Online Appendix

This appendix provides proofs of ancillary results stated without proof in the paper. Section 1 provides several combinatorial lemmas used in the proof of Proposition 1. Section 2 establishes two additional properties of Shapley values. Section 3 establishes a payoff equivalence and related results. Section 4 establishes that the two equilibrium bid functions given in Proposition 5 are equivalent.

1 Some Combinatorial Results

In this section we provide the combinatorial results used in the proof of Proposition 1.

Claim 1: *Let n and i positive integers and m be a non-negative integer such that $n > i > m \geq 0$, then*

$$\sum_{s=1}^{n-i+m} \frac{\binom{n-i+m}{s-1}}{\binom{n-1}{s-1}} = \frac{n}{i-m}.$$

Claim 1 is a known combinatorial identity. See Graham, Knuth, and Patashnik (1990), p. 173, for a proof.

Claim 2: *For positive integers n and x, we have*

$$\sum_{k=0}^{n} (-1)^k \frac{1}{x+k} = \frac{1}{x\binom{x+n}{n}}.$$

Claim 2 is also a known combinatorial identity. See Graham, Knuth, and Patashnik (1990), p. 188, for a proof.

Claim 3: *For positive integers $n, i, s,$ and t such that $n > i$, $s \geq t$, and $n - i > s - t$, we have for $p = 0, ..., \bar{p}$, we have that*

$$\binom{n-i}{s-t} = \sum_{m=0}^{p} (-1)^{m+p} \binom{p}{m} \binom{n-i+m}{s-t+p}.$$

Proof: The claim is clearly true for $p = 0$. Assume the theorem is true for $p = k$, then

$$\binom{n-i}{s-t} = \sum_{m=0}^{k} (-1)^{m+k} \binom{k}{m} \binom{n-i+m}{s-t+k}.$$

Now, by Pascal's Recursion,

$$\binom{n-i+m}{s-t+k} = \binom{n-i+(m+1)}{s-t+(k+1)} - \binom{n-i+m}{s-t+(k+1)}.$$

Hence,

$$\binom{n-i}{n-t} = \sum_{m=0}^{k} (-1)^{m+k} \binom{k}{m} \left[\binom{n-i+(m+1)}{s-t+(k+1)} - \binom{n-i+m}{s-t+(k+1)} \right].$$

Now, collecting terms, we have

$$\begin{aligned}
\binom{n-i}{S-t} &= (-1)^{0+(k+1)} \binom{k}{0} \binom{n-i}{s-t+(k+1)} \\
&\quad +(-1)^{1+(k+1)} \left[\binom{k}{0} + \binom{k}{1} \right] \binom{n-i+1}{s-t+(k+1)} \\
&\quad + \\
&\quad \ldots \\
&\quad +(-1)^{k+(k+1)} \left[\binom{k}{k-1} + \binom{k}{k} \right] \binom{n-i+k+1}{s-t+(k+1)} \\
&\quad +(-1)^{(k+1)+(k+1)} \binom{k+1}{k+1} \binom{n-i+k+1}{s-t+(k+1)}.
\end{aligned}$$

Using Pascal's Recursion

$$\binom{k+1}{m} = \binom{k}{m-1} + \binom{k}{m}$$

we have

$$\binom{n-i}{s-t} = \sum_{m=0}^{k+1} (-1)^{m+(k+1)} \binom{k+1}{m} \binom{n-i+m}{s-t+(k+1)}. \square$$

Next, we establish an interesting combinatorial identity that is useful in computing the Shapley Value for the problem of assigning players to positions.

Claim 4: *For positive integers n, i, and t such that $n \geq i$, we have that*

$$\frac{1}{n}\sum_{s=t}^{n-i+t}\frac{\binom{i-1}{t-1}\binom{n-i}{s-t}}{\binom{n-1}{s-1}} = \frac{1}{i}.$$

Proof: From Claim 3, by setting $p = t-1$, we have

$$\begin{aligned} & \frac{1}{n}\sum_{s=t}^{n-i+t}\frac{\binom{i-1}{t-1}\binom{n-i}{s-t}}{\binom{n-1}{s-1}} \\ = & \frac{1}{n}\sum_{s=t}^{N-i+t}\frac{\binom{i-1}{t-1}\left[\sum_{m=0}^{t-1}(-1)^{m+t-1}\binom{t-1}{m}\binom{n-i+m}{s-1}\right]}{\binom{n-1}{s-1}}. \end{aligned}$$

Now, for all $s < t$,

$$\binom{n-i}{s-t} = \sum_{m=0}^{t-1}(-1)^{m+t-1}\binom{t-1}{m}\binom{n-i+m}{s-1} = 0$$

by Pascal's recursion. So, we have

$$\begin{aligned} & \frac{1}{n}\sum_{s=t}^{n-i+t}\frac{\binom{i-1}{t-1}\left[\sum_{m=0}^{t-1}(-1)^{m+t-1}\binom{t-1}{m}\binom{n-i+m}{s-1}\right]}{\binom{n-1}{s-1}} \\ = & \\ & \frac{1}{n}\sum_{s=1}^{n-i+t}\frac{\binom{i-1}{t-1}\left[\sum_{m=0}^{t-1}(-1)^{m+t-1}\binom{t-1}{m}\binom{n-i+m}{s-1}\right]}{\binom{n-1}{s-1}} \end{aligned}$$

This is equal to

$$= \frac{1}{n}\sum_{m=0}^{t-1}(-1)^{m+t-1}\binom{t-1}{m}\binom{i-1}{t-1}\sum_{s=1}^{n-i+t}\frac{\binom{n-i+m}{s-1}}{\binom{n-1}{s-1}}.$$

Next,

$$\begin{aligned} \sum_{s=1}^{n-i+m}\frac{\binom{n-i+m}{s-1}}{\binom{n-1}{s-1}} &= \frac{n-1+1}{n-1-(n-i+m)+1} \\ &= \frac{n}{i-m} \end{aligned}$$

This follows from Claim 1. Thus, our expression reduces to

$$\frac{1}{n}\sum_{s=t}^{n-i+t}\frac{\binom{i-1}{t-1}\binom{n-i}{s-t}}{\binom{n-1}{s-1}} = \sum_{m=0}^{t-1}(-1)^{m+t-1}\binom{t-1}{m}\binom{i-1}{t-1}\frac{1}{i-m}.$$

The RHS of the above equation is related to the combinatorial identity presented in Claim 2 – i.e., $\frac{1}{x\binom{x+n}{n}} = \sum_{k=0}^{n}(-1)^k\frac{1}{x+k}$. In particular,

$$\begin{aligned} & \sum_{m=0}^{t-1}(-1)^{m+t-1}\binom{t-1}{m}\binom{i-1}{t-1}\frac{1}{i-m} \\ = & \binom{i-1}{n}\sum_{m'=0}^{n}(-1)^{m'}\binom{n}{m'}\frac{1}{i-n+m'} \end{aligned}$$

This is seen to be equivalent by setting $n = t-1$, $m' = m-n$, using the symmetry of the binomial coefficient $\binom{t-1}{m} = \binom{t-1}{m'}$, and observing $(-1)^{m+n} = (-1)^{m'}$. Using Claim 2 and then some manipulation, we have

$$\begin{aligned} & \sum_{m=0}^{t-1}(-1)^{m+t-1}\binom{t-1}{m}\binom{i-1}{t-1}\frac{1}{i-m} \\ = & \binom{i-1}{n}\sum_{m'=0}^{n}(-1)^{m'}\binom{n}{m'}\frac{1}{i-n+m'} \\ = & \binom{i-1}{n}\frac{1}{(i-n)\binom{i-n+n}{n}} \\ = & \frac{\binom{i-1}{n}}{(i-n)\binom{i}{n}} \\ = & \frac{\frac{(i-1)!}{n!(i-1-n)!}}{(i-n)\frac{i!}{n!(i-n)!}} \\ = & \frac{1}{i}. \ \square \end{aligned}$$

2 Additional Properties of the Shapley Value

Two properties of the Shapley value in the position allocation problem are stated on page 11, which we now prove.

Claim 5: *Suppose $x_1 \geq \cdots \geq x_N$, then the Shapley value of player k is greater than the Shapley value of player $k+1$, i.e., $\phi_k \geq \phi_{k+1}$.*

Proof: By Corollary 1, player k's Shapley value is

$$\begin{aligned}
\phi_k &= \alpha_k x_k + \tau_k \\
&= \frac{1}{k}\left[(\alpha_1 + \cdots + \alpha_k)\, x_k - \tau_{k+1} - \sum_{i=k+2}^{N} \tau_i\right] \\
&= \frac{1}{k}\left[\begin{array}{c} (\alpha_1 + \cdots + \alpha_k)\, x_k + \alpha_{k+1}x_{k+1} - \frac{1}{k+1}(\alpha_1 + \cdots + \alpha_{k+1})\, x_{k+1} \\ + \frac{1}{k+1}\sum_{i=k+2}^{N}\tau_i - \sum_{i=k+2}^{N}\tau_i. \end{array}\right]
\end{aligned}$$

Since $x_k \geq x_{k+1}$ then

$$\begin{aligned}
\phi_k &\geq \frac{1}{k}\left[(\alpha_1 + \cdots + \alpha_k)\, x_{k+1} + \alpha_{k+1}x_{k+1} - \frac{1}{k+1}(\alpha_1 + \cdots + \alpha_{k+1})\, x_{k+1} - \frac{k}{k+1}\sum_{i=k+2}^{N}\tau_i\right] \\
&= \frac{1}{k}\left[\frac{k}{k+1}(\alpha_1 + \cdots + \alpha_k)\, x_{k+1} + \frac{k}{k+1}\alpha_{k+1}x_{k+1} - \frac{k}{k+1}\sum_{i=k+2}^{N}\tau_i\right] \\
&= \frac{1}{k+1}\left[(\alpha_1 + \cdots + \alpha_{k+1})\, x_{k+1} - \sum_{i=k+2}^{N}\tau_i\right] \\
&= \phi_{k+1}.
\end{aligned}$$

□

Claim 6: *The Shapley value of player i is at least his value for an average position, i.e.,*

$$\phi_i \geq \underline{\phi}_i = \frac{\alpha_1 + \ldots + \alpha_N}{N} x_i.$$

Proof: From the formula for ϕ_i in Proposition 1 it is clear that player i's Shapley value is decreasing in x_j for $j < i$. In particular, the Shapley value ϕ_i generated by values

$$x_1 \geq \cdots \geq x_i \geq x_{i+1} \geq \cdots \geq x_N$$

and the Shapley value ϕ_i' generated by values

$$x_1 \geq \cdots \geq x_i \geq x_{i+1}' \geq \cdots \geq x_N'$$

where $x'_j \geq x_j$ for $j = i+1, ..., N$, satisfies $\phi_i \geq \phi'_i$. Hence, the Shapley value $\underline{\phi}_i$ generated by values

$$x_1 \geq \cdots \geq x_i = x_{i+1} = \cdots = x_N$$

satisfies $\phi_i \geq \underline{\phi}_i$. Since $x_i = x_{i+1} = \cdots = x_N$, then

$$\underline{\phi}_i = \phi_N = \frac{\alpha_1 + \cdots + \alpha_N}{N} x_i.$$

□

3 Payoff Equivalence

We establish that when bidders are risk neutral, then all symmetric, efficient, and budget-balanced mechanisms are payoff equivalent. The AGV mechanism, in particular, is payoff equivalent to any compensated position auction.

Claim 7: *Suppose that bidders are risk neutral. Let Σ and Σ' be two symmetric budget-balanced mechanisms for the position allocation problem with symmetric Bayes Nash equilibria, β_Σ and $\beta_{\Sigma'}$, that allocate positions efficiently. Then the interim equilibrium expected payoff of a bidder with value x is the same in Σ and Σ'.*

Proof: By the Revelation Principle, we can restrict attention to direct mechanisms in which truth telling is a Bayes Nash equilibrium. Let Σ be an arbitrary direct, incentive compatible, symmetric, and efficient mechanism. In the truth telling equilibrium, the payoff to a bidder with value x who reports r is

$$\pi(r; x) = P(r)x - m_\Sigma(r),$$

where

$$P(r) = \Pr(r > Z_{N-1}^{(N-1)})\alpha_1 + \Pr(Z_{N-1}^{(N-1)} > r > Z_{N-2}^{(N-1)})\alpha_2 + \cdots + \Pr(Z_1^{(N-1)} > r)\alpha_N$$

and $m_\Sigma(r)$ is the bidder's expected transfer.

We now characterize m_Σ, showing that it is determined by P. The marginal payoff to the bidder of increasing his report is

$$\frac{d\pi(r; x)}{dr} = P'(r)x - \frac{dm_\Sigma(r)}{dr}.$$

Since truth telling is a Bayes Nash equilibrium, then at $r = x$ we have

$$\frac{dm_\Sigma(x)}{dr} = P'(x)x,$$

which holds for all values of x. From the Fundamental Theorem of Calculus

$$m_\Sigma(x) = m_\Sigma(0) + \int_0^x zP'(z)dz,$$

where $m_\Sigma(0)$ is a constant.

The ex-ante expected transfer of the bidder is

$$\int_0^{\bar{x}} m_\Sigma(q)f(q)dq = m_\Sigma(0) + \int_0^{\bar{x}} \int_0^q zP'(z)f(q)dzdq = 0,$$

where the second equality follows from budget balancedness. Hence,

$$m_\Sigma(0) = -\int_0^{\bar{x}} \int_0^q zP'(z)f(q)dzdq$$

and, thus,

$$m_\Sigma(x) = -\int_0^{\bar{x}} \int_0^q zP'(z)f(q)dzdq + \int_0^x zP'(z)dz$$

is determined by P. Since the right hand side is independent of Σ, the result follows. □

A common critique of the AGV mechanism is that participation need not be individual rational. Since the AGV is payoff equivalent to a compensated position auction when bidders are risk neutral, and since a position auction is individually rational regardless of risk attitudes (see the Discussion section), then the following is immediate and appears in footnote 15.

Claim 8: *When bidders are risk neutral, the AGV mechanism is individually rational for the position allocation problem.*

In section 5, p. 21, we use the following result.

Claim 9: *Suppose bidders are risk neutral, and* $\alpha_1 = 1$*, and* $\alpha_2 = ... = \alpha_N = 0$*. Then the interim equilibrium expected payoff of a bidder with value* 0 *is* $E[Z_{N-1}^{(N)}]/N > 0$*.*

Proof: Let Σ be an arbitrary direct, incentive compatible, symmetric, and efficient mechanism. In the truth telling equilibrium, the payoff to a bidder with value x who reports r is

$$\pi(r;x) = P(r)x - m_\Sigma(r),$$

where, since $\alpha_1 = 1$ and $\alpha_2 = ... = \alpha_N = 0$,

$$P(r) = \Pr(r > Z_{N-1}^{(N-1)}) = F_{N-1}^{(N-1)}(r)$$

and $m_\Sigma(r)$ is the bidder's expected transfer.

Note the derivative of P is the density of the highest order statistic when there are $N-1$ draws from F -i.e.,

$$P'(r) = f_{N-1}^{(N-1)}(r) = (N-1)f(r)F(r)^{N-2}.$$

Finally, a bidder with value zero has an expected payoff of $-m_\Sigma(0)$. From Payoff Equivalence, we know

$$\begin{aligned} -m_\Sigma(0) &= \int_0^{\bar{x}}\int_0^q zP'(z)f(q)dzdq \\ &= \int_0^{\bar{x}}\int_z^{\bar{x}} zP'(z)f(q)dqdz \\ &= \int_0^{\bar{x}} zP'(z)[1-F(z)]dz \\ &= \frac{1}{N}\int_0^{\bar{x}} z\frac{N!}{(N-2)!}f(z)F(z)^{N-2}[1-F(z)]dz \\ &= \frac{1}{N}E[Z_{N-1}^{(N)}]. \end{aligned}$$

□

4 Equivalent Representation of Bid Function in Proposition 5

Finally, we establish that the two equilibrium bid functions given in Proposition 5 are equivalent.

Claim 10: *The risk neutral bid function is, for* $t = 1, \ldots, N-1$, *given by*

$$\begin{aligned}
\beta_t^0(x) &= \frac{N-t}{N-t+1} E\left[\left(\alpha_{N-t} - \alpha_{N-t+1}\right) Z_t^{(N)} + \beta_{t+1}^0(Z_t^{(N)}) | Z_t^{(N)} > x > Z_{t-1}^{(N)}\right] \\
&= \sum_{m=1}^{N-t} \frac{m}{N-t+1} E\left[\left(\alpha_m - \alpha_{m+1}\right) Z_{N-m}^{(N)} | Z_t^{(N)} > x > Z_{t-1}^{(N)}\right],
\end{aligned}$$

where $\beta_N^0 \equiv 0$.

Proof: The result is clearly true for $t = N-1$ since $\beta_N^0 \equiv 0$. Assume

$$\beta_{t+1}^0(x) = \sum_{m=1}^{N-t-1} \frac{m}{N-t} E\left[\left(\alpha_m - \alpha_{m+1}\right) Z_{N-m}^{(N)} | Z_{t+1}^{(N)} > x > Z_t^{(N)}\right].$$

In round t, we have

$$\begin{aligned}
\beta_t^0(x) &= \frac{N-t}{N-t+1} E\left[\left(\alpha_{N-t} - \alpha_{N-t+1}\right) Z_t^{(N)} + \beta_{t+1}^0(Z_t^{(N)}) | Z_t^{(N)} > x > Z_{t-1}^{(N)}\right] \\
&= \frac{N-t}{N-t+1} E\left[\left(\alpha_{N-t} - \alpha_{N-t+1}\right) Z_t^{(N)} | Z_t^{(N)} > x > Z_{t-1}^{(N)}\right] \\
&\quad + \sum_{m=1}^{N-t-1} \frac{m}{N-t+1} E\left[E\left[\left(\alpha_m - \alpha_{m+1}\right) Z_{N-m}^{(N)} | Z_{t+1}^{(N)} > Z_t^{(N)} > Z_t^{(N)}\right] | Z_t^{(N)} > x > Z_{t-1}^{(N)}\right] \\
&= \frac{N-t}{N-t+1} E\left[\left(\alpha_{N-t} - \alpha_{N-t+1}\right) Z_t^{(N)} | Z_t^{(N)} > x > Z_{t-1}^{(N)}\right] \\
&\quad + \sum_{m=1}^{N-t-1} \frac{m}{N-t} E\left[\left(\alpha_m - \alpha_{m+1}\right) Z_{N-m}^{(N)} | Z_t^{(N)} > x > Z_{t-1}^{(N)}\right] \\
&= \sum_{m=1}^{N-t} \frac{m}{N-t+1} E\left[\left(\alpha_m - \alpha_{m+1}\right) Z_{N-m}^{(N)} | Z_t^{(N)} > x > Z_{t-1}^{(N)}\right].
\end{aligned}$$

where the second equality follows from induction and the third equality follows since for each m, $E\left[E\left[Z_{N-m}^{(N)}|Z_{t+1}^{(N)}>Z_t^{(N)}>Z_t^{(N)}\right]|Z_t^{(N)}>x>Z_{t-1}^{(N)}\right]$

$$
\begin{aligned}
&= \int_x^{\bar{x}}\int_{z_t}^{\bar{x}} s\frac{(N-t)!\,[F(s)-F(z_t)]^{N-t-1-m}\,f(s)[1-F(s)]^m}{(N-t-1-m)!m![1-F(z_t)]^{N-t}}\frac{(N-t+1)!f(z_t)[1-F(z_t)]^{N-t}}{(N-t)![1-F(x)]^{N-t+1}}dsdz_t \\
&= \int_x^{\bar{x}}\int_x^{s} s\frac{(N-t+1)!\,[F(s)-F(z_t)]^{N-t-1-m}\,f(s)[1-F(s)]^m}{(N-t-1-m)!m![1-F(x)]^{N-t+1}}f(z_t)dz_tds \\
&= \int_x^{\bar{x}} s\frac{(N-t+1)!\,[F(s)-F(x)]^{N-t-m}\,f(s)[1-F(s)]^m}{(N-t-m)!m![1-F(x)]^{N-t+1}}ds \\
&= E\left[Z_{N-m}^{(N)}|Z_t^{(N)}>x>Z_{t-1}^{(N)}\right].\ \square
\end{aligned}
$$