategy $\underline{\beta}$ has a natural fairness interpretation akin to the solution to the contested garment problem. Recall that the solution to the contest garment problem called for giving each participant the uncontested portion of his demand and splitting the contested portion equally. The maxmin bid function likewise calls for a bidder to demand an equal share of the incremental benefits of contested positions.

Consider the bid function in Example 2. At round 1, positions 1 though $N-1$ are contested and position N, the worst position, is uncontested. There are $N-1$ bidders who will be allocated position $N-1$ or better and who will each enjoy an incremental benefit of $\alpha_{N-1} - \alpha_N$ times their value. A bidder i with value x demands an equal share, $1/N$-th, of this total benefit as he himself values it, i.e., he demands $\frac{N-1}{N}(\alpha_{N-1} - \alpha_N)\,x$. There are $N-2$ bidders who will obtain position $N-2$ or better and who will each enjoy an incremental benefit of $\alpha_{N-2} - \alpha_{N-1}$ times their value. Bidder i demands an equal share of this total benefit too, i.e., he demands $\frac{N-2}{N}(\alpha_{N-2} - \alpha_{N-1})\,x$. Continuing in this fashion, one bidder will obtain position 1 and enjoy an incremental benefit of $\alpha_1 - \alpha_2$ times his value. Bidder i demands an equal share. Adding up these shares of incremental benefits for the contested positions yields $\underline{\beta}_1(x)$, bidder i's demand for compensation at round 1, as

$$\frac{1}{N}(\alpha_1 - \alpha_2)\,x + \cdots + \frac{N-2}{N}(\alpha_{N-2} - \alpha_{N-1})\,x + \frac{N-1}{N}(\alpha_{N-1} - \alpha_N)\,x.$$

The round t maxmin bid function $\underline{\beta}_t$ has an interpretation analogous to $\underline{\beta}_1$, where equal shares are relative to the $N-t+1$ bidders and unallocated positions remaining in the auction.

For general cost sharing rules, to make the interpretation of the bid function more transparent, we can rewrite the expression for $\underline{\beta}_t(x; \mathbf{d}_{t-1})$ as

$$\underline{\beta}_t(x; \mathbf{d}_{t-1}) = \sum_{m=1}^{N-t} \frac{m}{N-t+1} \left[(\alpha_m - \alpha_{m+1})x - \sum_{i=1}^{t-1} (\lambda_m^i - \lambda_{m+1}^i) d_i \right].$$

Written this way, one can see that bidders demand in round t an equal share of incremental "net" benefits of contested positions, where positions differ in both in their inherent value and in their liabilities for compensation.

Proposition 3, which follows, provides the decision theoretic foundation of the Shapley value in compensated position auctions. If all bidders follow the maxmin perfect bidding strategy, then payoffs are the Shapley value payoffs. Surprisingly, this result is independent of the cost sharing rule.

Proposition 3: *If each bidder follows the maxmin perfect bidding strategy, then each bidder obtains his Shapley value.*

The following example illustrates Proposition 3 and the irrelevance of the cost sharing rule when there are three bidders.

Example 3: Suppose, as in Example 1 that $x_1 > x_2 > x_3$. By Proposition 2 the maxmin perfect bidding strategy is

$$\begin{aligned}
\underline{\beta}_1(x) &= \frac{1}{3}(\alpha_1 - \alpha_2)\,x + \frac{2}{3}(\alpha_2 - \alpha_3)\,x \\
\underline{\beta}_2(x; d_1) &= \frac{1}{2}(\alpha_1 - \alpha_2)\,x - \frac{1}{2}(\lambda_1^1 - \lambda_2^1)d_1.
\end{aligned}$$

Since $\lambda_1^1 + \lambda_2^1 = 1$ we can write

$$\underline{\beta}_2(x; d_1) = \frac{1}{2}(\alpha_1 - \alpha_2)\,x + \left[\lambda_2^1 - \frac{1}{2}\right] d_1.$$

Bidder 3's round 1 demand $d_1 = \underline{\beta}_1(x_3)$ is smallest, he wins position 3, and he exits after receiving compensation $\underline{\beta}_1(x_3)$. His payoff is

$$\alpha_3 x_3 + \underline{\beta}_1(x_3) = \frac{1}{3}(\alpha_1 + \alpha_2 + \alpha_3)\,x_3 = \phi_3.$$

Bidder 2's demand $\underline{\beta}_2(x_2; d_1)$ is smallest at round 2, he wins position 2, and he exits after receiving compensation $\underline{\beta}_2(x_2; d_1)$ and paying compensation $\lambda_2^1 d_1 = \lambda_2^1 \underline{\beta}_1(x_3)$. His payoff is

$$\begin{aligned}
& \alpha_2 x_2 + \underline{\beta}_2(x_2; d_1) - \lambda_2^1 d_1 \\
= & \frac{1}{2}(\alpha_1 + \alpha_2)\,x_2 - \frac{1}{6}(\alpha_1 - \alpha_2)\,x_3 - \frac{1}{3}(\alpha_2 - \alpha_3)\,x_3 \\
= & \phi_2.
\end{aligned}$$

Bidder 1 wins position 1 and pays total compensation of $d_2 + (1 - \lambda_2^1)d_1 = \underline{\beta}_2(x_2; d_1) + (1 - \lambda_2^1)\underline{\beta}_1(x_3)$. His payoff is

$$\begin{aligned}
& \alpha_1 x_1 - d_2 - (1 - \lambda_2^1)d_1 \\
= & \alpha_1 x_1 - \frac{1}{2}(\alpha_1 - \alpha_2)\,x_2 - \frac{1}{6}(\alpha_1 - \alpha_2)\,x_3 - \frac{1}{3}(\alpha_2 - \alpha_3)\,x_3 \\
= & \phi_1.
\end{aligned}$$

Thus, each bidder receives his Shapley value.

Intuitively, whatever the cost sharing rule, maxmin bids adjust so that bidders allocated better positions ultimately pay equal shares of the surplus received by bidder allocated worse positions. In Example 3, this is evident from the term $\left[\lambda_2^1 - \frac{1}{2}\right] d_1$ that appears in the bid function $\underline{\beta}_2(x; d_1)$. If λ_2^1 exceeds 1/2, then round 2 demands increase by an amount that exactly offsets the compensation that the winner of position 2 pays in excess of an equal share of Bidder 1's demand.[7]

5 Equilibrium

In this section we study compensated position auctions in a standard independent private values setting. To reduce notation we focus on the compensated position auction in which demands are paid equally by bidders obtaining better positions, i.e., we assume at each round t that $\lambda_j^t = \frac{1}{N-t}$ for $j = 1, \ldots, N-t$.[8] Propositions 4 through 7 are stated for this cost sharing rule. Importantly, Proposition 8, which shows that the equilibrium bid function converges uniformly to the maximin bid function as bidders become infinitely risk averse, holds for any cost sharing rule.

Independent Private Values

The bidders' preference intensities (hereafter "values") are independently and identically distributed according to cumulative distribution function F with support $[0, \bar{x}]$, where $\bar{x} < \infty$ and $f \equiv F'$ is continuous and positive on $[0, \bar{x}]$. Let $X_1, \ldots, X_N$ be N independent draws from F. Let $Z_1^{(N)}, \ldots, Z_N^{(N)}$ be a rearrangement of the X_i's such that $Z_1^{(N)} \leq Z_2^{(N)} \leq \ldots \leq Z_N^{(N)}$. The joint density of $Z_1^{(N)}, \ldots, Z_N^{(N)}$ is

$$g_{1,\ldots,N}^{(N)}(z_1, \ldots, z_N) = N! \prod\nolimits_{i=1}^{N} f(z_i)$$

for $z_1 \leq z_2 \leq \ldots \leq z_N$ and $g_{1,\ldots,N}^{(N)}(z_1, \ldots, z_N) = 0$ otherwise. Let $G_t^{(N)}$ denote the *c.d.f.* of $Z_t^{(N)}$, i.e., $G_t^{(N)}$ is the distribution of the t-th lowest of N draws.

[7] Since it is not apparent in $N = 3$ example, we note that for general N if a bidder following the maxmin perfect bidding strategy is allocated position i (at round $N - i + 1$), then his payoff is independent of λ_i^{N-i} (the share of the cost d_{N-i} that he pays) but not $\lambda_i^1, \ldots, \lambda_i^{N-i-1}$.

[8] When $\lambda_j^t = \frac{1}{N-t}$ for $j = 1, \ldots, N-t$, one can show that equilibrium demands are always positive. The auction can then equivalently be framed as one in which at each round the bid ascends from zero. The first bidder to drop receives the worst unallocated position and receives compensation equal to the price at which he drops.

The conditional density of $Z_{t+1}^{(N)}$ given $Z_1^{(N)} = z_1, \ldots, Z_t^{(N)} = z_t$ is

$$g_{t+1}^{(N)}(z_{t+1}|z_t) = (N-t)f(z_{t+1})\frac{[1-F(z_{t+1})]^{N-(t+1)}}{[1-F(z_t)]^{N-t}}$$

if $0 \leq z_1 \leq \ldots \leq z_{t+1}$ and is zero otherwise. Define

$$\Gamma_t^N(z) \equiv g_{t+1}^{(N)}(z|z) = (N-t)\frac{f(z)}{1-F(z)}$$

to be the hazard function.

Necessary Conditions for Equilibrium

Proposition 4 provides necessary conditions for β to be a symmetric equilibrium in strictly increasing and differentiable bidding strategies. These conditions are also sufficient for risk neutral and CARA bidders, as we establish in Propositions 5 and 6.

Proposition 4: *Any symmetric equilibrium β in increasing and differentiable bidding strategies satisfies the following system of $N-1$ differential equations:*

$$u'\left(\alpha_{N-t+1}x + \beta_t(x;\mathbf{d}_{t-1}) - \sum_{j=1}^{t-1}\frac{d_j}{N-j}\right)\beta_t'(x;\mathbf{d}_{t-1})$$
$$= -\left[\begin{array}{c} u\left(\alpha_{N-t}x + \beta_{t+1}(x;\mathbf{d}_{t-1},\beta_t(x;\mathbf{d}_{t-1})) - \frac{1}{N-t}\beta_t(x;\mathbf{d}_{t-1}) - \sum_{j=1}^{t-1}\frac{d_j}{N-j}\right) \\ -u\left(\alpha_{N-t+1}x + \beta_t(x;\mathbf{d}_{t-1}) - \sum_{j=1}^{t-1}\frac{d_j}{N-j}\right)\end{array}\right]\Gamma_t^N(x),$$

for each $t \in \{1, \ldots, N-1\}$ where $\beta_N(x;d_{N-1}) \equiv 0$.

Risk Neutral Bidders

Proposition 5 identifies the equilibrium when bidders are risk neutral. We write β_t^0 for the equilibrium bid function.

Proposition 5: *Suppose that bidders are risk neutral. The unique symmetric equilibrium in increasing and differentiable strategies is given, for $t = 1, \ldots, N-1$, by*

$$\beta_t^0(x) = \frac{N-t}{N-t+1}E\left[\left(\alpha_{N-t} - \alpha_{N-t+1}\right)Z_t^{(N)} + \beta_{t+1}^0(Z_t^{(N)})|Z_t^{(N)} > x > Z_{t-1}^{(N)}\right]$$

where $\beta^0_N \equiv 0$. *Equivalently, it is given by*

$$\beta^0_t(x) = \sum_{m=1}^{N-t} \frac{m}{N-t+1} E\left[(\alpha_m - \alpha_{m+1})\, Z^{(N)}_{N-m} | Z^{(N)}_t > x > Z^{(N)}_{t-1}\right].$$

Equilibrium demands at each round are independent of the smallest demands at prior rounds.

Observe from the second expression for β^0_t that if at some round t all the remaining positions have the same α's, i.e., $\alpha_1 = \ldots = \alpha_{N-t+1}$, then bids are zero at round t and every subsequent round. This is intuitive since when the remaining positions are identical and the number of positions is equal to the number of remaining bidders, then no position is contested.

The risk neutral bid function β^0_t, given in Proposition 5, has a similar form and interpretation to the maxmin perfect bid function $\underline{\beta}_t$ when demands are paid equally by players obtaining better positions. At round $t > 1$, positions are $1, \ldots, N-t$ are contested. The m-th term of $\underline{\beta}_t$,

$$\frac{m}{N-t+1} (\alpha_m - \alpha_{m+1})\, x,$$

is an equal share (among the $N-t+1$ active bidders at round t) of the total benefit obtained by the m bidders allocated position m or better, as a bidder with value x values it. The m-th term of β^0_t,

$$\frac{m}{N-t+1} (\alpha_m - \alpha_{m+1})\, E\left[Z^{(N)}_{N-m} | Z^{(N)}_t > x > Z^{(N)}_{t-1}\right],$$

is the same except that it is based on a value that is inflated relative to the bidder's true value. Risk neutral bidders demand more compensation, increasing their expected payoff at the expense of reducing their worst-case payoff.

Example 4: If $N = 3$ and values are distributed $U[0,1]$, then the equilibrium bid functions for risk neutral bidders are

$$\beta^0_1(x) = (\alpha_1 - \alpha_2)\left(\frac{1}{6}x + \frac{1}{6}\right) + (\alpha_2 - \alpha_3)\left(\frac{1}{2}x + \frac{1}{6}\right)$$

and

$$\beta^0_2(x) = (\alpha_1 - \alpha_2)\left(\frac{1}{3}x + \frac{1}{6}\right).$$

CARA Bidders

The next proposition characterizes equilibrium when bidders have constant absolute risk aversion (CARA), i.e., utility is given by

$$u^{\theta}(x) = \frac{1 - e^{-\theta x}}{\theta},$$

where $\theta > 0$ is the common index of risk aversion. Note that $\lim_{\theta \to 0} u^{\theta}(x) = x$, i.e., bidders are risk neutral in the limit as θ approaches zero. Denote by β_t^{θ} the equilibrium bid function in round t when bidders have CARA index of risk aversion θ.

Proposition 6: *Suppose that bidders are CARA risk averse with index of risk aversion $\theta > 0$. The unique symmetric equilibrium in increasing and differentiable strategies is given recursively, for $t = 1, \ldots, N-1$, by*

$$\beta_t^{\theta}(x) = -\frac{N-t}{(N-t+1)\,\theta} \ln \left\{ E\left[e^{-\theta[(\alpha_{N-t} - \alpha_{N-t+1}) Z_t^{(N)} + \beta_{t+1}^{\theta}(Z_t^{(N)})]} | Z_t^{(N)} > x > Z_{t-1}^{(N)} \right] \right\}$$

where $\beta_N^{\theta} \equiv 0$. Equilibrium bids at each round are independent of prior smallest demands.

Example 5: If $N = 3$ and values are distributed $U[0,1]$, then the equilibrium bid functions for CARA risk averse bidders are

$$\beta_1^{\theta}(x) = -\frac{2}{3\theta} \ln \left\{ \frac{\int_x^1 e^{-\theta\left[(\alpha_2 - \alpha_3) z + \beta_2^{\theta}(z)\right]} 3(1-z)^2 dz}{(1-x)^3} \right\}$$

and

$$\beta_2^{\theta}(x) = -\frac{1}{2\theta} \ln \left\{ \frac{\int_x^1 e^{-\theta(\alpha_1 - \alpha_2) z} 2(1-z) dz}{(1-x)^2} \right\}.$$

Bounds and Comparative Statics

Proposition 7 provides upper and lower bounds for the CARA equilibrium bid functions. The risk neutral bid function β_t^0 is an upper bound for the equilibrium bid function of a CARA risk averse bidder while $\underline{\beta}_t$ is a lower bound: a risk averse bidder submits a smaller demand, and thus accepts less compensation, than were he risk neutral, but submits a larger demand than his maxmin perfect demand.

Proposition 7: *Suppose that bidders are CARA risk averse with index of risk aversion $\theta > 0$ and $\alpha_1 > \alpha_2$. Then for each $t = 1, \ldots, N-1$ we have that*

$$\underline{\beta}_t(x) < \beta_t^\theta(x) < \beta_t^0(x) \text{ for } x < \bar{x},$$

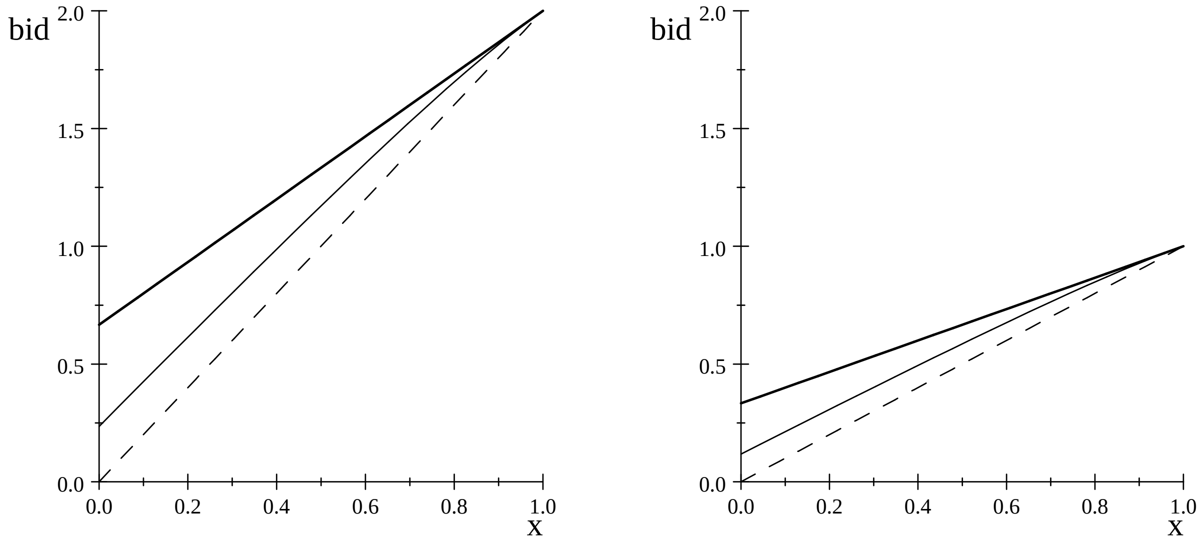
where

$$\underline{\beta}_t(x) = \sum_{m=1}^{N-t} \frac{m}{N-t+1} (\alpha_m - \alpha_{m+1}) x.$$

Proposition 8 shows that demands decrease as bidders become more risk averse, and that demands converge uniformly to the maxmin demands as bidders become infinitely risk averse.

Proposition 8: *Suppose that bidders are CARA risk averse with index of risk aversion $\theta > 0$. Then for each $t = 1, \ldots, N-1$ we have that $\beta_t^\theta(x)$ is decreasing in θ, and β_t^θ converges uniformly to $\underline{\beta}_t$ on $[0, \bar{x}]$ as $\theta \to \infty$.*

Figure 1 illustrates propositions 7 and 8 when $N = 3$, values are distributed $U[0,1]$, and $\alpha_1 = 6$, $\alpha_2 = 4$, and $\alpha_3 = 2$. In the figure, the bold solid lines are the risk neutral bid functions (i.e., $\theta = 0$) for rounds 1 and 2, which are the upper bounds for CARA risk averse bidders. The dashed lines give $\underline{\beta}_1$ and $\underline{\beta}_2$, the maximin bid functions. The thin solid lines are the bid functions when bidders have CARA index of risk aversion of $\theta = 10$.



Round 1: $\theta = 0$, 10, and ∞.

Round 2: $\theta = 0$, 10, and ∞.

Figure 1: CARA Bounds

6 Equilibrium, Maxmin, and the Shapley Value

By Proposition 8, as bidders become infinitely risk averse, the equilibrium bid function converges to the maxmin perfect bid function $\underline{\beta}$. By Proposition 2, when each bidder follows the maxmin perfect strategy, then each obtains his Shapley value allocation. Corollary 2 follows immediately.

Corollary 2: *As bidders become infinitely risk averse, the equilibrium allocation approaches the Shapley-value allocation.*

Since the Shapley value allocation is in the anti-core, these results imply that the compensated position auction produces allocations in the anti-core when bidders are sufficiently risk averse or when each bidder follows the maxmin perfect strategy. The next example and the associated figure illustrate Corollary 2, showing that the bidders' realized payoffs converge to their Shapley value payoffs as bidders become infinitely risk averse.

Example 6: Suppose demands are paid equally by players obtaining better positions. Figure 2 shows the equilibrium payoff of each bidder as a function of θ, when $\alpha_1 = 6$, $\alpha_2 = 4$, $\alpha_3 = 2$ and the bidders' values are $x_1 = 3/4$, $x_2 = 1/2$, and $x_3 = 1/4$. The payoff of bidder 3 is

$$y_3(\theta) := \alpha_3 x_3 + \beta_1^\theta(x_3),$$

of bidder 2 is

$$y_2(\theta) := \alpha_2 x_2 + \beta_2^\theta(x_2; \beta_1^\theta(x_3)) - \frac{1}{2}\beta_1^\theta(x_3),$$

of bidder 1 is

$$y_1(\theta) := \alpha_1 x_1 - \beta_2^\theta(x_2; \beta_1^\theta(x_3)) - \frac{1}{2}\beta_1^\theta(x_3).$$

The dashed lines are the bidders' Shapley values.

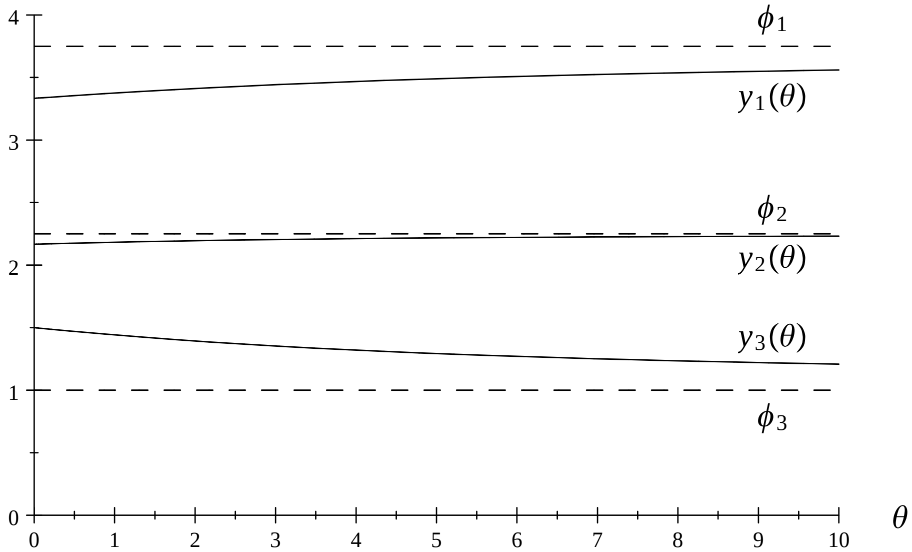


Figure 2: Equilibrium Payoffs as a Function of θ.

Since the auction is efficient, each bidder is allocated the same position he would receive in the Shapley allocation. As θ approaches infinity, each bidder also receives the same transfer that he would receive in the Shapley allocation: Bidder 3 receives compensation of $\underline{\beta}_1(1/4) = 1/2$. Bidder 2 receives compensation of $\underline{\beta}_2(1/2) = 1/2$ from Bidder 1, but pays compensation of $\frac{1}{2}\underline{\beta}_1(1/4)$ to Bidder 3, for a net transfer of $1/4$. Bidder 1 pays compensation of $\frac{1}{2}\underline{\beta}_1(1/4)$ to Bidder 3 and $\underline{\beta}_2(1/2)$ to Bidder 2, for a net transfer of $-3/4$. These are exactly the transfers identified in Example 1.

7 Discussion

This paper proposes compensated position auctions as a solution to the problem of fairly allocating heterogeneous items, priorities, positions, or rights among participants who have equal claims. Compensated position auctions are efficient and budget balanced. From a purely theoretical perspective these auctions are of interest since they provide decision theoretic and non-cooperative foundations for the Shapley value in an environment with incomplete information. Since the equilibrium allocation approaches the Shapley allocation as bidders become risk averse, then for sufficiently risk averse bidders the equilibrium allocation is (i) in the anti-core and (ii) transfers have the "top-down" property discussed in Section 2, when the Shapley payoffs are in the interior of the anti-core.

Participation in the auction is individually rational for a bidder when the (natural) alternative, since bidders have common claims, is the random assignment of positions. Following the maxmin perfect bidding strategy, by Proposition 2 a bidder with value x guarantees himself a payoff of at least $\bar{v}_1(x) = \frac{1}{N}\sum_{m=1}^{N} \alpha_m x$ and a utility of at least $u(\frac{1}{N}\sum_{m=1}^{N} \alpha_m x)$. His equilibrium expected utility is therefore at least $u(\frac{1}{N}\sum_{m=1}^{N} \alpha_m x)$ since it must exceed his maxmin utility. Concavity of u implies that

$$u(\frac{1}{N}\sum_{m=1}^{N} \alpha_m x) \geq \frac{1}{N}\sum_{m=1}^{N} u(\alpha_m x),$$

and thus bidders prefer the compensated position auction to the random allocation of positions.

There may be other auctions whose Bayes Nash equilibria converge to the Shapley value as bidders become infinitely risk averse and which generate Shapley value allocations under maxmin play. It is easy, however, to construct auctions that do not have these properties. Consider, for example, the auction in which all bidders simultaneously make sealed bids, the highest bidder gets the best position, the second highest bidder gets the second best position, and so on. Suppose further that only the highest bidder pays his bid and his bid is divided equally among all the bidders. If the auction has a symmetric equilibrium in increasing strategies, then the auction will be efficient and budget balanced. It cannot, however, generate the Shapley allocation as all the bidders (except the highest) receive the same net transfer, namely $1/N$-th of the highest bid. As Example 1 illustrates, the Shapley allocation requires different bidders receive different net transfers.

8 Appendix

The proof of Proposition 1 involves combinatorial arguments that play no role in the remaining proofs. It is included for completeness, but the reader is invited to skip it.

Proof of Proposition 1: We compute the Shapley value directly using that

$$\phi_i = \sum_{s=1}^{N} \frac{(N-s)!(s-1)!}{N!} \left[\sum_{B_i(s)} (v(S) - v(S\backslash\{i\})) \right]$$

where

$$B_i(s) = \{S | i \in S \text{ and } |S| = s\}.$$

We first compute the marginal contribution of player i to coalition S. If $i \in S$ has the j-th highest value in coalition S (i.e., $x_i = y_j^{(S)}$) then

$$\begin{aligned} v(S) - v(S\backslash\{i\}) &= \alpha_j y_j^{(S)} - \sum_{m=1}^{|S|-j} (\alpha_{j+m-1} - \alpha_{j+m})\, y_{j+m}^{(S)} \\ &= \alpha_j x_i - \sum_{m=1}^{|S|-j} (\alpha_{j+m-1} - \alpha_{j+m})\, y_{j+m}^{(S)}. \end{aligned}$$

This follows since in coalition S player i is assigned the j-th position, players in S with a smaller index than i stay in the same position they occupied in $S\backslash\{i\}$, and players with a higher index than i move down one position.

Player i's Shapley value can be written as

$$\phi_i = c^i x_i - \sum_{m=1}^{N-i} \delta^{im} x_{i+m},$$

where c^i is of the form

$$c^i = c_1^i \alpha_1 + \cdots + c_i^i \alpha_i,$$

and δ^{im} is of the form

$$\delta^{im} = \delta_1^{im}(\alpha_1 - \alpha_2) + \cdots + \delta_{i+m-1}^{im}(\alpha_{i+m-1} - \alpha_{i+m}).$$

The term c^i is the expected contribution of player i and $\delta^{im} x_{i+m}$ is the expected externality that i imposes on player $i+m$.

We now compute c_r^i for $1 \leq r \leq i$, which is the contribution of player i when allocated position r. For each coalition size s, we count the number of coalitions of size s where i is in position r and multiply this number by the appropriate Shapley weight. The coefficient c_r^i is the sum of these terms over all s.

The smallest coalitions where i is in position r are coalitions of size r, and consist of player i and $r-1$ players with a smaller index. The largest coalitions where i is in position r are coalitions of size $N-i+r$, and consist of player i, $r-1$ players with a smaller index, and $N-i$ players with a larger index. The number of coalitions of size s where i is placed in position r is

$$\binom{i-1}{r-1}\binom{N-i}{s-r},$$

where $\binom{i-1}{r-1}$ is the number of ways of choosing $r-1$ players with index smaller than i from $i-1$ players, and $\binom{N-i}{s-r}$ is the number of ways of choosing $s-r$ players with index larger than i from $N-i$ players. The Shapley weight for coalitions of size s is

$$\frac{(s-1)!(N-s)!}{N!},$$

and therefore

$$c_r^i = \sum_{s=r}^{N-i+r} \frac{(s-1)!(N-s)!}{N!}\binom{i-1}{r-1}\binom{N-i}{s-r}.$$

Summing across positions where player i can be placed yields

$$\begin{aligned} c^i &= \sum_{r=1}^{i}\left[\sum_{s=r}^{N-i+r} \frac{(s-1)!(N-s)!}{N!}\binom{i-1}{r-1}\binom{N-i}{s-r}\right]\alpha_r \\ &= \sum_{r=1}^{i}\left[\frac{1}{N}\sum_{s=r}^{N-i+r} \frac{\binom{i-1}{r-1}\binom{N-i}{s-r}}{\binom{N-1}{s-1}}\right]\alpha_r \\ &= \frac{1}{i}\sum_{r=1}^{i}\alpha_r, \end{aligned}$$

where the last equality holds by Claim 4 in the Supplemental Appendix.

Next, we compute δ_r^{im} for $0 < m \leq N - i$ and $1 \leq r < i + m$. The term $\delta_r^{im}(\alpha_r - \alpha_{r+1})x_{i+m}$ will be the expected externality player i imposes on player $i + m$ by pushing player $i + m$ from r to $r + 1$. For each player $i + m$, position r, and coalition size s, we count the number of coalitions of size s where player i pushes player $i + m$ from position r to position $r + 1$ and we multiply this number by the appropriate Shapley weight. The coefficient δ_r^{im} is the sum of these terms over all s.

The smallest coalitions where i pushes $i + m$ from position r to position $r + 1$ are coalitions of size $r + 1$, and consist of player i, player $i + m$, and $r - 1$ other players with smaller index than $i + m$. The largest coalitions where i pushes $i + m$ from position r to position $r + 1$ are coalitions of size $r + 1 + N - (i + m)$, and consist of player i, player $i + m$, $r - 1$ other players with index smaller than $i + m$, and the $N - (i + m)$ players with an index larger than $i + m$. The number of coalitions of size s where i pushes $i + m$ from position r to position $r + 1$ is

$$\binom{i+m-2}{r-1}\binom{N-(i+m)}{s-(r+1)},$$

where $\binom{i+m-2}{r-1}$ is the number of ways of choosing $r - 1$ players (excluding player i) with index smaller than $i + m$, and $\binom{N-(i+m)}{s-(r+1)}$ is the number of ways of choosing $s - (r + 1)$ players with index larger than $i + m$ from $N - (i + m)$ players. The Shapley weight for coalitions of size s is

$$\frac{(s-1)!(N-s)!}{N!},$$

and therefore,

$$\delta_r^{im} = \sum_{s=r+1}^{r+1+N-(i+m)} \frac{(s-1)!(N-s)!}{N!}\binom{i+m-2}{r-1}\binom{N-(i+m)}{s-(r+1)}.$$

Summing across positions where player $i + m$ can be placed yields

$$\begin{aligned}
\delta^{im} &= \sum_{r=1}^{i+m-1}\left[\sum_{s=r+1}^{r+1+N-(i+m)} \frac{(s-1)!(N-s)!}{N!}\binom{i+m-2}{r-1}\binom{N-(i+m)}{s-(r+1)}\right](\alpha_r - \alpha_{r+1}) \\
&= \sum_{r=1}^{i+m-1}\left[\frac{1}{N}\sum_{s=r+1}^{r+1+N-(i+m)} \frac{\binom{i+m-2}{r-1}\binom{N-(i+m)}{s-(r+1)}}{\binom{N-1}{S-1}}\right](\alpha_r - \alpha_{r+1}).
\end{aligned}$$

The identity in Claim 4 holds for all $i \leq N$. Replacing i with $i+m$ and r with $r+1$ in this identity, and noting that $i+m \leq N$ also, we obtain the following new identity

$$\frac{1}{N} \sum_{s=(r+1)}^{N+(r+1)-(i+m)} \binom{(i+m)-1}{(r+1)-1} \frac{\binom{N-(i+m)}{s-(r+1)}}{\binom{N-1}{s-1}} = \frac{1}{i+m}.$$

Applying this new identity to δ^{im} yields

$$\begin{aligned} \delta^{im} &= \sum_{r=1}^{i+m-1} \frac{\binom{i+m-2}{r-1}}{\binom{(i+m)-1}{(r+1)-1}} \left[\frac{1}{N} \sum_{s=r+1}^{r+1+N-(i+m)} \frac{\binom{(i+m)-1}{(r+1)-1} \binom{N-(i+m)}{s-(r+1)}}{\binom{N-1}{S-1}} \right] (\alpha_r - \alpha_{r+1}) \\ &= \frac{1}{i+m} \sum_{r=1}^{i+m-1} \frac{\binom{i+m-2}{r-1}}{\binom{i+m-1}{r}} (\alpha_r - \alpha_{r+1}). \end{aligned}$$

The total expected externality that player i imposes on the other players is

$$\begin{aligned} \sum_{m=1}^{N-i} \delta^{im} x_{i+m} &= \sum_{m=1}^{N-i} \left[\frac{1}{i+m} \sum_{r=1}^{i+m-1} \frac{\binom{i+m-2}{r-1}}{\binom{i+m-1}{r}} (\alpha_r - \alpha_{r+1}) \right] x_{i+m} \\ &= \sum_{m=1}^{N-i} \frac{1}{i+m-1} \left[\frac{i+m-1}{i+m} \sum_{r=1}^{i+m-1} \frac{\binom{i+m-2}{r-1}}{\binom{i+m-1}{r}} (\alpha_r - \alpha_{r+1}) \right] x_{i+m}. \end{aligned}$$

Noting that

$$\begin{aligned} (i+m-1) \frac{\binom{i+m-2}{r-1}}{\binom{i+m-1}{r}} &= (i+m-1) \frac{\frac{(i+m-2)!}{(i+m-2-(r-1))!(r-1)!}}{\frac{(i+m-1)!}{(i+m-1-r))!r!}} \\ &= (i+m-1) \frac{\frac{(i+m-2)!}{(i+m-1-r)!(r-1)!}}{\frac{(i+m-1)!}{(i+m-1-r))!r!}} \\ &= r, \end{aligned}$$

we can write

$$\sum_{m=1}^{N-i} \delta^{im} x_{i+m} = \sum_{m=1}^{N-i} \frac{1}{i+m-1} \left[\sum_{r=1}^{i+m-1} \frac{r}{i+m} (\alpha_r - \alpha_{r+1}) x_{i+m} \right].$$

Collecting terms, the Shapley value of player i is

$$\phi_i = \frac{1}{i}\left(\sum_{m=1}^{i} \alpha_m\right) x_i - \sum_{m=1}^{N-i} \frac{1}{i+m-1}\left[\sum_{r=1}^{i+m-1} \frac{r}{i+m}\left(\alpha_r - \alpha_{r+1}\right) x_{i+m}\right].$$

□

Proof of Corollary 1: We first show that

$$\tau_k = \left[\frac{\alpha_1 + \cdots + \alpha_{k-1}}{k} - \frac{k-1}{k}\alpha_k\right] x_k - \sum_{j=k+1}^{N} \frac{1}{j-1}\left[\frac{\alpha_1 + \cdots + \alpha_{j-1}}{j} - \frac{j-1}{j}\alpha_j\right] x_j.$$

Clearly true for $k = N$. Assume that it is true for $k = k' + 1$. We show it is true for k'.

Subclaim: We first establish the following: If for $j = k'+1, ..., N$ we have

$$\tau_j = s_j - \sum_{m=j+1}^{N} \frac{1}{m-1} s_m,$$

then

$$\tau_N + \cdots + \tau_{k'+1} = s_{k'+1} + \frac{k'}{k'+1} s_{k'+2} + \cdots + \frac{k'}{N-2} s_{N-1} + \frac{k'}{N-1} s_N.$$

We have $\tau_N = s_N$. Assume that the claim is true for $\tau_N + \cdots + \tau_{k'+2}$. We show it is true for $\tau_N + \cdots + \tau_{k'+1}$. We have

$$\tau_N + \cdots + \tau_{k'+2} = s_{k'+2} + \cdots + \frac{k'+1}{N-2} s_{N-1} + \frac{k'+1}{N-1} s_N$$

and

$$\tau_{k'+1} = s_{k'+1} - \frac{1}{k'+1} s_{k'+2} - \cdots - \frac{1}{N-2} s_{N-1} - \frac{1}{N-1} s_N.$$

Adding these equations gives us the result.

Define

$$s_j = \left[\frac{\alpha_1 + \cdots + \alpha_{j-1}}{j} - \frac{j-1}{j}\alpha_j\right] x_j.$$

Simple algebra shows that

$$s_j = \sum_{m=1}^{j-1} \frac{m}{j}(\alpha_m - \alpha_{m+1}) x_j$$

We show that

$$\tau_{k'} = \left[\frac{\alpha_1 + \cdots + \alpha_{k'-1}}{k'} - \frac{k'-1}{k'} \alpha_{k'} \right] x_{k'} - \sum_{j=k'+1}^{N} \frac{1}{j-1} \left[\frac{\alpha_1 + \cdots + \alpha_{j-1}}{j} - \frac{j-1}{j} \alpha_j \right] x_j,$$

which we can write as

$$\tau_{k'} = s_{k'} - \sum_{j=k'+1}^{N} \frac{1}{j-1} s_j.$$

We have

$$\begin{aligned} \tau_{k'} &= \frac{1}{k'} \left[(\alpha_1 + \cdots + \alpha_{k'})\, x_{k'} - \sum_{i=k'+1}^{N} \tau_i \right] - \alpha_{k'} x_{k'} \\ &= \left[\frac{\alpha_1 + \cdots + \alpha_{k'-1}}{k'} - \frac{k'-1}{k'} \alpha_{k'} \right] x_{k'} - \frac{1}{k'} \sum_{i=k'+1}^{N} \tau_i \end{aligned}$$

By the subclaim

$$\frac{1}{k'} \sum_{i=k'+1}^{N} \tau_i = \frac{1}{k'} s_{k'+1} + \frac{1}{k'+1} s_{k'+2} + \cdots + \frac{1}{N-2} s_{N-1} + \frac{1}{N-1} s_N.$$

Hence,

$$\tau_{k'} = s_{k'} - \left(\frac{1}{k'} s_{k'+1} + \frac{1}{k'+1} s_{k'+2} + \cdots + \frac{1}{N-2} s_{N-1} + \frac{1}{N-1} s_N \right),$$

which establishes the claim.

Next we show that $\phi_{k'} = \alpha_{k'} x_{k'} + \tau_{k'}$, which establishes the Corollary.

$$\begin{aligned} \alpha_{k'} x_{k'} + \tau_{k'} &= \frac{\alpha_1 + \cdots + \alpha_{k'}}{k'} x_{k'} - \left(\frac{1}{k'} s_{k'+1} + \frac{1}{k'+1} s_{k'+2} + \cdots + \frac{1}{N-2} s_{N-1} + \frac{1}{N-1} s_N \right) \\ &= \frac{\alpha_1 + \cdots + \alpha_{k'}}{k'} x_{k'} - \sum_{m=1}^{N-k'} \frac{1}{k'+m-1} \left[\sum_{r=1}^{k'+m-1} \frac{r}{k'+m} (\alpha_r - \alpha_{r+1}) x_{k'+m} \right], \end{aligned}$$

where we use

$$s_{k'+m} = \sum_{r=1}^{k'+m-1} \frac{r}{k'+m} (\alpha_r - \alpha_{r+1}) x_{k'+m}.$$

Hence $\alpha_{k'} x_{k'} + \tau_{k'} = \phi_{k'}$ as given in Proposition 1. This establishes the Corollary. □

Proof of Proposition 2: We first show that following $\underline{\beta}$ guarantees a bidder with value x a payoff at round t of at least

$$\bar{v}_t(x; \mathbf{d}_{t-1}) = \left(\frac{\alpha_1 + \cdots + \alpha_{N-t+1}}{N-t+1} \right) x - \sum_{i=1}^{t-1} \left[\frac{\lambda_1^i + \cdots + \lambda_{N-t+1}^i}{N-t+1} \right] d_i,$$

when d_{t-1} is the sequence of smallest demands at prior rounds.

The proof is by induction. Consider round $N-1$. A bidder with value x whose demand is d_{N-1} either (i) has the smallest demand and obtains a payoff of

$$\alpha_2 x + d_{N-1} - \sum_{i=1}^{N-2} \lambda_2^i d_i$$

or (ii) his rival has the smallest demand $b \leq d_{N-1}$ and he obtains a payoff of $\alpha_1 x - b - \sum_{i=1}^{N-2} \lambda_1^i d_i$. In the second case, his payoff is at least

$$\alpha_1 x - d_{N-1} - \sum_{i=1}^{N-2} \lambda_1^i d_i.$$

The bidder maximizes his minimum payoff when d_{N-1} satisfies

$$\alpha_2 x + d_{N-1} - \sum_{i=1}^{N-2} \lambda_2^i d_i = \alpha_1 x - d_{N-1} - \sum_{i=1}^{N-2} \lambda_1^i d_i,$$

i.e.,

$$d_{N-1} = \frac{\alpha_1 - \alpha_2}{2} x - \sum_{i=1}^{N-2} \frac{\lambda_1^i - \lambda_2^i}{2} d_i.$$

Hence at round $N-1$ the bidder guarantees himself a payoff of at least

$$\bar{v}_{N-1}(x; \mathbf{d}_{N-2}) = \frac{\alpha_1 + \alpha_2}{2} x - \sum_{i=1}^{N-2} \frac{\lambda_1^i + \lambda_2^i}{2} d_i$$

by following

$$\underline{\beta}_{N-1}(x; \mathbf{d}_{N-2}) = \frac{\alpha_1 - \alpha_2}{2} x - \sum_{i=1}^{N-2} \frac{\lambda_1^i - \lambda_2^i}{2} d_i.$$

Suppose that at round $t+1$, given smallest demands d_t, a bidder with value x can guarantee himself at least

$$\bar{v}_{t+1}(x;\mathbf{d}_t) = \sum_{m=1}^{N-t} \frac{\alpha_m}{N-t} x - \sum_{i=1}^{t} \frac{\lambda_1^i + \ldots + \lambda_{N-t}^i}{N-t} d_i,$$

by following

$$\underline{\beta}_s(x;\mathbf{d}_{s-1}) = \sum_{m=1}^{N-s} \frac{m}{N-s+1}(\alpha_m - \alpha_{m+1})x - \sum_{i=1}^{s-1}\left[\sum_{m=1}^{N-s} \frac{m}{N-s+1}(\lambda_m^i - \lambda_{m+1}^i)d_i\right]$$

for $s = t+1, \ldots, N-1$. We show that at round t he can guarantee himself at least

$$\bar{v}_t(x;\mathbf{d}_{t-1}) = \left(\frac{\alpha_1 + \cdots + \alpha_{N-t+1}}{N-t+1}\right) x - \sum_{i=1}^{t-1}\left[\frac{\lambda_1^i + \cdots + \lambda_{N-t+1}^i}{N-t+1}\right] d_i,$$

by following

$$\underline{\beta}_s(x;\mathbf{d}_{s-1}) = \sum_{m=1}^{N-s} \frac{m}{N-s+1}(\alpha_m - \alpha_{m+1})x - \sum_{i=1}^{s-1}\left[\sum_{m=1}^{N-s} \frac{m}{N-s+1}(\lambda_m^i - \lambda_{m+1}^i)d_i\right]$$

for $s = t, \ldots, N-1$.

A bidder with value x whose demand is d_t at round t either (i) has the smallest demand and obtains a payoff of

$$\alpha_{N-t+1}x + d_t - \sum_{i=1}^{t-1} \lambda_{N-t+1}^i d_i,$$

or (ii) a rival has the smallest demand $b \leq d_t$ and he obtains a payoff of at least

$$\begin{aligned}
\bar{v}_{t+1}(x;(\mathbf{d}_{t-1},b)) &= \left(\frac{\alpha_1 + \cdots + \alpha_{N-t}}{N-t}\right) x - \frac{1}{N-t} b - \sum_{i=1}^{t-1}\left[\frac{\lambda_1^i + \cdots + \lambda_{N-t}^i}{N-t}\right] d_i \\
&\geq \left(\frac{\alpha_1 + \cdots + \alpha_{N-t}}{N-t}\right) x - \frac{1}{N-t} d_t - \sum_{i=1}^{t-1}\left[\frac{\lambda_1^i + \cdots + \lambda_{N-t}^i}{N-t}\right] d_i,
\end{aligned}$$

where the first equality holds since $\lambda_1^t + \cdots + \lambda_{N-t}^t = 1$.

The bidder maximizes his minimum payoff when d_t satisfies

$$\alpha_{N-t+1}x+d_t-\sum_{i=1}^{t-1}\lambda^i_{N-t+1}d_i = \left(\frac{\alpha_1+\cdots+\alpha_{N-t}}{N-t}\right)x-\frac{1}{N-t}d_t-\sum_{i=1}^{t-1}\left[\frac{\lambda^i_1+\cdots+\lambda^i_{N-t}}{N-t}\right]d_i$$

i.e.,

$$\begin{aligned} d_t &= \left(\frac{\alpha_1+\cdots+\alpha_{N-t}}{N-t+1}-\frac{N-t}{N-t+1}\alpha_{N-t+1}\right)x \\ &\quad -\sum_{i=1}^{t-1}\left[\frac{\lambda^i_1+\cdots+\lambda^i_{N-t}}{N-t+1}-\frac{N-t}{N-t+1}\lambda^i_{N-t+1}\right]d_i. \end{aligned}$$

Hence at round t the bidder guarantees himself a payoff of at least

$$\begin{aligned} \bar{v}_t(x;\mathbf{d}_{t-1}) &= \alpha_{N-t+1}x+d_t-\sum_{i=1}^{t-1}\lambda^i_{N-t+1}d_i \\ &= \left(\frac{\alpha_1+\cdots+\alpha_{N-t+1}}{N-t+1}\right)x-\sum_{i=1}^{t-1}\left[\frac{\lambda^i_1+\cdots+\lambda^i_{N-t+1}}{N-t+1}\right]d_i, \end{aligned}$$

by following

$$\begin{aligned} \underline{\beta}_t(x;\mathbf{d}_{t-1}) &= \left(\frac{\alpha_1+\cdots+\alpha_{N-t}}{N-t+1}-\frac{N-t}{N-t+1}\alpha_{N-t+1}\right)x \\ &\quad -\sum_{i=1}^{t-1}\left[\frac{\lambda^i_1+\cdots+\lambda^i_{N-t}}{N-t+1}-\frac{N-t}{N-t+1}\lambda^i_{N-t+1}\right]d_i \\ &= \sum_{m=1}^{N-t}\frac{m}{N-t+1}(\alpha_m-\alpha_{m+1})x-\sum_{i=1}^{t-1}\left[\sum_{m=1}^{N-t}\frac{m}{N-t+1}(\lambda^i_m-\lambda^i_{m+1})d_i\right]. \end{aligned}$$

Next, we show that $\bar{v}_t(x;d_{t-1})$ is the largest payoff a bidder with value x can guarantee at round t given smallest demands d_{t-1}. Suppose to the contrary he can guarantee himself $v'_t > \bar{v}_t(x;d_{t-1})$. If all active bidder have the same value x then, since the game is symmetric, each such bidder can guarantee himself at least v'_t and hence the total guaranteed payoffs of the active bidders is at least

$$\begin{aligned} (N-t+1)v'_t &> (N-t+1)\left[\left(\frac{\alpha_1+\cdots+\alpha_{N-t+1}}{N-t+1}\right)x-\sum_{i=1}^{t-1}\left[\frac{\lambda^i_1+\cdots+\lambda^i_{N-t+1}}{N-t+1}\right]d_i\right] \\ &= \sum_{m=1}^{N-t+1}\alpha_m x-\sum_{i=1}^{t-1}[\lambda^i_1+\cdots+\lambda^i_{N-t+1}]d_i, \end{aligned}$$

which is a contraction since the RHS is the total surplus that can be obtained by the active bidders at round t. The first term is the surplus realized from allocating positions 1 through $N-t+1$ to the active bidders, and the second term is the compensation they owe.

We have established that $\underline{\beta}$ is a maxmin perfect strategy. Next we show that $\underline{\beta}$ is the unique maxmin perfect strategy. As a first step, we establish at each round t that a bidder with value x can be held to a payoff $\bar{v}_t(x;d_{t-1})$ given smallest demands d_{t-1}.

Consider a bidder with value x at round $N-1$ given smallest demands d_{N-2}. Suppose his rival bids $\underline{\beta}_{N-1}(x;\mathbf{d}_{N-2})$. If the bidder demands $d_{N-1} < \underline{\beta}_{N-1}(x;\mathbf{d}_{N-2})$, then his payoff is

$$\alpha_2 x + d_{N-1} - \sum_{i=1}^{N-2} \lambda_2^i d_i < \alpha_2 x + \underline{\beta}_{N-1}(x;\mathbf{d}_{N-2}) - \sum_{i=1}^{N-2} \lambda_2^i d_i = \bar{v}_{N-1}(x;\mathbf{d}_{N-2}).$$

If he demands $d_{N-1} > \underline{\beta}_{N-1}(x;\mathbf{d}_{N-2})$ then his payoff is

$$\alpha_1 x - \underline{\beta}_{N-1}(x;\mathbf{d}_{N-2}) - \sum_{i=1}^{N-2} \lambda_2^i d_i = \bar{v}_{N-1}(x;\mathbf{d}_{N-2}).$$

In both cases, his payoff is at most $\bar{v}_{N-1}(x;d_{N-2})$, which establishes he is held to $\bar{v}_{N-1}(x;d_{N-2})$.

Suppose the claim is true for rounds $t+1,\ldots,N-1$. We show it holds for round t. Consider a bidder with value x at round t with smallest demands d_{t-1}. Suppose at each round $s=t,\ldots,N-1$ that each of his rivals demands $\underline{\beta}_s(x;\mathbf{d}_{s-1})$ at round s given smallest demands d_{s-1}. If at round t the bidder demands $d_t < \underline{\beta}_t(x;\mathbf{d}_{t-1})$ his payoff is

$$\alpha_{N-t+1}x + d_t - \sum_{i=1}^{t-1} \lambda_{N-t+1}^i d_i < \alpha_{N-t+1}x + \underline{\beta}_t(x;\mathbf{d}_{t-1}) - \sum_{i=1}^{t-1} \lambda_{N-t+1}^i d_i = \bar{v}_t(x;\mathbf{d}_{t-1}).$$

If he demands $d_t > \underline{\beta}_t(x;\mathbf{d}_{t-1})$, then he continues to round $t+1$ and by the induction hypothesis his rivals hold him to $\bar{v}_{t+1}(x;d_{t-1},\underline{\beta}_t(x;\mathbf{d}_{t-1}))$. Straight forward algebra establishes that

$$\bar{v}_{t+1}(x;\mathbf{d}_{t-1},\underline{\beta}_t(x;\mathbf{d}_{t-1})) = \bar{v}_t(x;\mathbf{d}_{t-1}).$$

This establishes the claim holds for all rounds.

Finally, we show that $\underline{\beta}$ is the unique maxmin perfect strategy. Suppose that there is another maxmin perfect strategy $\hat{\beta} \neq \underline{\beta}$. Then for some x, t, and d_{t-1} we have that $\hat{\beta}_t(x;d_{t-1}) \neq \underline{\beta}_t(x;\mathbf{d}_{t-1})$. Consider a bidder with value x at round t, given smallest demands d_{t-1}, who follows $\hat{\beta}$. Suppose that at each round $s = t, \ldots, N-1$ that his rivals bid $\underline{\beta}_s(x;\mathbf{d}_{s-1})$. If $\hat{\beta}_t(x;d_{t-1}) < \underline{\beta}_t(x;\mathbf{d}_{t-1})$ then bidder i drops out at round t and obtains the payoff

$$\alpha_{N-t+1}x+\hat{\beta}_t(x;\mathbf{d}_{t-1})-\sum_{i=1}^{t-1}\lambda^i_{N-t+1}d_i < \alpha_{N-t+1}x+\underline{\beta}_t(x;\mathbf{d}_{t-1})-\sum_{i=1}^{t-1}\lambda^i_{N-t+1}d_i = \bar{v}_t(x;\mathbf{d}_{t-1}).$$

If $\hat{\beta}_t(x;d_{t-1}) > \underline{\beta}_t(x;\mathbf{d}_{t-1})$ and his rivals bid $(\hat{\beta}_t(x;d_{t-1}) + \underline{\beta}_t(x;\mathbf{d}_{t-1}))/2$ at round t and bids $\underline{\beta}_s(x;\mathbf{d}_{s-1})$ at each round $s = t+1, \ldots, N-1$ then the bidder's payoff at round t is at most

$$\bar{v}_{t+1}(x;\mathbf{d}_{t-1}, \frac{1}{2}(\hat{\beta}_t(x;\mathbf{d}_{t-1}) + \underline{\beta}_t(x;\mathbf{d}_{t-1})))$$

by the immediately prior claim. Since

$$\begin{aligned}\bar{v}_{t+1}(x;\mathbf{d}_{t-1},d_t) &= \left(\frac{\alpha_1+\cdots+\alpha_{N-t}}{N-t}\right)x - \left[\frac{\lambda^t_1+\cdots+\lambda^t_{N-t}}{N-t}\right]d_t - \sum_{i=1}^{t-1}\left[\frac{\lambda^i_1+\cdots+\lambda^i_{N-t}}{N-t}\right]d_i \\ &= \left(\frac{\alpha_1+\cdots+\alpha_{N-t}}{N-t}\right)x - \frac{1}{N-t}d_t - \sum_{i=1}^{t-1}\left[\frac{\lambda^i_1+\cdots+\lambda^i_{N-t}}{N-t}\right]d_i\end{aligned}$$

is decreasing in d_t, we have

$$\bar{v}_{t+1}(x;\mathbf{d}_{t-1},\frac{1}{2}(\hat{\beta}_t(x;\mathbf{d}_{t-1})+\underline{\beta}_t(x;\mathbf{d}_{t-1}))) < \bar{v}_{t+1}(x;\mathbf{d}_{t-1},\underline{\beta}_t(x;\mathbf{d}_{t-1})) = \bar{v}_t(x;\mathbf{d}_{t-1}),$$

which contradicts that $\hat{\beta}$ is a maxmin perfect strategy. □

Proof of Proposition 3: It is convenient to define

$$s_t(x) = \sum_{m=1}^{N-t}\frac{m}{N-t+1}(\alpha_m - \alpha_{m+1})x,$$

which, when positions $1, \ldots, N-t+1$ remain to be allocated, can be interpreted as an equal share of incremental benefits of the contested positions to a bidder with value x. We can also express $s_t(x)$ as

$$s_t(x) = \left[\frac{\alpha_1 + \cdots + \alpha_{N-t+1}}{N-t+1} - \alpha_{N-t+1}\right] x.$$

It is straightforward to show that the Shapley transfers satisfy

$$\tau_{N-t+1} = s_t(x_{N-t+1}) - \sum_{i=1}^{t-1} \frac{1}{N-i} s_i(x_{N-i+1}).$$

Let $\mathbf{d}_{t-1}$ be the sequence of dropout prices at round t. When all bidders follow the maxmin bidding strategy, then at round t the active bidders have values $x_1, \ldots, x_{N-t+1}$. The bidder with value x_{N-t+1} submits the smallest demand, he receives position $N-t+1$ and, by the construction of the maxmin bid function (see the proof of Proposition 2), his payoff is equal to his value, i.e., $\bar{v}_t(x_{N-t+1}; \mathbf{d}_{t-1})$. In particular, his payoff is

$$\bar{v}_t(x_{N-t+1}; \mathbf{d}_{t-1}) = \left(\frac{\alpha_1 + \cdots + \alpha_{N-t+1}}{N-t+1}\right) x_{N-t+1} - \sum_{i=1}^{t-1} \left[\frac{\lambda_1^i + \cdots + \lambda_{N-t+1}^i}{N-t+1}\right] d_i.$$

We show that $\bar{v}_t(x_{N-t+1}; \mathbf{d}_{t-1}) = \phi_{N-t+1}$.

We first show that

$$\sum_{i=1}^{t-1} \left[\frac{\lambda_1^i + \cdots + \lambda_{N-t+1}^i}{N-t+1}\right] d_i = \sum_{i=1}^{t-1} \frac{1}{N-i} s_i(x_{N-i+1}).$$

Since $\lambda_1^{t-1} + \cdots + \lambda_{N-t+1}^{t-1} = 1$, we have

$$\sum_{i=1}^{t-1} \left[\frac{\lambda_1^i + \cdots + \lambda_{N-t+1}^i}{N-t+1}\right] d_i = \frac{1}{N-t+1} d_{t-1} + \sum_{i=1}^{t-2} \left[\frac{\lambda_1^i + \cdots + \lambda_{N-t+1}^i}{N-t+1}\right] d_i.$$

Since the maxmin perfect bid at round $t-1$ is

$$d_{t-1} = s_{t-1}(x_{N-t+2}) - \sum_{i=1}^{t-2} \left[\frac{\lambda_1^i + \cdots + \lambda_{N-t+1}^i}{N-t+2} - \frac{N-t+1}{N-t+2} \lambda_{N-t+2}^i\right] d_i,$$

we have $\sum_{i=1}^{t-1}\left[\frac{\lambda_1^i+\cdots+\lambda_{N-t+1}^i}{N-t+1}\right]d_i$

$$= \frac{s_{t-1}(x_{N-t+2})}{N-t+1} - \frac{1}{N-t+1}\sum_{i=1}^{t-2}\left[\frac{\lambda_1^i+\cdots+\lambda_{N-t+1}^i}{N-t+2} - \frac{N-t+1}{N-t+2}\lambda_{N-t+2}^i\right]d_i$$
$$+\sum_{i=1}^{t-2}\left[\frac{\lambda_1^i+\cdots+\lambda_{N-t+1}^i}{N-t+1}\right]d_i$$
$$= \frac{s_{t-1}(x_{N-t+2})}{N-t+1} - \frac{1}{N-t+1}\sum_{i=1}^{t-2}\left[\frac{\lambda_1^i+\cdots+\lambda_{N-t+1}^i+\lambda_{N-t+2}^i}{N-t+2} - \lambda_{N-t+2}^i\right]d_i$$
$$+\sum_{i=1}^{t-2}\left[\frac{\lambda_1^i+\cdots+\lambda_{N-t+1}^i}{N-t+1}\right]d_i$$
$$= \frac{s_{t-1}(x_{N-t+2})}{N-t+1} - \frac{1}{N-t+1}\sum_{i=1}^{t-2}\left[\frac{\lambda_1^i+\cdots+\lambda_{N-t+1}^i+\lambda_{N-t+2}^i}{N-t+2} - (\lambda_1^i+\cdots+\lambda_{N-t+2}^i)\right]d_i$$
$$= \frac{s_{t-1}(x_{N-t+2})}{N-t+1} + \frac{1}{N-t+1}\sum_{i=1}^{t-2}\left[\frac{\lambda_1^i+\cdots+\lambda_{N-t+1}^i+\lambda_{N-t+2}^i}{N-t+2}\right]d_i.$$

Hence we have

$$\sum_{i=1}^{t-1}\left[\frac{\lambda_1^i+\cdots+\lambda_{N-t+1}^i}{N-t+1}\right]d_i = \frac{s_{t-1}(x_{N-t+2})}{N-t+1}+\cdots+\frac{s_1(x_N)}{N-1} = \sum_{i=1}^{t-1}\frac{1}{N-i}s_i(x_{N-i+1}).$$

Substituting into the expression for $\bar{v}_t(x_{N-t+1};\mathbf{d}_{t-1})$ above, we obtain

$$\bar{v}_t(x_{N-t+1};\mathbf{d}_{t-1}) = \left(\frac{\alpha_1+\cdots+\alpha_{N-t+1}}{N-t+1}\right)x_{N-t+1} - \sum_{i=1}^{t-1}\frac{1}{N-i}s_i(x_{N-i+1}).$$

From above,

$$\frac{\alpha_1+\cdots+\alpha_{N-t+1}}{N-t+1}x_{N-t+1} = \alpha_{N-t+1}x_{N-t+1} + s_t(x_{N-t+1})$$

and

$$-\sum_{i=1}^{t-1}\frac{1}{N-i}s_i(x_{N-i+1}) = \tau_{N-t+1} - s_t(x_{N-t+1}),$$

and thus the bidder with value x_{N-t+1} exits at round t with his Shapley value payoff

$$\bar{v}_t(x_{N-t+1};\mathbf{d}_{t-1}) = \alpha_{N-t+1}x_{N-t+1} + \tau_{N-t+1} = \phi_{N-t+1}.$$

□

Proof of Proposition 4: Let $\beta = (\beta_1, \ldots, \beta_{N-1})$ be a symmetric equilibrium in increasing and differentiable strategies. Since equilibrium is in increasing strategies, the sequence of smallest demands $(d_1, \ldots, d_{t-1})$ at round t reveals the $t-1$ lowest values $(z_1, \ldots, z_{t-1})$. In the proof it is convenient to write the round t equilibrium bid as a function of the prior dropout values rather than as a function of the prior smallest demands. In particular, we write $\beta_t(x|\mathbf{z}_{t-1})$ rather than $\beta_t(x;d_{t-1})$.

For each $t < N$, let $\pi_t(y, x|\mathbf{z}_{t-1})$ be the expected payoff to a bidder with value x who in round t deviates from equilibrium and bids as though his value is y (i.e., he bids $\beta_t(y|\mathbf{z}_{t-1})$), when $\mathbf{z}_{t-1}$ is the profile of values of the $t-1$ bidders to drop so far. In this case we will sometimes say the bidder "bids y". Let

$$\Pi_t(x|\mathbf{z}_{t-1}) = \pi_t(x, x|\mathbf{z}_{t-1})$$

be the equilibrium payoff of a bidder in round t when his value is x and $\mathbf{z}_{t-1}$ is the profile of values of the $t-1$ bidders to drop in prior rounds.

We now derive the necessary conditions in Proposition 4. Let $\mathbf{z}_{t-1}$ be arbitrary. Consider a bid y. If $z_t \in [z_{t-1}, y]$ the bidder continues to round $t+1$ and has an expected payoff of $\Pi_{t+1}(x|\mathbf{z}_{t-1}, z_t)$. If $z_t \geq y$ he obtains a payoff of $\alpha_{N-t+1}x + \beta_t(y|\mathbf{z}_{t-1}) - \Sigma_{j=1}^{t-1}\frac{1}{N-j}d_j$ in round t. Hence his expected payoff is

$$\begin{aligned}\pi_t(y, x|\mathbf{z}_{t-1}) = \quad & \int_{z_{t-1}}^{y} \Pi_{t+1}(x|\mathbf{z}_{t-1}, z_t) g_t^{(N-1)}(z_t|z_{t-1}) dz_t \\ & + \int_y^{\bar{x}} u\left(\alpha_{N-t+1}x + \beta_t(y|\mathbf{z}_{t-1}) - \Sigma_{j=1}^{t-1}\frac{1}{N-j}d_j\right) g_t^{(N-1)}(z_t|z_{t-1}) dz_t.\end{aligned}$$

Differentiating with respect to y yields

$$\begin{aligned}\frac{\partial \pi_t(y, x|\mathbf{z}_{t-1})}{\partial y} = \quad & [\Pi_{t+1}(x|\mathbf{z}_{t-1}, y) - u\left(\alpha_{N-t+1}x + \beta_t(y|\mathbf{z}_{t-1}) - \Sigma_{j=1}^{t-1}\frac{1}{N-j}d_j\right)] g_t^{(N-1)}(y|z_{t-1}) \\ & + u'\left(\alpha_{N-t+1}x + \beta_t(y|\mathbf{z}_{t-1}) - \Sigma_{j=1}^{t-1}\frac{1}{N-j}d_j\right) \beta_t'(y|\mathbf{z}_{t-1})(1 - G_t^{(N-1)}(y|z_{t-1})).\end{aligned}$$

A necessary condition for equilibrium is that $\partial \pi_t(y, x|\mathbf{z}_{t-1})/\partial y|_{y=x} = 0$, i.e.,

$$\begin{aligned}& [\Pi_{t+1}(x|\mathbf{z}_{t-1}, x) - u\left(\alpha_{N-t+1}x + \beta_t(x|\mathbf{z}_{t-1}) - \Sigma_{j=1}^{t-1}\frac{1}{N-j}d_j\right)] g_t^{(N-1)}(x|z_{t-1}) \\ & + u'\left(\alpha_{N-t+1}x + \beta_t(x|\mathbf{z}_{t-1}) - \Sigma_{j=1}^{t-1}\frac{1}{N-j}d_j\right) \beta_t'(x|\mathbf{z}_{t-1})(1 - G_t^{(N-1)}(x|z_{t-1})) = 0.\end{aligned}$$

Since

$$\begin{aligned}\Pi_{t+1}(x|\mathbf{z}_{t-1},x) &= \pi_{t+1}(x,x|\mathbf{z}_{t-1},x)\\ &= u\left(\alpha_{N-t}x+\beta_{t+1}(x|\mathbf{z}_{t-1},x)-\frac{1}{N-t}\beta_t(x|\mathbf{z}_{t-1})-\Sigma_{j=1}^{t-1}\frac{1}{N-j}d_j\right)\end{aligned}$$

the necessary condition can be written as

$$\begin{aligned} & u'\left(\alpha_{N-t+1}x+\beta_t(x|\mathbf{z}_{t-1})-\Sigma_{j=1}^{t-1}\frac{1}{N-j}d_j\right)\beta_t'(x|\mathbf{z}_{t-1})\\ = & -\left[\begin{array}{c} u\left(\alpha_{N-t}x+\beta_{t+1}(x|\mathbf{z}_{t-1},x)-\frac{1}{N-t}\beta_t(x|\mathbf{z}_{t-1})-\Sigma_{j=1}^{t-1}\frac{1}{N-j}d_j\right)\\ -u\left(\alpha_{N-t+1}x+\beta_t(x|\mathbf{z}_{t-1})-\Sigma_{j=1}^{t-1}\frac{1}{N-j}d_j\right)\end{array}\right]\Gamma_t^N(x),\end{aligned}$$

where $\beta_N(x;\mathbf{z}_{N-1})\equiv 0$. Replacing $\mathbf{z}_{t-1}$ with d_{t-1} and the x in $\beta_{t+1}(x|\mathbf{z}_{t-1},x)$ with $\beta_t(x|d_{t-1})$ yields the differential equation given in the Proposition for round t. $\square$

Proof of Proposition 5: We first show that the bidding functions in Proposition 5 satisfies the system of differential equations in Proposition 4. The proof is by induction. Consider round $N-1$. The differential equation for round $N-1$ is

$$\beta_{N-1}^{0\prime}(x|\mathbf{z}_{N-2})=-[(\alpha_1-\alpha_2)x-2\beta_{N-1}^0(x|\mathbf{z}_{N-2})]\Gamma_{N-1}^N(x). \tag{1}$$

The unique solution is

$$\begin{aligned}\beta_{N-1}^0(x) &= \frac{1}{2}\frac{\int_x^{\bar{x}}(\alpha_1-\alpha_2)\,z2f(z)(1-F(z))dz}{(1-F(x))^2}\\ &= \frac{1}{2}E\left[(\alpha_1-\alpha_2)Z_{N-1}^{(N)}|Z_{N-1}^{(N)}>x>Z_{N-2}^{(N)}\right],\end{aligned}$$

which is $\beta_{N-1}^0(x)$, as given in Proposition 5.

Suppose $\beta_{t+1}^0,\ldots,\beta_{N-1}^0$ are as given in Proposition 5 for round $t+1,\ldots,N-1$. Consider round t. The differential equation in Proposition 4 for round t, using the notation from the proof of Proposition 4, is

$$\beta_t^{0\prime}(x|\mathbf{z}_{t-1})=-\left[(\alpha_{N-t}-\alpha_{N-t+1})x+\beta_{t+1}^0(x|\mathbf{z}_{t-1},x)-\frac{N-t+1}{N-t}\beta_t^0(x|\mathbf{z}_{t-1})\right]\Gamma_t^N(x).$$

Since $\beta_{t+1}^{0}(x|\mathbf{z}_{t-1},x)$ is independent of $(\mathbf{z}_{t-1},x)$, we can write

$$\beta_t^{0\prime}(x) = -\left[(\alpha_{N-t}-\alpha_{N-t+1})x + \beta_{t+1}^0(x) - \frac{N-t+1}{N-t}\beta_t^0(x)\right]\Gamma_t^N(x).$$

The unique solution is

$$\begin{aligned}\beta_t^0(x) &= \frac{N-t}{N-t+1}\int_x^{\bar{x}} \frac{\left((\alpha_{N-t}-\alpha_{N-t+1})z+\beta_{t+1}^0(z)\right)(N-t+1)f(z)(1-F(z))^{N-t}}{(1-F(x))^{N-t+1}}dz \\ &= \frac{N-t}{N-t+1}E\left[(\alpha_{N-t}-\alpha_{N-t-1})\,Z_t^{(N)}+\beta_{t+1}^0(Z_t^{(N)})|Z_t^{(N)}>x>Z_{t-1}^{(N)}\right] \\ &= \sum_{m=1}^{N-t}\frac{m}{N-t+1}E\left[(\alpha_m-\alpha_{m+1})\,Z_{N-m}^{(N)}|Z_t^{(N)}>x>Z_{t-1}^{(N)}\right],\end{aligned}$$

where the second equality restates the first equality as an expected value. The third equality is established as Claim 5 in the Supplemental Appendix. This establishes the result for round t and hence, by induction, the result for all t.

Next we establish that the bidding strategies are an equilibrium. It is sufficient to show that the following three-part claim holds for every t:

1. If $x \geq z_{t-1}$ then $x \in \arg\max_y \pi_t(y,x|\mathbf{z}_{t-1})$, i.e., it is optimal for a bidder with value x to bid $\beta_t^0(x)$ in round t.
2. If $x < z_{t-1}$ then $z_{t-1} \in \arg\max_y \pi_t(y,x|\mathbf{z}_{t-1})$.
3. $\frac{d\Pi_t(x|\mathbf{z}_{t-1})}{dx} \geq \alpha_{N-t+1}$.

Parts 2 and 3 are ancillary results needed to establish Part 1 for rounds prior to the last round. Part 2 is necessary to evaluate the consequence at round t of a bid y greater than the equilibrium bid x. In this case, a rival bidder with value $z_t > x$ may drop out before the bidder, and we need to evaluate the consequence for his optimal bid in round $t+1$. Part 2 shows that in this event it is optimal for the bidder to bid z_t (rather than x) in round $t+1$.

The proof is by induction. Consider round $N-1$. Any bid below z_{N-2} is strictly dominated by a bid of z_{N-2} since both bids result in the same position while a bid of z_{N-2} yields higher compensation. Suppose $y \geq z_{N-2}$. When bidders are risk neutral we have

$$\begin{aligned}\pi_{N-1}(y,x|\mathbf{z}_{N-2}) = & \int_{z_{N-2}}^{y} \left(\alpha_1 x - \beta^0_{N-1}(z_{N-1}) - \Sigma_{j=1}^{N-2}\tfrac{1}{N-j}d_j\right) g^{(N-1)}_{N-1}(z_{N-1}|z_{N-2})dz_{N-1} \\ & + \int_{y}^{\bar{x}} \left(\alpha_2 x + \beta^0_{N-1}(y) - \Sigma_{j=1}^{N-2}\tfrac{1}{N-j}d_j\right) g^{(N-1)}_{N-1}(z_{N-1}|z_{N-2})dz_{N-1}.\end{aligned}$$

Differentiating with respect to y yields $\partial\pi_{N-1}(y,x|\mathbf{z}_{N-2})/\partial y =$

$$\begin{aligned} & (\alpha_1 x - \beta^0_{N-1}(y) - \Sigma_{j=1}^{N-2}\tfrac{1}{N-j}d_j)g^{(N-1)}_{N-1}(y|z_{N-2}) \\ - & (\alpha_2 x + \beta^0_{N-1}(y) - \Sigma_{j=1}^{N-2}\tfrac{1}{N-j}d_j)g^{(N-1)}_{N-1}(y|z_{N-2}) \\ + & \beta^{0\prime}_{N-1}(y)(1 - G^{(N-1)}_{N-1}(y|z_{N-2})).\end{aligned}$$

Substituting the differential equation (1)

$$\beta^{0\prime}_{N-1}(y) = -[(\alpha_1-\alpha_2)y - 2\beta^0_{N-1}(y)]\Gamma^N_{N-1}(y) \tag{2}$$

into the expression for $\partial\pi_{N-1}(y,x|\mathbf{z}_{N-2})/\partial y$ yields

$$\partial\pi_{N-1}(y,x|\mathbf{z}_{N-2})/\partial y = (\alpha_1-\alpha_2)(x-y)g^{(N-1)}_{N-1}(y|z_{N-2}).$$

If $y < x$ then $\partial\pi_{N-1}(y,x|\mathbf{z}_{N-2})/\partial y > 0$, and if $y > x$ then $\partial\pi_{N-1}(y,x|\mathbf{z}_{N-2})/\partial y < 0$. Thus $x \geq z_{N-2}$ implies $x \in \arg\max_y \pi_{N-1}(y,x|\mathbf{z}_{N-2})$, which establishes Part 1.

If $x < z_{N-2}$, then any bid below z_{N-2} is strictly dominated. For any bid $y \geq z_{N-2}$ then $y > x$ and the above argument establishes $\partial\pi_{N-1}(y,x|\mathbf{z}_{N-2})/\partial y < 0$ for all $y \geq z_{N-2}$, i.e., $z_{N-2} \in \arg\max_y \pi_{N-1}(y,x|\mathbf{z}_{N-2})$. This establishes Part 2.

By the Envelope Theorem

$$\begin{aligned}\frac{d\Pi_{N-1}(x|\mathbf{z}_{N-2})}{dx} &= \left.\frac{\partial\pi_{N-1}(y,x|\mathbf{z}_{N-2})}{\partial x}\right|_{y=x} \\ &= \alpha_1 G^{(N-1)}_{N-1}(x|z_{N-2}) + \alpha_2(1 - G^{(N-1)}_{N-1}(x|z_{N-2})) \\ &\geq \alpha_2,\end{aligned}$$

which establishes Part 3. This completes the claim for round $N-1$.

Assume the three-part claim is true for rounds $t+1$ through $N-1$. We show it is true for round t. Let $\mathbf{z}_{t-1}$ be a sequence of dropout values. Suppose

$x \geq z_{t-1}$. Consider an active bidder in the t-th round whose value is x and who bids y. A bid below z_{t-1} is dominated. Since his payoff function differs in each case, we need to distinguish (i) $y \in [z_{t-1}, x]$ and (ii) $y > x$. In what follows, we denote the payoff to a bid of y as $\pi_t^L(y, x|\mathbf{z}_{t-1})$ if $y \in [z_{t-1}, x]$ and as $\pi_t^H(y, x|\mathbf{z}_{t-1})$ if $y \geq x$.

Case (i): Consider a bid $y \in [z_{t-1}, x]$. If the next highest value of a rival bidder is $z_t \in [z_{t-1}, y]$, then the bidder continues to round $t+1$ where, by the induction hypothesis, he optimally bids x and he has an expected payoff of $\Pi_{t+1}(x|\mathbf{z}_{t-1}, z_t)$. If $z_t \geq y$ he obtains a payoff of $\alpha_{N-t+1}x + \beta_t^0(y) - \Sigma_{j=1}^{t-1}\frac{1}{N-j}d_j$ in round t. Hence his payoff is

$$\begin{aligned}\pi_t^L(y, x|\mathbf{z}_{t-1}) = \; & \int_{z_{t-1}}^{y} \Pi_{t+1}(x|\mathbf{z}_{t-1}, z_t) g_t^{(N-1)}(z_t|z_{t-1})dz_t \\ & + \int_y^{\bar{x}} \left(\alpha_{N-t+1}x + \beta_t^0(y) - \Sigma_{j=1}^{t-1}\frac{1}{N-j}d_j\right) g_t^{(N-1)}(z_t|z_{t-1})dz_t.\end{aligned}$$

Differentiating with respect to y yields

$$\begin{aligned}\frac{\partial \pi_t^L(y, x|\mathbf{z}_{t-1})}{\partial y} = \; & [\Pi_{t+1}(x|\mathbf{z}_{t-1}, y) - \left(\alpha_{N-t+1}x + \beta_t^0(y) - \Sigma_{j=1}^{t-1}\frac{1}{N-j}d_j\right)] g_t^{(N-1)}(y|z_{t-1}) \\ & + \beta_t^{0\prime}(y)(1 - G_t^{(N-1)}(y|z_{t-1})).\end{aligned}$$

By the induction hypothesis we have

$$\frac{\partial^2 \pi_t^L(y, x|\mathbf{z}_{t-1})}{\partial x \partial y} = \left(\frac{d\Pi_{t+1}(x|\mathbf{z}_{t-1}, y)}{dx} - \alpha_{N-t+1}\right) g_t^{(N-1)}(y|z_{t-1}) \geq 0.$$

Case (ii): Consider a bid $y \geq x$. If the next highest value of a rival bidder is $z_t \in [z_{t-1}, x]$, then the bidder continues to round $t+1$ and, by the induction hypothesis, he bids x and obtains $\Pi_{t+1}(x|\mathbf{z}_{t-1}, z_t)$. If $z_t \in [x, y]$, then he continues to round $t+1$ and, by the part 2 of the induction hypothesis, he optimally bids z_t, he wins position $N-t$, and obtains compensation $\beta_{t+1}^0(z_t)$. His payoff is $\alpha_{N-t}x+$ $\beta_{t+1}^0(z_t) - \frac{1}{N-t}\beta_t^0(z_t) - \Sigma_{j=1}^{t-1}\frac{1}{N-j}d_j$. If $z_t > y$, then in round t his payoff is $\alpha_{N-t+1}x+$ $\beta_t^0(y|\mathbf{z}_{t-1}) - \Sigma_{j=1}^{t-1}\frac{1}{N-j}d_j$. Thus his expected payoff at round t is

$$\begin{aligned}\pi_t^H(y, x|\mathbf{z}_{t-1}) = \; & \int_{z_{t-1}}^{x} \Pi_{t+1}(x|\mathbf{z}_{t-1}, z_t) g_t^{(N-1)}(z_t|z_{t-1})dz_t \\ & + \int_x^{y} \left(\alpha_{N-t}x + \beta_{t+1}^0(z_t) - \frac{1}{N-t}\beta_t^0(z_t) - \Sigma_{j=1}^{t-1}\frac{1}{N-j}d_j\right) g_t^{(N-1)}(z_t|z_{t-1})dz_t \\ & + \int_y^{\bar{x}} \left(\alpha_{N-t+1}x + \beta_t^0(y) - \Sigma_{j=1}^{t-1}\frac{1}{N-j}d_j\right) g_t^{(N-1)}(z_t|z_{t-1})dz_t.\end{aligned}$$

Differentiating with respect to y yields

$$\begin{aligned}\frac{\partial \pi_t^H(y,x|\mathbf{z}_{t-1})}{\partial y} &= \left[\left((\alpha_{N-t}-\alpha_{N-t+1})x+\beta_{t+1}^0(y)-\frac{N-t+1}{N-t}\beta_t^0(y)\right)\right] g_t^{(N-1)}(y|z_{t-1}) \\ &\quad +\beta_t^{0\prime}(y)(1-G_t^{(N-1)}(y|z_{t-1})).\end{aligned}$$

Since $\alpha_{N-t}-\alpha_{N-t+1}\geq 0$ then

$$\frac{\partial^2 \pi_t^H(y,x|\mathbf{z}_{t-1})}{\partial x\partial y} = (\alpha_{N-t}-\alpha_{N-t+1})g_t^{(N-1)}(y|z_{t-1})\geq 0.$$

We have shown that

$$\left.\frac{\partial \pi_t^H(y,x|\mathbf{z}_{t-1})}{\partial y}\right|_{y=x} = \left.\frac{\partial \pi_t^L(y,x|\mathbf{z}_{t-1})}{\partial y}\right|_{y=x} = 0$$

and

$$\frac{\partial^2 \pi_t^L(y,x|\mathbf{z}_{t-1})}{\partial x\partial y}\geq 0 \text{ for } y\leq x \text{ and } \frac{\partial^2 \pi_t^H(y,x|\mathbf{z}_{t-1})}{\partial x\partial y}\geq 0 \text{ for } y\leq x.$$

hence by Lemma 0 in Van Essen and Wooders (2016) we have $x\in\arg\max_{y\in[z_{t-1},\bar{x}]}\pi_t(y,x|\mathbf{z}_{t-1})$. This establishes Part 1 for round t.

Suppose $x<z_{t-1}$. Any $y<z_{t-1}$ is strictly dominated by a bid of z_{t-1}. For $y\geq z_{t-1}$ we can write

$$\begin{aligned}\pi_t(y,x|\mathbf{z}_{t-1}) = &\int_{z_{t-1}}^{y}\left(\alpha_{N-t}x+\beta_{t+1}^0(z_t)-\tfrac{1}{N-t}\beta_t^0(z_t)-\Sigma_{j=1}^{t-1}\tfrac{1}{N-j}d_j\right)g_t^{(N-1)}(z_t|z_{t-1})dz_t \\ &+\int_{y}^{\bar{x}}\left(\alpha_{N-t+1}x+\beta_t^0(y)-\Sigma_{j=1}^{t-1}\tfrac{1}{N-j}d_j\right)g_t^{(N-1)}(z_t|z_{t-1})dz_t.\end{aligned}$$

Differentiating with respect to y and replacing $\beta_t^{0'}(y)$ with the equilibrium differential equation yields

$$\frac{\partial \pi_t(y,x|\mathbf{z}_{t-1})}{\partial y} = (\alpha_{N-t}-\alpha_{N-t+1})\,(x-y)\,g_t^{(N-1)}(y|z_{t-1})\leq 0$$

since $y>x$ and $\alpha_{N-t}-\alpha_{N-t+1}\geq 0$. Hence, if $x<z_{t-1}$ then $z_{t-1}\in\arg\max_y\pi_t(y,x|\mathbf{z}_{t-1})$. This establishes Part 2 for round t.

Finally, by the Envelope Theorem, we have

$$\begin{aligned}\frac{d\Pi_t(x|\mathbf{z}_{t-1})}{dx} &= \left.\frac{\partial \pi_t^L(y,x|\mathbf{z}_{t-1})}{\partial x}\right|_{y=x} = \left.\frac{\partial \pi_t^H(y,x|\mathbf{z}_{t-1})}{\partial x}\right|_{y=x} \\ &= \int_{z_{t-1}}^{x}\frac{d\Pi_{t+1}(x|\mathbf{z}_{t-1},z_t)}{dx}g_t^{(N-1)}(z_t|z_{t-1})dz_t+\alpha_{N-t+1}(1-G_t^{(N-1)}(x|z_{t-1}) \\ &\geq \alpha_{N-t}G_t^{(N-1)}(x|z_{t-1})+\alpha_{N-t+1}(1-G_t^{(N-1)}(x|z_{t-1}) \\ &\geq \alpha_{N-t+1}\end{aligned}$$

where the first inequality follows from the induction hypothesis and the second inequality follows since $\alpha_{N-t} \geq \alpha_{N-t+1}$. This establishes Part 3 for round t, and completes the proof by induction. □

Proof of Proposition 6: We first show that the bidding functions given in Proposition 6 are the unique solution to the system of differential equations in Proposition 4 when bidders have CARA utility. The proof is by induction. Consider round $N-1$. The differential equation for round $N-1$ is

$$-\theta e^{-\theta[\alpha_2 x+\beta^{\theta}_{N-1}(x|\mathbf{z}_{N-2})-\Sigma_{j=1}^{N-2}\frac{1}{N-j}d_j]}\beta^{\theta\prime}_{N-1}(x|\mathbf{z}_{N-2}) =$$
$$[e^{-\theta[\alpha_2 x+\beta^{\theta}_{N-1}(x|\mathbf{z}_{N-2})-\Sigma_{j=1}^{N-2}\frac{1}{N-j}d_j]} - e^{-\theta[\alpha_1 x-\beta^{\theta}_{N-1}(x|\mathbf{z}_{N-2})-\Sigma_{j=1}^{N-2}\frac{1}{N-j}d_j]}]\Gamma^N_{N-1}(x).$$

Dividing both sides by $e^{-\theta[\alpha_2 x-\beta^{\theta}_{N-1}(x|\mathbf{z}_{N-2})-\Sigma_{j=1}^{N-2}\frac{1}{N-j}d_j]}$ yields

$$-\theta e^{-2\theta\beta^{\theta}_{N-1}(x|\mathbf{z}_{N-2})}\beta^{\theta\prime}_{N-1}(x|\mathbf{z}_{N-2}) = [e^{-2\theta\beta^{\theta}_{N-1}(x|\mathbf{z}_{N-2})} - e^{-\theta(\alpha_1-\alpha_2)x}]\Gamma^N_{N-1}(x).$$

Multiplying both sides by $2(1-F(x))^2$ yields

$$\begin{aligned} & -2(1-F(x))^2\theta e^{-2\theta\beta^{\theta}_{N-1}(x|\mathbf{z}_{N-2})}\beta^{\theta\prime}_{N-1}(x|\mathbf{z}_{N-2}) \\ = \; & 2f(x)(1-F(x))[e^{-2\theta\beta^{\theta}_{N-1}(x|\mathbf{z}_{N-2})} - e^{-\theta(\alpha_1-\alpha_2)x}]. \end{aligned}$$

Rearranging

$$\begin{aligned} & -2\theta(1-F(x))^2 e^{-2\theta\beta^{\theta}_{N-1}(x|\mathbf{z}_{N-2})}\beta^{\theta\prime}_{N-1}(x|\mathbf{z}_{N-2}) - 2f(x)(1-F(x))e^{-2\theta\beta^{\theta}_{N-1}(x|\mathbf{z}_{N-2})} \\ = \; & -e^{-\theta(\alpha_1-\alpha_2)x}2f(x)(1-F(x)). \end{aligned}$$

or

$$\frac{d}{dx}\left(e^{-2\theta\beta^{\theta}_{N-1}(x|\mathbf{z}_{N-2})}(1-F(x))^2\right) = -e^{-\theta(\alpha_1-\alpha_2)x}2f(x)(1-F(x)).$$

By the Fundamental Theorem of Calculus

$$e^{-2\theta\beta^{\theta}_{N-1}(x|\mathbf{z}_{N-2})}(1-F(x))^2 = \int_0^x -e^{-\theta(\alpha_1-\alpha_2)s}2f(s)(1-F(s))ds + C.$$

Since the LHS is zero at $x=\bar{x}$ then

$$C = \int_0^{\bar{x}} e^{-\theta(\alpha_1-\alpha_2)s}2f(s)(1-F(s))ds.$$

The unique solution $\beta^{\theta}_{N-1}(x|\mathbf{z}_{N-2})$ therefore satisfies

$$e^{-2\theta\beta^{\theta}_{N-1}(x|\mathbf{z}_{N-2})}(1-F(x))^2 = \int_x^{\bar{x}} e^{-\theta(\alpha_1-\alpha_2)s} 2f(s)(1-F(s))ds.$$

Rearranging yields

$$\begin{aligned}\beta^{\theta}_{N-1}(x) &= -\frac{1}{2\theta}\ln\left(\frac{\int_x^{\bar{x}} e^{-\theta[(\alpha_1-\alpha_2)s]}2f(s)(1-F(s))ds}{(1-F(x))^2}\right) \\ &= -\frac{1}{2\theta}\ln\left(E\left[e^{-\theta(\alpha_1-\alpha_2)Z^{(N)}_{N-1}}|Z^{(N)}_{N-1} > x > Z^{(N)}_{N-2}\right]\right),\end{aligned}$$

which is $\beta^{\theta}_{N-1}(x)$, as given in Proposition 6.

Suppose $\beta^{\theta}_{t+1},\ldots,\beta^{\theta}_{N-1}$ are as given in Proposition 6 for rounds $t+1,\ldots,N-1$. Consider round t. The differential equation in the proof of Proposition 4 for round t is

$$\begin{aligned} & u'\left(\alpha_{N-t+1}x+\beta^{\theta}_t(x|\mathbf{z}_{t-1})-\sum_{j=1}^{t-1}\frac{1}{N-j}d_j\right)\beta^{\theta\prime}_t(x|\mathbf{z}_{t-1}) \\ = & -\left[\begin{array}{c} u\left(\alpha_{N-t}x+\beta^{\theta}_{t+1}(x)-\frac{1}{N-t}\beta^{\theta}_t(x|\mathbf{z}_{t-1})-\sum_{j=1}^{t-1}\frac{1}{N-j}d_j\right) \\ -u\left(\alpha_{N-t+1}x+\beta^{\theta}_t(x|\mathbf{z}_{t-1})-\sum_{j=1}^{t-1}\frac{1}{N-j}d_j\right)\end{array}\right]\Gamma^N_t(x),\end{aligned}$$

where we have used that $\beta^{\theta}_{t+1}(x)$ is independent of $\mathbf{z}_t$ by the induction hypothesis. We have

$$\begin{aligned} & \theta e^{-\theta[\alpha_{N-t+1}x+\beta^{\theta}_t(x|\mathbf{z}_{t-1})-\sum_{j=1}^{t-1}\frac{1}{N-j}d_j]}\beta^{\theta\prime}_t(x|\mathbf{z}_{t-1}) \\ = & -\left[e^{-\theta[\alpha_{N-t+1}x+\beta^{\theta}_t(x|\mathbf{z}_{t-1})-\sum_{j=1}^{t-1}\frac{1}{N-j}d_j]} - e^{-\theta[\alpha_{N-t}x+\beta^{\theta}_{t+1}(x)-\frac{1}{N-t}\beta^{\theta}_t(x|\mathbf{z}_{t-1})-\sum_{j=1}^{t-1}\frac{1}{N-j}d_j]}\right]\Gamma^N_t(x).\end{aligned}$$

Dividing both sides by $e^{-\theta[\alpha_{N-t+1}x-\frac{1}{N-t}\beta^{\theta}_t(x|\mathbf{z}_{t-1})-\sum_{j=1}^{t-1}\frac{1}{N-j}d_j]}$ yields

$$\begin{aligned} & -\theta e^{-\theta\frac{N-t+1}{N-t}\beta^{\theta}_t(x|\mathbf{z}_{t-1})}\beta^{\theta\prime}_t(x|\mathbf{z}_{t-1}) \\ = & [e^{-\theta\frac{N-t+1}{N-t}\beta^{\theta}_t(x|\mathbf{z}_{t-1})} - e^{-\theta[(\alpha_{N-t}-\alpha_{N-t+1})x+\beta^{\theta}_{t+1}(x)]}](N-t)\frac{f(x)}{1-F(x)}.\end{aligned}$$

Multiplying both sides by $\frac{N-t+1}{N-t}(1-F(x))^{N-t+1}$ yields

$$\begin{aligned} & -\theta\frac{N-t+1}{N-t}(1-F(x))^{N-t+1}e^{-\theta\frac{N-t+1}{N-t}\beta^{\theta}_t(x|\mathbf{z}_{t-1})}\beta^{\theta\prime}_t(x|\mathbf{z}_{t-1}) \\ = & [e^{-\theta\frac{N-t+1}{N-t}\beta^{\theta}_t(x|\mathbf{z}_{t-1})} - e^{-\theta[(\alpha_{N-t}-\alpha_{N-t+1})x+\beta^{\theta}_{t+1}(x)]}](N-t+1)(1-F(x))^{N-t}f(x).\end{aligned}$$

This equation can be rewritten as

$$\begin{aligned} & -\theta\frac{N-t+1}{N-t}(1-F(x))^{N-t+1}e^{-\theta\frac{N-t+1}{N-t}\beta_t^\theta(x|\mathbf{z}_{t-1})}\beta_t^{\theta\prime}(x|\mathbf{z}_{t-1}) \\ & -e^{-\theta\frac{N-t+1}{N-t}\beta_t^\theta(x|\mathbf{z}_{t-1})}(N-t+1)(1-F(x))^{N-t}f(x) \\ = & -e^{-\theta[(\alpha_{N-t}-\alpha_{N-t+1})x+\beta_{t+1}^\theta(x)]}(N-t+1)(1-F(x))^{N-t}f(x), \end{aligned}$$

i.e.,

$$\frac{d}{dx}((1-F(x))^{N-t+1}e^{-\theta\frac{N-t+1}{N-t}\beta_t^\theta(x|\mathbf{z}_{t-1})}) = -e^{-\theta[(\alpha_{N-t}-\alpha_{N-t+1})x+\beta_{t+1}^\theta(x)]}(N-t+1)(1-F(x))^{N-t}f(x).$$

By the Fundamental Theorem of Calculus

$$\begin{aligned} & (1-F(x))^{N-t+1}e^{-\theta\frac{N-t+1}{N-t}\beta_t^\theta(x|\mathbf{z}_{t-1})} \\ = & \int_0^x -e^{-\theta[(\alpha_{N-t}-\alpha_{N-t+1})s+\beta_{t+1}^\theta(s)]}(N-t+1)(1-F(s))^{N-t}f(s)ds+C. \end{aligned}$$

Since the LHS is zero at $x=\bar{x}$ then

$$C=\int_0^{\bar{x}} e^{-\theta[(\alpha_{N-t}-\alpha_{N-t+1})s+\beta_{t+1}^\theta(s)]}(N-t+1)(1-F(s))^{N-t}f(s)ds.$$

Hence the unique solution $\beta_t^\theta(x|\mathbf{z}_{t-1})$ satisfies

$$(1-F(x))^{N-t+1}e^{-\theta\frac{N-t+1}{N-t}\beta_t^\theta(x|\mathbf{z}_{t-1})} = \int_x^{\bar{x}} e^{-\theta[(\alpha_{N-t}-\alpha_{N-t+1})s+\beta_{t+1}^\theta(s)]}(N-t+1)(1-F(s))^{N-t}f(s)ds.$$

Thus

$$\begin{aligned} \beta_t^\theta(x) &= -\frac{N-t}{(N-t+1)\theta}\ln\left(\int_x^{\bar{x}} e^{-\theta[(\alpha_{N-t}-\alpha_{N-t+1})s+\beta_{t+1}^\theta(s)]}(N-t+1)\frac{(1-F(s))^{N-t}}{(1-F(x))^{N-t+1}}f(s)ds\right) \\ &= -\frac{N-t}{(N-t+1)\theta}\ln\left\{E\left[e^{-\theta\left((\alpha_{N-t}-\alpha_{N-t+1})Z_t^{(N)}+\beta_{t+1}^\theta(Z_t^{(N)})\right)}|Z_t^{(N)}>x>Z_{t-1}^{(N)}\right]\right\}, \end{aligned}$$

which establishes the result for round t and hence, by induction, the result for all t.

Next we establish that the bidding strategies are an equilibrium. It is sufficient to show that the following two-part claim holds for every t:

1. If $x \geq z_{t-1}$ then $x \in \arg\max_y \pi_t(y,x|\mathbf{z}_{t-1})$, i.e., it is optimal for a bidder with value x to bid $\beta_t^\theta(x)$ in round t.

2. If $x < z_{t-1}$ then $z_{t-1} \in \arg\max_y \pi_t(y, x|\mathbf{z}_{t-1})$.

The proof is by induction. Consider round $N-1$. Suppose that $x \geq z_{N-2}$. Consider an active bidder whose value is x and who bids y. Any bid below z_{N-2} is strictly dominated by a bid of z_{N-2} since both bids result in the same position while a bid of z_{N-2} yields higher compensation. Hence consider bids $y \geq z_{N-2}$.

With a bid of y the bidder wins Position 1 and obtains $\alpha_1 x - \beta^{\theta}_{N-1}(z_{N-1}) - \Sigma_{j=1}^{N-2}\frac{1}{N-j}d_j$ if $y > z_{N-1}$, and he obtains $\alpha_2 x + \beta^{\theta}_{N-1}(y) - \Sigma_{j=1}^{N-2}\frac{1}{N-j}d_j$ if $y < z_{N-1}$. Hence

$$\begin{aligned}\pi_{N-1}(y, x|\mathbf{z}_{N-2}) = & \int_{z_{N-2}}^{y} u\left(\alpha_1 x - \beta^{\theta}_{N-1}(z_{N-1}) - \Sigma_{j=1}^{N-2}\frac{1}{N-j}d_j\right) g^{(N-1)}_{N-1}(z_{N-1}|z_{N-2})dz_{N-1} \\ & + \int_{y}^{\bar{x}} u\left(\alpha_2 x + \beta^{\theta}_{N-1}(y) - \Sigma_{j=1}^{N-2}\frac{1}{N-j}d_j\right) g^{(N-1)}_{N-1}(z_{N-1}|z_{N-2})dz_{N-1}.\end{aligned}$$

Differentiating with respect to y yields $\partial\pi_{N-1}(y, x|\mathbf{z}_{N-2})/\partial y =$

$$\begin{aligned} & u(\alpha_1 x - \beta^{\theta}_{N-1}(y) - \Sigma_{j=1}^{N-2}\tfrac{1}{N-j}d_j)g^{(N-1)}_{N-1}(y|z_{N-2}) \\ - & u(\alpha_2 x + \beta^{\theta}_{N-1}(y) - \Sigma_{j=1}^{N-2}\tfrac{1}{N-j}d_j)g^{(N-1)}_{N-1}(y|z_{N-2}) \\ + & u'(\alpha_2 x + \beta^{\theta}_{N-1}(y) - \Sigma_{j=1}^{N-2}\tfrac{1}{N-j}d_j)\beta^{\theta\prime}_{N-1}(y)(1 - G^{(N-1)}_{N-1}(y|z_{N-2})). \end{aligned} \tag{3}$$

The necessary condition given in Proposition 4 for the general utility function u is

$$\begin{aligned} & u'\left(\alpha_2 y + \beta^{\theta}_{N-1}(y) - \sum_{j=1}^{N-2}\frac{1}{N-j}d_j\right)\beta^{\theta\prime}_{N-1}(y) \\ = & -\left[\begin{array}{c} u\left(\alpha_1 y - \beta^{\theta}_{N-1}(y) - \sum_{j=1}^{N-2}\frac{1}{N-j}d_j\right) \\ -u\left(\alpha_2 y + \beta^{\theta}_{N-1}(y) - \sum_{j=1}^{N-2}\frac{1}{N-j}d_j\right)\end{array}\right]\Gamma^{N}_{N-1}(y). \end{aligned}$$

Substituting this expression into $\partial\pi_{N-1}(y, x|\mathbf{z}_{N-2})/\partial y$ yields

$$\begin{aligned} & u(\alpha_1 x - \beta^{\theta}_{N-1}(y) - \Sigma_{j=1}^{N-2}\tfrac{1}{N-j}d_j)g^{(N-1)}_{N-1}(y|z_{N-2}) \\ - & u(\alpha_2 x + \beta^{\theta}_{N-1}(y) - \Sigma_{j=1}^{N-2}\tfrac{1}{N-j}d_j)g^{(N-1)}_{N-1}(y|z_{N-2}) \\ - & \frac{u'(\alpha_2 x + \beta^{\theta}_{N-1}(y) - \Sigma_{j=1}^{N-2}\frac{1}{N-j}d_j)}{u'(\alpha_2 y + \beta^{\theta}_{N-1}(y) - \Sigma_{j=1}^{N-2}\frac{1}{N-j}d_j)}\left[\begin{array}{c} u\left(\alpha_1 y - \beta^{\theta}_{N-1}(y) - \sum_{j=1}^{N-2}\frac{1}{N-j}d_j\right) \\ -u\left(\alpha_2 y + \beta^{\theta}_{N-1}(y) - \sum_{j=1}^{N-2}\frac{1}{N-j}d_j\right)\end{array}\right] g^{(N-1)}_{N-1}(y|z_{N-2}). \end{aligned}$$

This derivative has the same sign as

$$
\begin{aligned}
& u(\alpha_1 x - \beta^\theta_{N-1}(y) - \Sigma_{j=1}^{N-2}\tfrac{1}{N-j}d_j) - u(\alpha_2 x + \beta^\theta_{N-1}(y) - \Sigma_{j=1}^{N-2}\tfrac{1}{N-j}d_j) \\
- & \frac{u'(\alpha_2 x + \beta^\theta_{N-1}(y) - \Sigma_{j=1}^{N-2}\frac{1}{N-j}d_j)}{u'(\alpha_2 y + \beta^\theta_{N-1}(y) - \Sigma_{j=1}^{N-2}\frac{1}{N-j}d_j)} \left[\begin{array}{c} u\left(\alpha_1 y - \beta^\theta_{N-1}(y) - \sum_{j=1}^{N-2}\frac{1}{N-j}d_j\right) \\ -u\left(\alpha_2 y + \beta^\theta_{N-1}(y) - \sum_{j=1}^{N-2}\frac{1}{N-j}d_j\right) \end{array} \right].
\end{aligned}
$$

Using that $u(x)$ has CARA we can write

$$
\frac{u'(\alpha_2 x + \beta^\theta_{N-1}(y) - \Sigma_{j=1}^{N-2}\frac{1}{N-j}d_j)}{u'(\alpha_2 y + \beta^\theta_{N-1}(y) - \Sigma_{j=1}^{N-2}\frac{1}{N-j}d_j)} = e^{-\theta\alpha_2(x-y)}.
$$

We can write

$$
u(\alpha_1 x - \beta^\theta_{N-1}(y) - \Sigma_{j=1}^{N-2}\frac{1}{N-j}d_j) - u(\alpha_2 x + \beta^\theta_{N-1}(y) - \Sigma_{j=1}^{N-2}\frac{1}{N-j}d_j)
$$

as

$$
\frac{e^{-\theta[\alpha_2 x + \beta^\theta_{N-1}(y) - \Sigma_{j=1}^{N-2}\frac{1}{N-j}d_j]} - e^{-\theta[\alpha_1 x - \beta^\theta_{N-1}(y) - \Sigma_{j=1}^{N-2}\frac{1}{N-j}d_j]}}{\theta}.
$$

Hence the sign of the derivative is the same as the sign of

$$
\begin{aligned}
& e^{-\theta[\alpha_2 x + \beta^\theta_{N-1}(y) - \Sigma_{j=1}^{N-2}\frac{1}{N-j}d_j]} - e^{-\theta[\alpha_1 x - \beta^\theta_{N-1}(y) - \Sigma_{j=1}^{N-2}\frac{1}{N-j}d_j]} \\
& - e^{-\theta\alpha_2(x-y)}\left(e^{-\theta[\alpha_2 y + \beta^\theta_{N-1}(y) - \Sigma_{j=1}^{N-2}\frac{1}{N-j}d_j]} - e^{-\theta[\alpha_1 y - \beta^\theta_{N-1}(y) - \Sigma_{j=1}^{N-2}\frac{1}{N-j}d_j]}\right).
\end{aligned}
$$

We can rewrite this as

$$
-e^{-\theta[\alpha_1 x - \beta^\theta_{N-1}(y) - \Sigma_{j=1}^{N-2}\frac{1}{N-j}d_j]} + e^{-\theta\alpha_2(x-y)} e^{-\theta[\alpha_1 y - \beta^\theta_{N-1}(y) - \Sigma_{j=1}^{N-2}\frac{1}{N-j}d_j]}
$$

which has the same sign as

$$
-e^{-\theta\alpha_1 x} + e^{-\theta\alpha_2(x-y)} e^{-\theta\alpha_1 y}
$$

which has the same sign as

$$
-e^{-\theta\alpha_1(x-y)} + e^{-\theta\alpha_2(x-y)}.
$$

Since $\alpha_1 > \alpha_2$, this expression is positive if $y < x$ and is negative if $y > x$. Thus $\partial\pi_{N-1}(y, x|\mathbf{z}_{N-2})/\partial y > 0$ if $y < x$ and $\partial\pi_{N-1}(y, x|\mathbf{z}_{N-2})/\partial y < 0$ if $y > x$.

We have shown if $x \geq z_{N-2}$ then $x \in \arg\max_y \pi_{N-1}(y, x|\mathbf{z}_{N-2})$, which establishes part 1 of the two-part claim. If $x < z_{N-2}$, then $y \geq z_{N-2}$ (since any bid below z_{N-2} is strictly dominated) implies $y \geq z_{N-2} > x$ and the above argument establishes bidding z_{N-2} is optimal since $\partial\pi_{N-1}(y,x|\mathbf{z}_{N-2})/\partial y < 0$ for all $y \geq z_{N-2}$, i.e., $z_{N-2} \in \arg\max_y \pi_{N-1}(y,x|\mathbf{z}_{N-2})$. This establishes part 2 of the two-part claim for round $N-1$.

Assume the two-part claim is true for rounds $t+1$ through $N-1$. We show it is true for round t. Let $\mathbf{z}_{t-1}$ be arbitrary. Suppose $x \geq z_{t-1}$. Consider an active bidder in the t-th round whose value is x and who bids as though his value is $y \geq z_{t-1}$. A bid below z_{t-1} is not optimal. We need to distinguish between two cases: (i) $y \in [z_{t-1}, x]$ and (ii) $y > x$, since his payoff function differs in each case. In what follows, we denote the payoff to a bid of y as $\pi_t^L(y,x|\mathbf{z}_{t-1})$ if $y \in [z_{t-1}, x]$ and as $\pi_t^H(y,x|\mathbf{z}_{t-1})$ if $y \geq x$.

Case (i): Consider a bid $y \in [z_{t-1}, x]$. If $z_t \in [z_{t-1}, y]$ the bidder continues to round $t+1$ where, by the induction hypothesis, he optimally bids x and he has an expected payoff of $\Pi_{t+1}(x|\mathbf{z}_{t-1}, z_t)$. If $z_t \geq y$ he obtains a payoff of $\alpha_{N-t+1}x + \beta_t^\theta(y) - \Sigma_{j=1}^{t-1}\frac{1}{N-j}d_j$ in round t. Hence his payoff is

$$\begin{aligned}\pi_t^L(y,x|\mathbf{z}_{t-1}) = \quad & \textstyle\int_{z_{t-1}}^{y} \Pi_{t+1}(x|\mathbf{z}_{t-1}, z_t) g_t^{(N-1)}(z_t|z_{t-1})dz_t \\ & + \textstyle\int_{y}^{\bar{x}} u\left(\alpha_{N-t+1}x + \beta_t^\theta(y) - \Sigma_{j=1}^{t-1}\frac{1}{N-j}d_j\right) g_t^{(N-1)}(z_t|z_{t-1})dz_t.\end{aligned}$$

Differentiating with respect to y yields

$$\begin{aligned}\frac{\partial\pi_t^L(y,x|\mathbf{z}_{t-1})}{\partial y} = \quad & [\Pi_{t+1}(x|\mathbf{z}_{t-1}, y) - u\left(\alpha_{N-t+1}x + \beta_t^\theta(y) - \Sigma_{j=1}^{t-1}\tfrac{1}{N-j}d_j\right)]g_t^{(N-1)}(y|z_{t-1}) \\ & + u'\left(\alpha_{N-t+1}x + \beta_t^\theta(y) - \Sigma_{j=1}^{t-1}\tfrac{1}{N-j}d_j\right)\beta_t^{\theta\prime}(y)(1 - G_t^{(N-1)}(y|z_{t-1})).\end{aligned}$$

Rewriting

$$\begin{aligned}\frac{\partial\pi_t^L(y,x|\mathbf{z}_{t-1})}{\partial y} = \quad & [\Pi_{t+1}(x|\mathbf{z}_{t-1}, y) - \tfrac{1-e^{-\theta[\alpha_{N-t+1}x + \beta_t^\theta(y) - \Sigma_{j=1}^{t-1}\frac{1}{N-j}d_j]}}{\theta}]g_t^{(N-1)}(y|z_{t-1}) \\ & + e^{-\theta[\alpha_{N-t+1}x + \beta_t^\theta(y) - \Sigma_{j=1}^{t-1}\frac{1}{N-j}d_j]}\beta_t^{\theta\prime}(y)(1 - G_t^{(N-1)}(y|z_{t-1})).\end{aligned}$$

Using the expression for $\beta_t^{\theta\prime}(y)$ from the necessary condition for equilibrium from Proposition 4 for round t and substituting yields

$$\begin{aligned}\frac{\partial\pi_t^L(y,x|\mathbf{z}_{t-1})}{\partial y} = \quad & [\Pi_{t+1}(x|\mathbf{z}_{t-1}, y) - \tfrac{1-e^{-\theta[\alpha_{N-t+1}x + \beta_t^\theta(y) - \Sigma_{j=1}^{t-1}\frac{1}{N-j}d_j]}}{\theta}]g_t^{(N-1)}(y|z_{t-1}) \\ & - e^{-\theta\alpha_{N-t+1}(x-y)}\frac{1}{\theta}\begin{bmatrix} e^{-\theta[\alpha_{N-t+1}y + \beta_t^\theta(y) - \sum_{j=1}^{t-1}\frac{1}{N-j}d_j]} \\ -e^{-\theta[\alpha_{N-t}y + \beta_{t+1}^\theta(y) - \frac{1}{N-t}\beta_t^\theta(y) - \sum_{j=1}^{t-1}\frac{1}{N-j}d_j]}\end{bmatrix} g_t^{(N-1)}(y|z_{t-1}).\end{aligned}$$

Simplifying yields $\partial \pi_t^L(y, x|\mathbf{z}_{t-1})/\partial y$ as

$$\left(\Pi_{t+1}(x|\mathbf{z}_{t-1}, y) - \frac{1}{\theta}\left[1 - e^{-\theta[\alpha_{N-t}y + \alpha_{N-t+1}(x-y) + \beta_{t+1}^\theta(y) - \frac{1}{N-t}\beta_t^\theta(y) - \sum_{j=1}^{t-1}\frac{1}{N-j}d_j]}\right]\right) g_t^{(N-1)}(y|z_{t-1}).$$

We show that $\partial \pi_t^L(y, x|\mathbf{z}_{t-1})/\partial y > 0$ for $y < x$. If the bid at round t is y, then $\Pi_{t+1}(x|\mathbf{z}_{t-1}, y)$ is the equilibrium payoff at round $t+1$ of a bidder with value x. If he were to deviate from equilibrium and bid y at round $t+1$, then he obtains position $N-t$ (since y is the smallest value of a rival bidder) and he receives $\beta_{t+1}^\theta(y)$ at round $t+1$ and pays $\frac{1}{N-t}\beta_t^\theta(y) + \sum_{j=1}^{t-1}\frac{1}{N-j}d_j$. By the induction hypothesis, this payoff is less than his equilibrium payoff, i.e.,

$$\Pi_{t+1}(x|\mathbf{z}_{t-1}, y) > \frac{1}{\theta}\left[1 - e^{-\theta[\alpha_{N-t}x + \beta_{t+1}^\theta(y) - \frac{1}{N-t}\beta_t^\theta(y) - \sum_{j=1}^{t-1}\frac{1}{N-j}d_j]}\right].$$

Since $\alpha_{N-t} > \alpha_{N-t+1}$ and $x > y$ we have

$$\alpha_{N-t}x > \alpha_{N-t}y + \alpha_{N-t+1}(x-y)$$

and hence

$$\begin{aligned} & \frac{1}{\theta}\left[1 - e^{-\theta[\alpha_{N-t}x + \beta_{t+1}^\theta(y) - \frac{1}{N-t}\beta_t^\theta(y) - \sum_{j=1}^{t-1}\frac{d_j}{N-j}]}\right] \\ > \; & \frac{1}{\theta}\left[1 - e^{-\theta[\alpha_{N-t}y + \alpha_{N-t+1}(x-y) + \beta_{t+1}^\theta(y) - \frac{1}{N-t}\beta_t^\theta(y) - \sum_{j=1}^{t-1}\frac{d_j}{N-j}]}\right]. \end{aligned}$$

Thus $\Pi_{t+1}(x|\mathbf{z}_{t-1}, y)$ is greater than the RHS of this inequality and hence $\partial \pi_t^L(y, x|\mathbf{z}_{t-1})/\partial y > 0$ for $y < x$.

Case (ii): Consider a bid $y \geq x$. If $z_t \in [z_{t-1}, x]$, then the bidder continues to round $t+1$ and, by part 1 of induction hypothesis, he bids x and obtains $\Pi_{t+1}(x|\mathbf{z}_{t-1}, z_t)$. If $z_t \in [x, y]$, then he continues to round $t+1$ and, by part 2 of the induction hypothesis, he bids z_t and obtains a payoff of

$$\alpha_{N-t}x + \beta_{t+1}^\theta(z_t) - \frac{1}{N-t}\beta_t^\theta(z_t) - \Sigma_{j=1}^{t-1}\frac{1}{N-j}d_j.$$

If $z_t > y$ then in round t he obtains position $N-t+1$ and his payoff is $\alpha_{N-t+1}x +$ $\beta_t^\theta(y) - \Sigma_{j=1}^{t-1}\frac{1}{N-j}d_j$. Thus his expected payoff at round t is

$$\begin{aligned} \pi_t^H(y, x|\mathbf{z}_{t-1}) = \; & \int_{z_{t-1}}^x \Pi_{t+1}(x|\mathbf{z}_{t-1}, z_t) g_t^{(N-1)}(z_t|z_{t-1}) dz_t \\ & + \int_x^y u(\alpha_{N-t}x + \beta_{t+1}^\theta(z_t) - \tfrac{1}{N-t}\beta_t^\theta(z_t) - \Sigma_{j=1}^{t-1}\tfrac{1}{N-j}d_j) g_t^{(N-1)}(z_t|z_{t-1}) dz_t, \\ & + \int_y^{\bar{x}} u\left(\alpha_{N-t+1}x + \beta_t^\theta(y) - \Sigma_{j=1}^{t-1}\tfrac{1}{N-j}d_j\right) g_t^{(N-1)}(z_t|z_{t-1}) dz_t. \end{aligned}$$

Differentiating with respect to y yields

$$\frac{\partial \pi_t^H(y,x|\mathbf{z}_{t-1})}{\partial y} = \begin{bmatrix} u\left(\alpha_{N-t}x+\beta_{t+1}^\theta(y)-\frac{1}{N-t}\beta_t^\theta(y)-\Sigma_{j=1}^{t-1}\frac{1}{N-j}d_j\right) \\ -u\left(\alpha_{N-t+1}x+\beta_t^\theta(y)-\Sigma_{j=1}^{t-1}\frac{1}{N-j}d_j\right) \end{bmatrix} g_t^{(N-1)}(y|z_{t-1})$$
$$+u'\left(\alpha_{N-t+1}x+\beta_t^\theta(y)-\Sigma_{j=1}^{t-1}\frac{1}{N-j}d_j\right)\beta_t^{\theta\prime}(y)(1-G_t^{(N-1)}(y|z_{t-1})).$$

Using the expression for $\beta_t^{\theta\prime}(y)$ from the necessary condition for equilibrium from Proposition 4 for round t and substituting gives $\partial\pi_t^H(y,x|\mathbf{z}_{t-1})/\partial y$ as

$$\begin{bmatrix} u(\alpha_{N-t}x+\beta_{t+1}^\theta(y)-\frac{1}{N-t}\beta_t^\theta(y)-\Sigma_{j=1}^{t-1}\frac{1}{N-j}d_j) \\ -u(\alpha_{N-t+1}x+\beta_t^\theta(y)-\Sigma_{j=1}^{t-1}\frac{1}{N-j}d_j) \end{bmatrix} g_t^{(N-1)}(y|z_{t-1})$$
$$-\frac{u'(\alpha_{N-t+1}x+\beta_t^\theta(y)-\Sigma_{j=1}^{t-1}\frac{1}{N-j}d_j)}{u'(\alpha_{N-t+1}y+\beta_t^\theta(y)-\Sigma_{j=1}^{t-1}\frac{1}{N-j}d_j)}$$
$$\times\begin{bmatrix} u(\alpha_{N-t}y+\beta_{t+1}^\theta(y)-\frac{1}{N-t}\beta_t^\theta(y)-\Sigma_{j=1}^{t-1}\frac{1}{N-j}d_j) \\ -u(\alpha_{N-t+1}y+\beta_t^\theta(y)-\Sigma_{j=1}^{t-1}\frac{1}{N-j}d_j) \end{bmatrix} g_t^{(N-1)}(y|z_{t-1}).$$

Since bidders have CARA preferences, then $\partial\pi_t^H(y,x|\mathbf{z}_{t-1})/\partial y$ is

$$\frac{1}{\theta}\begin{bmatrix} e^{-\theta[\alpha_{N-t+1}x+\beta_t^\theta(y)-\Sigma_{j=1}^{t-1}\frac{1}{N-j}d_j]} \\ -e^{-\theta[\alpha_{N-t}x+\beta_{t+1}^\theta(y)-\frac{1}{N-t}\beta_t^\theta(y)-\Sigma_{j=1}^{t-1}\frac{1}{N-j}d_j]} \end{bmatrix} g_t^{(N-1)}(y|z_{t-1})$$
$$-e^{-\theta\alpha_{N-t+1}(x-y)}\frac{1}{\theta}\begin{bmatrix} e^{-\theta[\alpha_{N-t+1}y+\beta_t^\theta(y)-\Sigma_{j=1}^{t-1}\frac{1}{N-j}d_j]} \\ -e^{-\theta[\alpha_{N-t}y+\beta_{t+1}^\theta(y)-\frac{1}{N-t}\beta_t^\theta(y)-\Sigma_{j=1}^{t-1}\frac{1}{N-j}d_j]} \end{bmatrix} g_t^{(N-1)}(y|z_{t-1}).$$
$$= \frac{1}{\theta}\begin{bmatrix} e^{-\theta[\alpha_{N-t}y+\alpha_{N-t+1}(x-y)+\beta_{t+1}^\theta(y)-\frac{1}{N-t}\beta_t^\theta(y)-\Sigma_{j=1}^{t-1}\frac{1}{N-j}d_j]} \\ -e^{-\theta[\alpha_{N-t}x+\beta_{t+1}^\theta(y)-\frac{1}{N-t}\beta_t^\theta(y)-\Sigma_{j=1}^{t-1}\frac{1}{N-j}d_j]} \end{bmatrix} g_t^{(N-1)}(y|z_{t-1}).$$

This has the same sign as

$$e^{-\theta\alpha_{N-t+1}(x-y)}-e^{-\theta\alpha_{N-t}(x-y)},$$

which is negative since $\alpha_{N-t}>\alpha_{N-t+1}$ and $y>x$.

We have shown if $x\geq z_{t-1}$ then $x\in\arg\max_y \pi_t(y,x|\mathbf{z}_{t-1})$. If $x<z_{t-1}$, then $y\geq z_{t-1}$ (since any bid below z_{t-1} is strictly dominated) implies $y\geq z_{t-1}>x$ and the above argument establishes bidding z_{t-1} is optimal since $\partial\pi_t(y,x|\mathbf{z}_{t-1})/\partial y<0$ for all $y\geq z_{t-1}$, i.e., $z_{t-1}\in\arg\max_y \pi_t(y,x|\mathbf{z}_{t-1})$.

This establishes the two-part claim for round t, and completes the proof by induction. □

Proof of Proposition 7: We first show that for each t we have that $\beta_t^0(x) > \beta_t^\theta(x)$ for $\theta > 0$ and $x < \bar{x}$. Consider round $t = N-1$. Since e^{-x} is convex, Jensen's inequality implies

$$e^{-E\left[\theta(\alpha_1-\alpha_2)Z_{N-1}^{(N)}|Z_{N-1}^{(N)}>x>Z_{N-2}^{(N)}\right]} < E\left[e^{-\theta(\alpha_1-\alpha_2)Z_{N-1}^{(N)}}|Z_{N-1}^{(N)} > x > Z_{N-2}^{(N)}\right].$$

Taking the log of both sides and then dividing both sides by -2θ yields

$$\begin{aligned}\beta_{N-1}^0(x) &= \frac{1}{2}E\left[(\alpha_1-\alpha_2)\, Z_{N-1}^{(N)}|Z_{N-1}^{(N)} > x > Z_{N-2}^{(N)}\right]\\ &> -\frac{1}{2\theta}\ln\left\{E\left[e^{-\theta(\alpha_1-\alpha_2)Z_{N-1}^{(N)}}|Z_{N-1}^{(N)} > x > Z_{N-2}^{(N)}\right]\right\}\\ &= \beta_{N-1}^\theta(x).\end{aligned}$$

Assume that $\beta_{t+1}^0(x) > \beta_{t+1}^\theta(x)$ for $x < \bar{x}$. We show that $\beta_t^0(x) > \beta_t^\theta(x)$ for $x < \bar{x}$. For $z < \bar{x}$ we have

$$(\alpha_{N-t} - \alpha_{N-t+1})\, z + \beta_{t+1}^0(z) > (\alpha_{N-t} - \alpha_{N-t+1})\, z + \beta_{t+1}^\theta(z).$$

Multiplying through by $-\theta$ and applying the exponential function to both sides gives

$$e^{-\theta\left[(\alpha_{N-t}-\alpha_{N-t+1})z+\beta_{t+1}^0(z)\right]} < e^{-\theta\left[(\alpha_{N-t}-\alpha_{N-t+1})z+\beta_{t+1}^\theta(z)\right]}.$$

Hence

$$\begin{aligned}& E\left[e^{-\theta\left[(\alpha_{N-t}-\alpha_{N-t+1})Z_t^{(N)}+\beta_{t+1}^0(Z_t^{(N)})\right]}|Z_t^{(N)} > x > Z_{t-1}^{(N)}\right]\\ <\; & E\left[e^{-\theta\left[(\alpha_{N-t}-\alpha_{N-t+1})Z_t^{(N)}+\beta_{t+1}^\theta(Z_t^{(N)})\right]}|Z_t^{(N)} > x > Z_{t-1}^{(N)}\right].\end{aligned}$$

By Jensen's inequality, we have

$$\begin{aligned}& e^{-\theta E\left[(\alpha_{N-t}-\alpha_{N-t+1})Z_t^{(N)}+\beta_{t+1}^0(Z_t^{(N)})|Z_t^{(N)}>x>Z_{t-1}^{(N)}\right]}\\ <\; & E\left[e^{-\theta\left[(\alpha_{N-t}-\alpha_{N-t+1})Z_t^{(N)}+\beta_{t+1}^0(Z_t^{(N)})\right]}|Z_t^{(N)} > x > Z_{t-1}^{(N)}\right],\end{aligned}$$

and thus

$$
\begin{aligned}
& e^{-\theta E\left[(\alpha_{N-t}-\alpha_{N-t+1})Z_t^{(N)}+\beta_{t+1}^0(Z_t^{(N)})|Z_t^{(N)}>x>Z_{t-1}^{(N)}\right]} \\
< \; & E\left[e^{-\theta\left[(\alpha_{N-t}-\alpha_{N-t+1})Z_t^{(N)}+\beta_{t+1}^\theta(Z_t^{(N)})\right]}|Z_t^{(N)} > x > Z_{t-1}^{(N)}\right].
\end{aligned}
$$

Taking the log of both sides and then multiplying both sides by $-(N-t)/((N-t+1)\,\theta)$ yields $\beta_t^0(x) > \beta_t^\theta(x)$. We have shown for each t that $\beta_t^0(x) > \beta_t^\theta(x)$ for $\theta > 0$ and $x < \bar{x}$.

Next we show that for each t we have that $\beta_t^\theta(x) > \underline{\beta}_t(x)$ for $\theta > 0$ and $x < \bar{x}$. Consider $t = N-1$. For $z > x$ we have

$$
e^{-\theta(\alpha_1-\alpha_2)z} < e^{-\theta(\alpha_1-\alpha_2)x},
$$

and hence

$$
E\left[e^{-\theta(\alpha_1-\alpha_2)Z_{N-1}^{(N)}}|Z_{N-1}^{(N)} > x > Z_{N-2}^{(N)}\right] < e^{-\theta(\alpha_1-\alpha_2)x}.
$$

Taking the log of both sides and then dividing both sides by -2θ yields

$$
\beta_{N-1}^\theta(x) = -\frac{1}{2\theta}\ln\left\{E\left[e^{-\theta(\alpha_1-\alpha_2)Z_{N-1}^{(N)}}|Z_{N-1}^{(N)} > x > Z_{N-2}^{(N)}\right]\right\} > \frac{\alpha_1-\alpha_2}{2}x = \underline{\beta}_{N-1}(x).
$$

Assume that $\beta_{t+1}^\theta(x) > \underline{\beta}_{t+1}(x)$ for $x < \bar{x}$. We show that $\beta_t^\theta(x) > \underline{\beta}_t(x)$ for $x < \bar{x}$. For $z > x$ we have

$$
e^{-\theta\left[(\alpha_{N-t}-\alpha_{N-t+1})z+\beta_{t+1}^\theta(z)\right]} < e^{-\theta\left[(\alpha_{N-t}-\alpha_{N-t+1})x+\underline{\beta}_{t+1}(x)\right]},
$$

and hence

$$
E\left[e^{-\theta\left[(\alpha_{N-t}-\alpha_{N-t+1})Z_t^{(N)}+\beta_{t+1}^\theta(Z_t^{(N)})\right]}|Z_t^{(N)} > x > Z_{t-1}^{(N)}\right] < e^{-\theta\left[(\alpha_{N-t}-\alpha_{N-t+1})x+\underline{\beta}_{t+1}(x)\right]}.
$$

By the analogous argument as above, we obtain

$$
\begin{aligned}
\beta_t^\theta(x) &= -\frac{N-t}{(N-t+1)\,\theta}\ln\left\{E\left[e^{-\theta[(\alpha_{N-t}-\alpha_{N-t+1})Z_t^{(N)}+\beta_{t+1}^\theta(Z_t^{(N)})]}|Z_t^{(N)} > x > Z_{t-1}^{(N)}\right]\right\} \\
&> \frac{N-t}{N-t+1}\left[(\alpha_{N-t}-\alpha_{N-t+1})\,x+\underline{\beta}_{t+1}(x)\right] \\
&= \underline{\beta}_t(x),
\end{aligned}
$$

where the last equality follows since

$$\begin{aligned}
& \frac{N-t}{N-t+1}\left[\left(\alpha_{N-t}-\alpha_{N-t+1}\right)x+\underline{\beta}_{t+1}(x)\right] \\
= & \frac{N-t}{N-t+1}\left(\alpha_{N-t}-\alpha_{N-t+1}\right)x+\frac{N-t}{N-t+1}\left(\sum_{m=1}^{N-t-1}\frac{1}{N-t}\alpha_m-\frac{N-t-1}{N-t}\alpha_{N-t}\right)x \\
= & \left(\sum_{m=1}^{N-t-1}\frac{1}{N-t+1}\alpha_m-\frac{N-t-1}{N-t+1}\alpha_{N-t}+\frac{N-t}{N-t+1}\left(\alpha_{N-t}-\alpha_{N-t+1}\right)\right)x \\
= & \left(\sum_{m=1}^{N-t}\frac{1}{N-t+1}\alpha_m-\frac{N-t}{N-t+1}\alpha_{N-t+1}\right)x \\
= & \underline{\beta}_t(x).
\end{aligned}$$

□

Proof of Proposition 8: We first show that for each t we have $\beta_t^{\theta'}(x) < \beta_t^{\theta}(x)$ for $\theta' > \theta$ and $x < \bar{x}$. Consider round $t = N-1$. Since $f(s) = s^{\frac{\theta}{\theta'}}$ is concave, by Jensen's inequality we have

$$\left(E\left[e^{-\theta'(\alpha_1-\alpha_2)Z_{N-1}^{(N)}}|Z_{N-1}^{(N)} > x > Z_{N-2}^{(N)}\right]\right)^{\frac{\theta}{\theta'}} > E\left[e^{-\theta(\alpha_1-\alpha_2)Z_{N-1}^{(N)}}|Z_{N-1}^{(N)} > x > Z_{N-2}^{(N)}\right].$$

Taking the log of both sides and then dividing both sides by -2θ yields

$$\begin{aligned}
\beta_{N-1}^{\theta}(x) &= -\frac{1}{2\theta}\ln\left(E\left[e^{-\theta(\alpha_1-\alpha_2)Z_{N-1}^{(N)}}|Z_{N-1}^{(N)} > x > Z_{N-2}^{(N)}\right]\right) \\
&> -\frac{1}{2\theta'}\ln\left(E\left[e^{-\theta'(\alpha_1-\alpha_2)Z_{N-1}^{(N)}}|Z_{N-1}^{(N)} > x > Z_{N-2}^{(N)}\right]\right) \\
&= \beta_{N-1}^{\theta'}(x).
\end{aligned}$$

Assume that $\beta_{t+1}^{\theta'}(x) < \beta_{t+1}^{\theta}(x)$ for $x < \bar{x}$. We show that $\beta_t^{\theta'}(x) < \beta_t^{\theta}(x)$ for $x < \bar{x}$. By Jensen's inequality we have

$$\begin{aligned}
& E\left[e^{-\theta[(\alpha_{N-t}-\alpha_{N-t+1})Z_t^{(N)}+\beta_{t+1}^{\theta'}(Z_t^{(N)})]}|Z_t^{(N)} > x > Z_{t-1}^{(N)}\right] \\
< & \left(E\left[e^{-\theta'[(\alpha_{N-t}-\alpha_{N-t+1})Z_t^{(N)}+\beta_{t+1}^{\theta'}(Z_t^{(N)})]}|Z_t^{(N)} > x > Z_{t-1}^{(N)}\right]\right)^{\frac{\theta}{\theta'}}
\end{aligned}$$

and since $\beta_{t+1}^{\theta'}(x) < \beta_{t+1}^{\theta}(x)$ then

$$\begin{aligned}
& E\left[e^{-\theta[(\alpha_{N-t}-\alpha_{N-t+1})Z_t^{(N)}+\beta_{t+1}^{\theta}(Z_t^{(N)})]}|Z_t^{(N)} > x > Z_{t-1}^{(N)}\right] \\
< & E\left[e^{-\theta[(\alpha_{N-t}-\alpha_{N-t+1})Z_t^{(N)}+\beta_{t+1}^{\theta'}(Z_t^{(N)})]}|Z_t^{(N)} > x > Z_{t-1}^{(N)}\right].
\end{aligned}$$

Hence

$$\begin{aligned} & E\left[e^{-\theta[(\alpha_{N-t}-\alpha_{N-t+1})Z_t^{(N)}+\beta_{t+1}^{\theta}(Z_t^{(N)})]}|Z_t^{(N)}>x>Z_{t-1}^{(N)}\right] \\ < \quad & \left(E\left[e^{-\theta'[(\alpha_{N-t}-\alpha_{N-t+1})Z_t^{(N)}+\beta_{t+1}^{\theta'}(Z_t^{(N)})]}|Z_t^{(N)}>x>Z_{t-1}^{(N)}\right]\right)^{\frac{\theta}{\theta'}}. \end{aligned}$$

Taking the log of both sides and multiplying both sides by $-(N-t)/((N-t+1)\,\theta)$ yields

$$\begin{aligned} \beta_t^{\theta'}(x) &= -\frac{N-t}{(N-t+1)\,\theta'}\ln\left\{E\left[e^{-\theta'[(\alpha_{N-t}-\alpha_{N-t+1})Z_t^{(N)}+\beta_{t+1}^{\theta'}(Z_t^{(N)})]}|Z_t^{(N)}>x>Z_{t-1}^{(N)}\right]\right\} \\ &< -\frac{N-t}{(N-t+1)\,\theta}\ln\left\{E\left[e^{-\theta[(\alpha_{N-t}-\alpha_{N-t+1})Z_t^{(N)}+\beta_{t+1}^{\theta}(Z_t^{(N)})]}|Z_t^{(N)}>x>Z_{t-1}^{(N)}\right]\right\} \\ &= \beta_t^{\theta}(x). \end{aligned}$$

Next we show that for each t we have $\lim_{\theta\to\infty}\beta_t^{\theta}(x)=\underline{\beta}_t(x)$ for all x. For $t=N-1$ the limit is obtained directly. Specifically, after applying l'Hopital's rule, we see that

$$\lim_{\theta\to\infty}\beta_{N-1}^{\theta}(x)=\frac{1}{2}(\alpha_1-\alpha_2)\lim_{\theta\to\infty}\frac{\int_x^{\bar{x}} ze^{-\theta(\alpha_1-\alpha_2)z}g_{N-1}^{(N)}(z|x)dz}{\int_x^{\bar{x}} e^{-\theta(\alpha_1-\alpha_2)z}g_{N-1}^{(N)}(z|x)dz}$$

where $g_{N-1}^{(N)}(z|x)=2f(z)(1-F(z))/(1-F(x))^2$. Van Essen and Wooders (2016, p. 239) established that

$$\lim_{\theta\to\infty}\frac{\int_x^{\bar{x}} ze^{-\theta z}g_{N-1}^{(N)}(z|x)dz}{\int_x^{\bar{x}} e^{-\theta z}g_{N-1}^{(N)}(z|x)dz}=x,$$

which implies that

$$\lim_{\theta\to\infty}\frac{\int_x^{\bar{x}} ze^{-\theta(\alpha_1-\alpha_2)z}g_{N-1}^{(N)}(z|x)dz}{\int_x^{\bar{x}} e^{-\theta(\alpha_1-\alpha_2)z}g_{N-1}^{(N)}(z|x)dz}=x,$$

Hence,

$$\lim_{\theta\to\infty}\beta_{N-1}^{\theta}(x)=\frac{1}{2}(\alpha_1-\alpha_2)\,x=\underline{\beta}_{N-1}(x).$$

Observe that $\beta_{N-1}^{\theta}(x)$ is continuous in x on the compact set $[0,\bar{x}]$ for each θ, it converges pointwise to $\underline{\beta}_{N-1}(x)$, which is continuous on $[0,\bar{x}]$, and it is decreasing in θ. Hence β_{N-1}^{θ} converges uniformly to $\underline{\beta}_{N-1}$ on $[0,\bar{x}]$ by Theorem 7.12 of Rudin (1976).

Assume that $\beta^{\theta}_{t+1}(x)$ converges uniformly to $\underline{\beta}_{t+1}(x)$ on $[0,\bar{x}]$. We show that $\beta^{\theta}_{t}(x)$ converges uniformly to $\underline{\beta}_{t}(x)$. The CARA bid function in round t is

$$\beta^{\theta}_{t}(x) = -\frac{N-t}{(N-t+1)\,\theta}\ln\left(\int_{x}^{\bar{x}} e^{-\theta[(\alpha_{N-t}-\alpha_{N-t+1})z+\beta^{\theta}_{t+1}(z)]}g^{(N)}_{t}(z|x)dz\right).$$

Let $\Delta>0$ be arbitrary. Since $\beta^{\theta}_{t}(x)$ is decreasing in θ and since $\beta^{\theta}_{t+1}\to\underline{\beta}_{t+1}$ uniformly as $\theta\to\infty$, then there is a $\bar{\theta}$ such that for all $\theta\geq\bar{\theta}$ we have

$$\beta^{\theta}_{t+1}(x)\leq\sum_{m=1}^{N-t-1}\frac{m}{N-t}\left(\alpha_m-\alpha_{m+1}\right)x+\Delta$$

for $x\in[0,\bar{x}]$. Define

$$\bar{\beta}^{\theta}_{t}(x)\equiv-\frac{N-t}{(N-t+1)\,\theta}\ln\left(\int_{x}^{\bar{x}} e^{-\theta[z(\alpha_{N-t}-\alpha_{N-t+1}+\sum_{m=1}^{N-t-1}\frac{m}{N-t}(\alpha_m-\alpha_{m+1}))+\Delta]}g^{(N)}_{t}(z|x)\right)dz.$$

Then $\beta^{\theta}_{t}(x)\leq\bar{\beta}^{\theta}_{t}(x)$ for $\theta\geq\bar{\theta}$ and $x\in[0,\bar{x}]$. By Proposition 7 we have $\underline{\beta}_{t}(x)\leq\beta^{\theta}_{t}(x)$ and thus

$$\underline{\beta}_{t}(x)\leq\beta^{\theta}_{t}(x)\leq\bar{\beta}^{\theta}_{t}(x)$$

for $\theta\geq\bar{\theta}$ and $x\in[0,\bar{x}]$.

We establish that $\beta^{\theta}_{t}(x)$ converges pointwise to $\underline{\beta}_{t}(x)$ for each $x\in[0,\bar{x}]$. Define

$$C=\alpha_{N-t}-\alpha_{N-t+1}+\sum_{m=1}^{N-t-1}\frac{m}{N-t}\left(\alpha_m-\alpha_{m+1}\right).$$

Applying L'Hopital's rule and using the same argument as for round $N-1$, we have

$$\begin{aligned}\lim_{\theta\to\infty}\bar{\beta}^{\theta}_{t}(x) &= \frac{N-t}{N-t+1}\lim_{\theta\to\infty}\frac{\int_{x}^{\bar{x}}(zC+\Delta)e^{-\theta(zC+\Delta)}g^{(N)}_{t}(z|x)dz}{\int_{x}^{\bar{x}}e^{-\theta(zC+\Delta)}g^{(N)}_{t}(z|x)dz}\\ &= \frac{N-t}{N-t+1}\left(C\lim_{\theta\to\infty}\frac{\int_{x}^{\bar{x}}ze^{-\theta zC}g^{(N)}_{t}(z|x)dz}{\int_{x}^{\bar{x}}e^{-\theta zC}g^{(N)}_{t}(z|x)dz}+\Delta\right)\\ &= \frac{N-t}{N-t+1}\left(Cx+\Delta\right),\end{aligned}$$

where the last inequality holds by Van Essen and Wooders (2016). Substituting for C and simplifying yields

$$\lim_{\theta\to\infty} \bar{\beta}_t^\theta(x) = \underline{\beta}_t(x) + \frac{N-t}{N-t+1}\Delta.$$

Since the inequality

$$\underline{\beta}_t(x) \leq \lim_{\theta\to\infty} \beta_t^\theta(x) \leq \lim_{\theta\to\infty} \bar{\beta}_t^\theta(x) = \underline{\beta}_t(x) + \frac{N-t}{N-t+1}\Delta$$

holds for arbitrary $\Delta > 0$, it follows that $\lim_{\theta\to\infty} \beta_t^\theta(x) = \underline{\beta}_t(x)$. By the same argument as for β_{N-1}^θ, we have that β_t^θ converges uniformly to $\underline{\beta}_t$ on $[0, \bar{x}]$. $\square$

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