1 Supplemental Appendix

In this appendix we present several combinatorial results used in the paper and the last step in establishing Proposition 3.

Claim 1: *Let n and i positive integers and m be a non-negative integer such that $n > i > m \geq 0$, then*

$$\sum_{s=1}^{n-i+m} \frac{\binom{n-i+m}{s-1}}{\binom{n-1}{s-1}} = \frac{n}{i-m}.$$

The expression in Claim 1 is a known combinatorial identity. See Graham, Knuth, and Patashnik (1990) – pg. 173 for a proof.

Claim 2: *For positive integers n and x, we have*

$$\sum_{k=0}^{n} (-1)^k \frac{1}{x+k} = \frac{1}{x\binom{x+n}{n}}.$$

The expression in Claim 2 is a known combinatorial identity. See Graham, Knuth, and Patashnik (1990) – pg. 188 for a proof.

Next, we have a repeated application of Pascal's Recursion.

Claim 3: *For positive integers $n, i, s,$ and t such that $n > i$, $s \geq t$, and $n - i > s - t$, we have for $p = 0, ..., \bar{p}$, we have that*

$$\binom{n-i}{s-t} = \sum_{m=0}^{p} (-1)^{m+p} \binom{p}{m} \binom{n-i+m}{s-t+p}.$$

Proof: The claim is clearly true for $p = 0$. Assume the theorem is true for $p = k$, then

$$\binom{n-i}{s-t} = \sum_{m=0}^{k} (-1)^{m+k} \binom{k}{m} \binom{n-i+m}{s-t+k}.$$

Now, by Pascal's Recursion,

$$\binom{n-i+m}{s-t+k} = \binom{n-i+(m+1)}{s-t+(k+1)} - \binom{n-i+m}{s-t+(k+1)}.$$

Hence,

$$\binom{n-i}{n-t} = \sum_{m=0}^{k}(-1)^{m+k}\binom{k}{m}\left[\binom{n-i+(m+1)}{s-t+(k+1)} - \binom{n-i+m}{s-t+(k+1)}\right].$$

Now, collecting terms, we have

$$\begin{aligned}\binom{n-i}{S-t} &= (-1)^{0+(k+1)}\binom{k}{0}\binom{n-i}{s-t+(k+1)} \\ &+(-1)^{1+(k+1)}\left[\binom{k}{0}+\binom{k}{1}\right]\binom{n-i+1}{s-t+(k+1)} \\ &+ \\ &\dots \\ &+(-1)^{k+(k+1)}\left[\binom{k}{k-1}+\binom{k}{k}\right]\binom{n-i+k+1}{s-t+(k+1)} \\ &+(-1)^{(k+1)+(k+1)}\binom{k+1}{k+1}\binom{n-i+k+1}{s-t+(k+1)}.\end{aligned}$$

Using Pascal's Recursion

$$\binom{k+1}{m} = \binom{k}{m-1} + \binom{k}{m}$$

we have

$$\binom{n-i}{s-t} = \sum_{m=0}^{k+1}(-1)^{m+(k+1)}\binom{k+1}{m}\binom{n-i+m}{s-t+(k+1)}.\square$$

Next, we establish an interesting combinatorial identity that is useful in computing the Shapley Value for the problem of assigning players to positions.

Claim 4: *For positive integers n, i, and t such that $n \geq i$, we have that*

$$\frac{1}{n}\sum_{s=t}^{n-i+t}\frac{\binom{i-1}{t-1}\binom{n-i}{s-t}}{\binom{n-1}{s-1}} = \frac{1}{i}.$$

Proof: From Claim 3, by setting $p = t-1$, we have

$$\begin{aligned} & \frac{1}{n}\sum_{s=t}^{n-i+t} \frac{\binom{i-1}{t-1}\binom{n-i}{s-t}}{\binom{n-1}{s-1}} \\ = \; & \frac{1}{n}\sum_{s=t}^{N-i+t} \frac{\binom{i-1}{t-1}\left[\sum_{m=0}^{t-1}(-1)^{m+t-1}\binom{t-1}{m}\binom{n-i+m}{s-1}\right]}{\binom{n-1}{s-1}}. \end{aligned}$$

Now, for all $s < t$,

$$\binom{n-i}{s-t} = \sum_{m=0}^{t-1}(-1)^{m+t-1}\binom{t-1}{m}\binom{n-i+m}{s-1} = 0$$

by Pascal's recursion. So, we have

$$\begin{aligned} & \frac{1}{n}\sum_{s=t}^{n-i+t} \frac{\binom{i-1}{t-1}\left[\sum_{m=0}^{t-1}(-1)^{m+t-1}\binom{t-1}{m}\binom{n-i+m}{s-1}\right]}{\binom{n-1}{s-1}} \\ = \; & \\ & \frac{1}{n}\sum_{s=1}^{n-i+t} \frac{\binom{i-1}{t-1}\left[\sum_{m=0}^{t-1}(-1)^{m+t-1}\binom{t-1}{m}\binom{n-i+m}{s-1}\right]}{\binom{n-1}{s-1}} \end{aligned}$$

This is equal to

$$= \frac{1}{n}\sum_{m=0}^{t-1}(-1)^{m+t-1}\binom{t-1}{m}\binom{i-1}{t-1}\sum_{s=1}^{n-i+t}\frac{\binom{n-i+m}{s-1}}{\binom{n-1}{s-1}}.$$

Next,

$$\begin{aligned} \sum_{s=1}^{n-i+m}\frac{\binom{n-i+m}{s-1}}{\binom{n-1}{s-1}} &= \frac{n-1+1}{n-1-(n-i+m)+1} \\ &= \frac{n}{i-m} \end{aligned}$$

This follows from Claim 1. Thus, our expression reduces to

$$\frac{1}{n}\sum_{s=t}^{n-i+t}\frac{\binom{i-1}{t-1}\binom{n-i}{s-t}}{\binom{n-1}{s-1}} = \sum_{m=0}^{t-1}(-1)^{m+t-1}\binom{t-1}{m}\binom{i-1}{t-1}\frac{1}{i-m}.$$

The RHS of the above equation is related to the combinatorial identity presented in Claim 2 – i.e., $\frac{1}{x\binom{x+n}{n}} = \sum_{k=0}^{n}(-1)^k \frac{1}{x+k}$. In particular,

$$\begin{aligned}
& \sum_{m=0}^{t-1}(-1)^{m+t-1}\binom{t-1}{m}\binom{i-1}{t-1}\frac{1}{i-m} \\
= \quad & \binom{i-1}{n}\sum_{m'=0}^{n}(-1)^{m'}\binom{n}{m'}\frac{1}{i-n+m'}
\end{aligned}$$

This is seen to be equivalent by setting $n = t-1$, $m' = m-n$, using the symmetry of the binomial coefficient $\binom{t-1}{m} = \binom{t-1}{m'}$, and observing $(-1)^{m+n} = (-1)^{m'}$. Using Claim 2 and then some manipulation, we have

$$\begin{aligned}
& \sum_{m=0}^{t-1}(-1)^{m+t-1}\binom{t-1}{m}\binom{i-1}{t-1}\frac{1}{i-m} \\
= \quad & \binom{i-1}{n}\sum_{m'=0}^{n}(-1)^{m'}\binom{n}{m'}\frac{1}{i-n+m'} \\
= \quad & \binom{i-1}{n}\frac{1}{(i-n)\binom{i-n+n}{n}} \\
= \quad & \frac{\binom{i-1}{n}}{(i-n)\binom{i}{n}} \\
= \quad & \frac{\frac{(i-1)!}{n!(i-1-n)!}}{(i-n)\frac{i!}{n!(i-n)!}} \\
= \quad & \frac{1}{i}. \square
\end{aligned}$$

Finally, we verify the last part of Proposition 3.

Claim 5: *For any t, the risk neutral bid function*

$$\begin{aligned}
\beta_t^0(x) &= \frac{N-t}{N-t+1}E\left[\left(\alpha_{N-t}-\alpha_{N-t+1}\right)Z_t^{(N)} + \beta_{t+1}^0(Z_t^{(N)})|Z_t^{(N)} > x > Z_{t-1}^{(N)}\right] \\
&= \sum_{m=1}^{N-t}\frac{m}{N-t+1}E\left[\left(\alpha_m-\alpha_{m+1}\right)Z_{N-m}^{(N)}|Z_t^{(N)} > x > Z_{t-1}^{(N)}\right].
\end{aligned}$$

where $\beta_N^0 \equiv 0$.

Proof: The result is clearly true for $t = N-1$ since $\beta_N^0 \equiv 0$. Assume

$$\beta_{t+1}^0(x) = \sum_{m=1}^{N-t-1} \frac{m}{N-t} E\left[(\alpha_m - \alpha_{m+1})\, Z_{N-m}^{(N)} | Z_{t+1}^{(N)} > x > Z_t^{(N)}\right].$$

In round t, we have

$$\begin{aligned}
\beta_t^0(x) &= \frac{N-t}{N-t+1} E\left[(\alpha_{N-t} - \alpha_{N-t+1})\, Z_t^{(N)} + \beta_{t+1}^0(Z_t^{(N)}) | Z_t^{(N)} > x > Z_{t-1}^{(N)}\right] \\
&= \frac{N-t}{N-t+1} E\left[(\alpha_{N-t} - \alpha_{N-t+1})\, Z_t^{(N)} | Z_t^{(N)} > x > Z_{t-1}^{(N)}\right] \\
&\quad + \sum_{m=1}^{N-t-1} \frac{m}{N-t+1} E\left[E\left[(\alpha_m - \alpha_{m+1})\, Z_{N-m}^{(N)} | Z_{t+1}^{(N)} > Z_t^{(N)} > Z_t^{(N)}\right] | Z_t^{(N)} > x > Z_{t-1}^{(N)}\right] \\
&= \frac{N-t}{N-t+1} E\left[(\alpha_{N-t} - \alpha_{N-t+1})\, Z_t^{(N)} | Z_t^{(N)} > x > Z_{t-1}^{(N)}\right] \\
&\quad + \sum_{m=1}^{N-t-1} \frac{m}{N-t} E\left[(\alpha_m - \alpha_{m+1})\, Z_{N-m}^{(N)} | Z_t^{(N)} > x > Z_{t-1}^{(N)}\right] \\
&= \sum_{m=1}^{N-t} \frac{m}{N-t+1} E\left[(\alpha_m - \alpha_{m+1})\, Z_{N-m}^{(N)} | Z_t^{(N)} > x > Z_{t-1}^{(N)}\right].
\end{aligned}$$

where the second equality follows from induction and the third equality follows since for each m, $E\left[E\left[Z_{N-m}^{(N)} | Z_{t+1}^{(N)} > Z_t^{(N)} > Z_t^{(N)}\right] | Z_t^{(N)} > x > Z_{t-1}^{(N)}\right]$

$$\begin{aligned}
&= \int_x^{\bar{x}} \int_{z_t}^{\bar{x}} s \frac{(N-t)!\,[F(s) - F(z_t)]^{N-t-1-m} f(s)[1-F(s)]^m}{(N-t-1-m)!m![1-F(z_t)]^{N-t}} \frac{(N-t+1)! f(z_t)[1-F(z_t)]^{N-t}}{(N-t)![1-F(x)]^{N-t+1}} ds dz_t \\
&= \int_x^{\bar{x}} \int_x^s s \frac{(N-t+1)!\,[F(s) - F(z_t)]^{N-t-1-m} f(s)[1-F(s)]^m}{(N-t-1-m)!m![1-F(x)]^{N-t+1}} f(z_t) dz_t ds \\
&= \int_x^{\bar{x}} s \frac{(N-t+1)!\,[F(s) - F(x)]^{N-t-m} f(s)[1-F(s)]^m}{(N-t-m)!m![1-F(x)]^{N-t+1}} ds \\
&= E\left[Z_{N-m}^{(N)} | Z_t^{(N)} > x > Z_{t-1}^{(N)}\right]. \square
\end{aligned}$$