Econ Theory (2009) 38:9–39
DOI 10.1007/s00199-006-0182-7

SYMPOSIUM

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Auctions with a buy price

Received: 25 May 2005 / Revised: 26 July 2006 / Published online: 23 December 2006

Abstract eBay and Yahoo allow sellers to list their auctions with a buy price at which a bidder may purchase the item immediately. On eBay, the buy-now option disappears once a bid is placed, while on Yahoo the buy-now option remains in effect throughout the auction. We show that when bidders are risk averse, both types of auctions raise seller revenue for a wide range of buy prices. The Yahoo format raises more revenue than the eBay format when bidders have either CARA or DARA. Bidders with DARA prefer the eBay auction, while bidders with CARA are indifferent between the two.

Keywords Auction · Buy price · Yahoo · eBay · Risk aversion

JEL Classification Numbers D44 · D82 · L86

1 Introduction

The expansion of commerce conducted over the Internet has sparked a surge of interest in auctions and the emergence of new auction forms.[1] A new twist in online auction formats appears in Yahoo and eBay auctions. In 1999 Yahoo introduced the *Buy-Now* feature into its ascending bid auctions. The Buy-Now feature allows a

Part of this work was completed while Reynolds was a visitor at Instituto de Analisys Economico in Barcelona and while Wooders was a visitor at Hong Kong University of Science and Technology. We are grateful to these institutions for their hospitality.

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[1] Lucking-Reiley (2000) describes the wide variety of online auction formats that were being used as of 1999.

seller to set a price, termed a buy price, at which any bidder may purchase the item at any time during the auction.[2] Since the buy price remains in effect throughout the auction, this feature allows the seller to post a maximum price for the item. In 2000 eBay introduced its own version of a fixed price feature into its online auctions via the *Buy It Now* option. In contrast to the Yahoo format, eBay permits bidders to select the buy price only at the opening of the auction, before any bids are submitted, or in the case of an auction with a (secret) reserve, before bids reach the reserve price.[3]

eBay.com is the dominant auction site in the United States and hosts millions of auctions each day, with around a *billion* listings over the course of last year. Yahoo! Auctions dominates in Japan. Both eBay's and Yahoo's buy-now auction formats have proven to be quite popular.[4] About 40% of eBay auctions are of the "buy-now" auction format that we study in our paper. At first glance, the popularity of buy-now auctions is puzzling. After all, an ascending bid auction is intended to elicit high bids from potential buyers. Putting a cap on these bids (as in a Yahoo buy-now auction) or offering a fixed price at the auction open (as in an eBay buy-now auction) would seem to limit the seller's expected revenue.

In this paper, focusing on the effects of bidder risk aversion, we analyze and then compare the Yahoo and eBay buy-now auctions. We characterize equilibrium bidding strategies for both auctions. We show that when bidders are risk averse, the introduction of a buy price raises seller revenue in both auctions for a wide range of buy prices. Intuitively, this is because a risk-averse bidder is willing to pay a buy price which includes a risk premium, rather than face uncertainty regarding whether he wins the auction and, if he wins, how much he pays. We also compare the eBay and Yahoo auction formats. When both auctions have the same reserve prices and buy prices, we demonstrate that the eBay and Yahoo buy-now auctions are payoff equivalent from the bidders' perspective if bidders have constant absolute risk aversion (CARA), the eBay auction is preferred if bidders have decreasing absolute risk aversion (DARA), and the Yahoo auction is preferred if bidders have increasing absolute risk aversion (IARA). The seller, however, obtains more revenue in the Yahoo auction provided that bidders have either constant or decreasing absolute risk aversion.

We utilize a symmetric independent private values framework with a continuous distribution of values for the n bidders. In both auction formats, at the auction open the bidders simultaneously choose whether to accept the buy price or wait. If a bidder accepts the buy price, then he pays the buy price and the auction ends. If all bidders wait then there is an ascending clock auction with the bid starting at the reserve. The eBay and Yahoo auctions differ in whether the buy-now option remains in effect during the ascending bid phase. On eBay the buy-now option is

[2] For more on this format see, http://www.auctions.yahoo.com/phtml/auc/us/promo/buy-now.html.

[3] See, http://www.pages.ebay.com/services/buyandsell/buyitnow.html.

[4] eBay's fixed price trading contributed approximately $3.4 billion or 32% of total gross merchandise sales during Q3–05, primarily from eBay's buy it now feature. Reynolds and Wooders (2003) find about 40% of eBay auctions and 66% of Yahoo auctions employ a buy price. Hof (2001) also cites a 40% figure for the fraction of eBay auctions that use the buy-now feature. Anderson et al. (2004) studied a sample of over 1,000 eBay auctions that resulted in a sale. They found that the seller offered a buy price in 49% of these auctions, and that the buy price was accepted 43% of the time when it was offered.

not present in the ascending bid phase and the auction ends when only one bidder remains; this bidder wins the auction and pays the current bid. We assume that during the ascending bid phase of the eBay auction each bidder remains active until the bid reaches his value. (Hence, the bidder with the highest value wins and pays the second highest value.) In the Yahoo auction the buy-now option remains in effect during the ascending bid phase; as the clock progresses, at any point a bidder may either remain in the auction, drop out of the auction, or accept the buy price. The auction ends when a bidder accepts the buy price or only one bidder remains. If a bidder accepts the buy price, then he pays the buy price. If only one bidder remains, this bidder wins the auction and pays the current bid. We assume that during the ascending bid phase of the Yahoo auction each bidder whose value is below the buy price remains active until the bid reaches his value; for bidders with values above the buy price, the bid at which they accept the buy price is determined in equilibrium.

In Sect. 3 we characterize equilibrium in the eBay buy-now auction, showing existence and uniqueness of a symmetric equilibrium when bidders are either risk neutral or have CARA. We show that the set of types that accept the buy price is decreasing in the buy price, increasing in the reserve price, and increasing as bidders become more risk averse. We also show that when bidders are risk averse then introducing a buy price into an eBay auction raises seller revenue for a wide range of buy prices. In particular, a buy price raises revenue if (i) it would be rejected by all bidder types if bidders where risk neutral, and (ii) it is accepted by some bidder types when bidders are risk averse. (The exact sufficient condition for a buy price to raise revenue is provided in the paper.) Moreover, for a given buy price, seller revenue is increasing as the bidders become more risk averse.

In Sect. 4 we turn to equilibrium in the Yahoo buy-now auction, showing existence and uniqueness of a symmetric equilibrium when bidders are either risk neutral or have CARA. Our analysis of the Yahoo auction proceeds by first establishing that the eBay and Yahoo buy-now auctions are utility equivalent from the bidders' perspective (under CARA) when the buy prices and the reserve prices are the same in both auctions. We then use this utility equivalence result to obtain the equilibrium bidding strategies in the Yahoo auction. We show that introducing a buy price into a Yahoo auction raises seller revenue when bidders are risk averse. Specifically, a buy price raises revenue so long as it would not be accepted by any risk-neutral bidder type at the auction open (if all bidders were risk neutral). Moreover, for any given buy price, if bidders were more risk averse then they would accept the buy price at lower bids and seller revenue would increase.

Our integrated framework allows us to compare the eBay and Yahoo auctions, and we are able to do so whether bidders have constant, increasing, or decreasing absolute risk aversion. Under CARA, the set of bidder types that accepts the buy-price immediately is the same for both auctions and bidders are indifferent between the two auction formats. Under DARA, we show that (i) more bidder types accept the buy price immediately in the Yahoo auction and (ii) bidders prefer the eBay auction. These results are reversed if bidders have IARA. We also show that the Yahoo auction yields more revenue to the seller than the eBay auction if bidders have either CARA or DARA. The auction formats are revenue equivalent if bidders are risk neutral.

1.1 Related literature

Durham et al. (2004) provide empirical evidence of the effect of a buy price. In a sample of 138 auctions of American silver dollars, they find that the 41 auctions listed with a buy price had an average selling price of $10.27, while the remaining auctions had an average selling price of $9.56, a statistically significant difference. This suggests that buy prices do, indeed, tend to raise seller revenue. (Of the 41 auctions listed with a buy price, 58% ended with a sale at the buy price, with an average sale price of $10.76.) They also find, consistent with our theoretical results, that lower buy prices are more likely to be accepted.

Several recent papers make important contributions to our understanding of auctions with a buy price. Mathews and Katzman (2006) model eBay buy-now auctions with risk-neutral bidders. They show that a risk-averse seller can raise his expected utility by setting a buy price. Mathews (2003a) explores the role of impatience in eBay buy-now auctions, in a model that allows for either impatient bidders or an impatient seller. He shows that a seller can increase his expected revenue by setting a buy price, thereby exploiting impatient bidders who are willing to pay a premium included in the buy price in order to end the auction early. Kirkegaard and Overgaard (2003) suggest another rationale for setting a buy price in eBay auctions. Their model has two risk-neutral bidders, each of whom demands two units. Two sellers sequentially offer a unit for sale in second-price auctions. Kirkegaard and Overgaard show that the first seller can raise his expected revenue by setting a buy price.

In contrast to these papers, our paper focuses on the consequences of bidder risk aversion on seller revenue. In addition to being of theoretical interest, there is substantial experimental evidence supporting bidder risk aversion in auctions. (See Chap. 7 of Kagel and Roth (1995) for a survey of experimental results.) This suggests that bidder risk aversion is likely of practical relevance in the field.[5]

Budish and Takeyama (2001) analyze a simple version of a Yahoo buy-now auction with two bidders and two possible valuations for each bidder—high or low. They demonstrate that when bidders are risk averse there is a buy price for which bidders with the high-value accept immediately, bidders with the low-value wait, and which yields more expected revenue to the seller than the ascending bid auction without a buy price.[6] Lopomo (1998), in a model with a general distribution for bidder values (which may be either independent or affiliated), studies a class of auctions he refers to as “simple sequential auctions”. This class includes the English ascending bid auction as well as other auctions, like the Yahoo buy-now auction, in which the item auctioned is available to bidders at a constant ask price throughout the course of the auction. Lopomo shows that if bidders are risk neutral, then the English auction is optimal within the class of all simple sequential auctions. Thus, when bidders are risk neutral, a Yahoo buy-now

[5] Theoretical work on the effect of bidder risk aversion on seller revenue goes back to Holt (1980). Recent work on the effect of bidder risk aversion on seller revenue includes work by Salmon and Iachini (2003), who study “pooled” auctions, Roberto Burguet (1999), who studies “right-to-choose” auctions, and Li and Tan (2000) who study secret reserve prices.

[6] In Budish and Takeyama the highest buy price with this property is denoted by B^*. They claim that this is the seller’s optimal buy price, without considering the revenue consequences of a buy price above B^*.

auction cannot yield more revenue for the seller than the English ascending bid auction.

In contrast to Budish and Takayama, our model of the Yahoo auction yields insights into the relationship between a bidder's value and the bid at which he accepts the buy price in the ascending bid phase of the auction. Our analysis, utilizing a model with n bidders with independent private values drawn from a general continuous distribution, also yields qualitatively different results. We find that the buy-now auction raises seller revenue even if the buy price is not accepted at the auction open by any bidder type. In contrast to Lopomo, we deal with bidder risk aversion in the Yahoo auction. We show that with bidder risk aversion, employing a buy price allows a seller to obtain more revenue than he could obtain using an English auction. In addition, we provide a characterization of the equilibrium bidding strategies in Yahoo auctions, whether bidders are risk averse or risk neutral.

Significantly, our analysis also differs from prior work in that we provide an integrated analysis of the eBay and Yahoo buy-now auctions. This allows us to compare the two auctions in terms of bidder payoffs, seller expected revenue, and the probability of a sale at the buy price.

Several recent papers have studied other novel features of online auctions. Roth and Ockenfels (2002) is an empirical analysis of the effect the auction closing rule—"hard" on eBay and "soft" on Amazon – has on bidding behavior and auction outcomes. Ariely et al. (2005) provide an experimental comparison of the effects of a hard and soft close. Peters and Severinov (2006) analyze the equilibrium strategies of sellers and bidders when many auctions are conducted simultaneously, as is often the case for online auction websites. Bajari and Hortacsu (2003) document empirical regularities in a sample of eBay coin auctions and estimate a structural model of bidding on eBay.

2 The model

There are $n \geq 2$ bidders for a single item whose values are independently and identically distributed according to cumulative distribution function F with support $[\underline{v}, \bar{v}]$, where F' is continuous and positive on $(\underline{v}, \bar{v})$. Let $G(v) = F(v)^{n-1}$ be the *c.d.f.* of the highest of $n-1$ values. Denote by v_i the value of bidder i. Let B denote the buy price set by the seller. We assume that $\underline{v} < B < \bar{v}$, since a seller would never wish to set $B \leq \underline{v}$, whereas if $B \geq \bar{v}$ then no bidder will ever take it. Denote by $r \in [\underline{v}, B)$ the minimum bid, or reserve, set by the seller.[7] If a bidder whose value is v wins the item and pays price p then his payoff is $u(v-p)$; he obtains a payoff of $u(0) = 0$ otherwise. In Sects. 3 and 4 we characterize the unique symmetric equilibrium in the eBay and Yahoo buy-now auctions, respectively, assuming bidders have constant absolute risk aversion, with $u(x) = (1-\mathrm{e}^{-\alpha x})/\alpha$ for $\alpha \geq 0$. Note $\lim_{\alpha \to 0} u(x) = x$, and hence $\alpha = 0$ corresponds to risk neutrality. In Sect. 5 we compare the auction formats under constant, decreasing, or increasing absolute risk aversion.

[7] If $r = \underline{v}$ then there is effectively no reserve, while if $r = B$ then the eBay and Yahoo buy-now auctions are both equivalent to a posted price of B.

2.1 eBay

At the open of the eBay buy-now auction the bidders simultaneously decide whether to "buy" or "wait". If some bidder chooses to buy, then he wins and he pays the seller B. (If more than one bidder chooses buy, then the winner is randomly assigned among these bidders.) The bidding process that follows if all the bidders wait is not explicitly modeled. Instead, we suppose that if all bidders wait, then the buy-now option disappears, and (i) if at least one bidder has a value of r or greater, then the bidder with the highest value wins the item and he pays the maximum of r and the second highest value, and (ii) if all bidders have a value less than r then the minimum bid is not met and item does not sell.[8] This is consistent with there being, for example, either an ascending clock auction (with the bid price starting at the minimum bid r) or a second-price sealed-bid auction, when all the bidders wait.

In the eBay buy-now auction a bidder's strategy tells him, for each possible value, whether to buy or wait. Under CARA there is no loss of generality in restricting attention to equilibria in "cutoff" strategies.[9] A cutoff strategy for a bidder is characterized by a value $c \in [B, \bar{v}]$ such that he chooses buy if his value exceeds c and chooses wait if his value is below c. Suppose that a bidder's value is $v > r$ and all his rivals employ the same cutoff c . The bidder's expected payoff if he chooses buy is

$$U^b(v, c) = u(v - B) \sum_{k=0}^{n-1} \binom{n-1}{k} \frac{1}{k+1} (1 - F(c))^k F(c)^{n-1-k},$$

where in this expression k is the number of other bidders who also choose to buy. If the bidder waits, then he wins the auction only if all his rivals also wait and he has the highest value. His expected payoff is

$$U^w(v, c) = \int_r^{\min\{v,c\}} u(v - y)\mathrm{d}G(y) + u(v - r)G(r).$$

A cutoff c^* is a **symmetric Bayes Nash equilibrium** if $U^w(v, c^*) > U^b(v, c^*)$ for all $v \in [\underline{v}, c^*)$ and $U^w(v, c^*) < U^b(v, c^*)$ for all $v \in (c^*, \bar{v}]$. That is, given that his rivals use the cutoff c^* then it is optimal for a bidder to wait if $v < c^*$ and it is optimal for a bidder to buy if $v > c^*$.

2.2 Yahoo buy-now auctions

We model the Yahoo buy-now auction as an ascending clock auction, in which the bid rises continuously from r to B. As the clock progresses, at any point a bidder

[8] In eBay auctions with a minimum bid (or reserve), the buy-now option disappears as soon as a bid is placed. eBay also allows sellers to set a "secret" reserve. In auctions with a secret reserve, the buy-now option remains active until a bid is placed that exceeds the secret reserve. We do not address the issue of a secret reserve in this paper.

[9] In particular, the best response to any profile of arbitrary strategies by rival bidders is a cutoff strategy.

may either drop out of the auction or accept the buy price. The auction ends when either all but one bidder has dropped out, or when a bidder accepts the buy price. In the former case, the remaining bidder wins and pays the current bid price. In the later case, the bidder who accepts the buy price wins and he pays B. (If all bidders drop at the bid of r then the auction ends without a winner.) As the clock progresses, bidders observe only the current bid, and not the number of remaining bidders.

Clearly a bidder whose value is less than B never accepts the buy price since by doing so he obtains a negative payoff, whereas he would obtain a payoff of zero by dropping out. We assume that such bidders simply drop out when the bid reaches their value, with bidders whose values are below r dropping out immediately. Similarly, a bidder whose value is above B never drops out since whatever the current bid is, he obtains a positive payoff accepting the buy price but would obtain zero by dropping out. Thus we focus on how bidders whose values are above B choose the bid at which to accept the buy price. A strategy for a bidder is a function which gives for each value v in $[B, \bar{v}]$ a threshold bid price $t(v)$ (in $[r, B]$) at which the bidder accepts the buy price. As we shall see, a threshold strategy may have a "jump down" at z with $t(v) > r$ for v in $[B, z]$ and $t(v) = r$ for v in $(z, \bar{v}]$. In this case a bidder with value $v > z$ accepts the buy price at the auction open.

To see how the bidders' payoffs are determined given a profile of threshold strategies, it is useful to consider Fig. 1 which shows the maximum of the other bidders' values (denoted by y) on the horizontal axis. Consider bidder 1 and suppose all

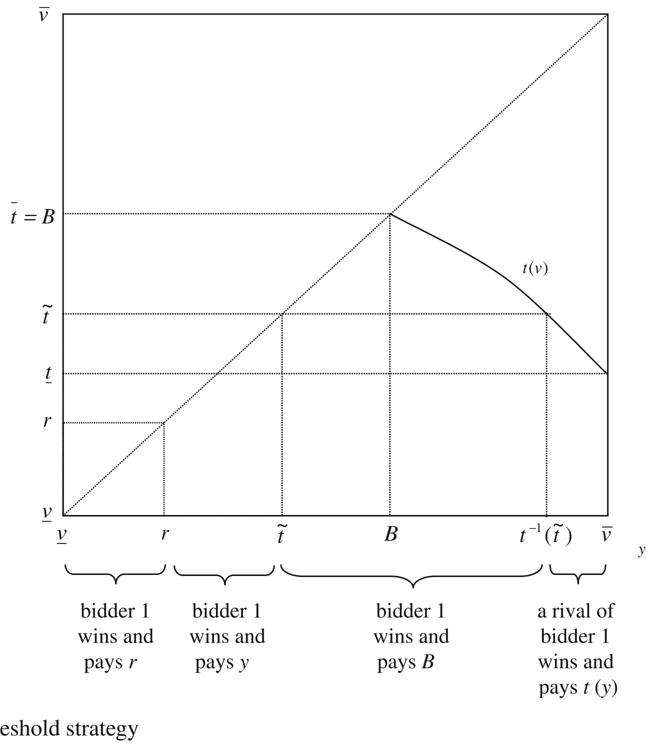

Fig. 1 A threshold strategy

the other bidders follow the threshold strategy $t(v)$. Let $[\underline{t}, \bar{t}]$ denote the range of threshold values for which $t(v)$ is strictly decreasing. (For $t(v)$ as in the figure, $\bar{t} = B$ and $\underline{t} = t^{-1}(\bar{v})$.) Suppose bidder 1 chooses the threshold $\tilde{t}$ (shown on the vertical axis). If y is less than r, then all the other bidders drop out at r, bidder 1 wins and he pays r. If y is between r and $\tilde{t}$, then bidder 1 is the last remaining bidder when the bid reaches y, bidder 1 wins and he pays y, the price at which the last of his rivals dropped out. If y is above $\tilde{t}$ but below $t^{-1}(\tilde{t})$, then bidder 1 accepts the buy price when the bid reaches $\tilde{t}$, he wins the auction, and he pays B. Finally, if y is above $t^{-1}(\tilde{t})$, then the bidder with value y accepts the buy price when the bid reaches $t(y)$, he wins the auction, and he pays B.

Hence, if a bidder's value is $v > r$, he chooses the threshold $\tilde{t}$, and the other bidders follow the threshold strategy t (one without a jump down) then the bidder's expected utility is

$$U(\tilde{t}, v; t) = \begin{cases} G(r)u(v-r) + \int_r^{\tilde{t}} u(v-y)\mathrm{d}G(y) + [G(t^{-1}(\tilde{t})) - G(\tilde{t})]u(v-B) \\ \quad \text{if } \tilde{t} \in [\underline{t}, \bar{t}] \\ \\ G(r)u(v-r) + \int_r^{\tilde{t}} u(v-y)\mathrm{d}G(y) + [1 - G(\tilde{t})]u(v-B) \\ \quad \text{if } \tilde{t} < \underline{t}. \end{cases}$$

Note that if $\tilde{t} < \underline{t}$ then the bidder wins for sure, paying r if the maximum value of a rival is less than r, paying the maximum of his rivals' values when this maximum is less than $\tilde{t}$, and paying B otherwise.

We say that a threshold strategy t is a **(symmetric) Bayes Nash equilibrium** if for each $v \in [B, \bar{v}]$ we have

$$U(t(v), v; t) \geq U(\tilde{t}, v; t) \quad \forall \tilde{t} \in [\underline{v}, B].$$

In other words, for each value v a bidder's optimal threshold is $t(v)$ when the other bidders follow the threshold strategy t.

We take both the eBay and Yahoo auctions without a buy price as being equivalent to an English ascending bid auction. In particular, the bidder with the highest value wins at a price equal to the maximum of the reserve and the second highest value.

While these models capture salient features of buy-now auctions as they are implemented on eBay and Yahoo, there are some differences. eBay auctions end at a predetermined time specified by the seller (i.e., they have a "hard" close). eBay and Yahoo auctions are not conducted as ascending clock auctions, but on eBay bidders submit "proxy" bids and on Yahoo bidders may bid either a fixed amount or may make proxy bids.[10] Last, here we have supposed that there is a fixed commonly known number of bidders, a condition that is unlikely to prevail in actual Internet auctions.

[10] See http://pages.ebay.com/help/buy/proxy-bidding.html for a description of proxy bidding.

3 eBay buy-now auctions

In this section we compare eBay auctions with and without buy prices. In characterizing equilibrium of the eBay buy-now auction with reserve r it is useful to first consider an eBay auction with the same reserve, but without a buy price. Consider a bidder whose value is v and who is either risk neutral or CARA risk averse with index of risk aversion $\alpha > 0$. If the bidder wins in the auction with no buy-price, he makes a (random) payment of $\max\{r, y\}$, where y denotes the maximum of his rivals' values. The certainty equivalent payment, denoted by $\delta_\alpha(v)$, is defined by

$$u(v - \delta_\alpha(v)) = E\left[u(v - \max\{r, y\})|\underline{v} \leq y \leq v\right], \tag{1}$$

where y is distributed according to G. In other words, a bidder with value v is indifferent between winning the auction (and making a random payment of $\max\{r, y\}$) and winning and paying the certain amount $\delta_\alpha(v)$. When bidders are risk neutral (i.e., $\alpha = 0$) then Eq. (1) reduces to

$$\delta_0(v) = E\left[\max\{r, y\}|\underline{v} \leq y \leq v\right].$$

To simplify notation we suppress the dependence of $\delta_\alpha(v)$ on the reserve price r and the distribution G.

The certainty equivalent payment $\delta_\alpha(v)$ has several important properties: $\delta_\alpha(r) = r$, $\delta_\alpha(v)$ is increasing in v, and $\delta_\alpha(v) < v$ for $v > r$. Furthermore, $\delta_\alpha(v)$ is increasing in α for $v > r$, i.e., as a bidder becomes more risk averse he is willing to pay more to avoid the uncertain payment of the auction.

Proposition 1 characterizes equilibrium in the eBay buy-now auction for risk-neutral and risk averse bidders.

Proposition 1 *Suppose bidders are risk neutral* $(\alpha = 0)$ *or CARA risk averse with index of risk aversion* $\alpha > 0$. *Consider an eBay auction with reserve price* r *and buy price* B.

(i) *If* $B \geq \delta_\alpha(\bar{v})$ *then the buy price is never accepted by a bidder in equilibrium, i.e., the unique symmetric equilibrium cutoff value is* $c^* = \bar{v}$.
(ii) *If* $B < \delta_\alpha(\bar{v})$ *then there is a unique symmetric equilibrium cutoff* $c^* \in (B, \bar{v})$ *that is implicitly defined by*

$$u(c^* - B)Q(F(c^*)) = u(c^* - \delta_\alpha(c^*))G(c^*),$$

where

$$Q(F(c^*)) = \left[\frac{1 - F(c^*)^n}{n(1 - F(c^*))}\right].$$

This cutoff value is increasing in B, *decreasing in* r, *and decreasing in* α. *The equilibrium is inefficient since the high-value bidder is awarded the item with probability less than one.*[11]

[11] By a simple application of the Revenue Equivalence Principle, when bidders are risk-neutral the equilibrium cutoff is the same so long as any standard auction follows when all bidders reject the buy price.

Proof Appendix.

Proposition 1(i) is intuitive. The certainty equivalent of the payment made by a bidder with value $\bar{v}$ is $\delta_\alpha(\bar{v})$, if he and all his rivals reject the buy price. Hence, if the buy price B exceeds $\delta_\alpha(\bar{v})$, then such a bidder prefers to reject the buy price when all his rivals also reject it. When a bidder with the highest value optimally rejects the buy price, then bidders with lower values optimally reject as well. Hence it is an equilibrium for all bidders to reject the buy price when $B \geq \delta_\alpha(\bar{v})$.

If $B < \delta_\alpha(\bar{v})$ then it is no longer an equilibrium for all bidders to reject the buy price. In particular, a bidder with value $\bar{v}$ would optimally accept the buy price if his rivals always rejected it. In equilibrium, a bidder with value c^* (defined implicitly in Proposition 1(ii)) is just indifferent between accepting the buy price and rejecting it, when his rivals follow the strategy of accepting the buy price if their value is above c^* and rejecting it otherwise. The buy-now auction is inefficient when the buy price is set low enough so that some bidder types accept it. The inefficiency is similar to the inefficiency that results when a single item is offered for sale at a fixed price to multiple buyers. If there is no mechanism to put the high-value buyer at the head of the queue of buyers, then there is a positive probability that the high-value buyer will not receive the item.

Proposition 1(ii) also establishes some intuitive comparative statics when $B < \delta_\alpha(\bar{v})$. Bidders are less likely to accept the buy price as the buy price increases (since c^* is increasing in B). Bidders are more likely to accept the buy price as the reserve price increases or as bidders become more risk averse. The increased willingness of more risk averse bidders to accept the buy price follows from two effects: First, acceptance of the buy price reduces the chance that a bidder will "lose", i.e., not be awarded the object, and have a zero surplus. (Acceptance of the buy price does not completely eliminate uncertainty for a bidder, because there is a chance another bidder will also accept; the object is randomly awarded in this case.)[12] A risk averse bidder is willing to trade off this reduced chance of losing for a lower expected surplus. Second, conditional on winning the auction, a more risk averse bidder is willing to make a higher certain payment in order to avoid the random payment he would make if he won in the ascending bid phase of the auction (i.e., $\delta_\alpha(v)$ is increasing in α).

3.1 Seller revenue

Myerson (1981) shows that when bidders are risk neutral, a first or second-price sealed-bid auction or an English ascending bid auction are each revenue-maximizing mechanisms, provided that the reserve price is set optimally.[13] Hence, when bidders are risk neutral ($\alpha = 0$), there is no advantage to a (risk-neutral) seller to setting a buy price. Indeed, taking the reserve price as given, it's easy to see from Myerson's characterization of the optimal mechanism that, in order to maximize

[12] Bidders with values $v > c^*$ accept the buy price and win with probability $Q(F(c^*))$. If such a bidder were to instead wait, then he wins in the ascending bid auction that follows with probability $F(c^*)^{n-1}$, i.e., he wins so long as none of his rivals accepts the buy price. In the proof of Proposition 1 it is established that $Q(F(c^*)) > F(c^*)^{n-1}$.

[13] This optimality result requires the regularity assumption that $J(v) = v - (1 - F(v))/F'(v)$ is increasing in v. The optimal reserve price satisfies $J(r) = 0$. See Burguet (2000) for a nice discussion of this result.

seller revenue, the object must be awarded to the bidder with the highest value (provided that this value exceeds the reserve). In an eBay buy-now auction with $B < \delta_0(\bar{v})$, with positive probability the object is not awarded to the bidder with the highest value. Thus, in this case, the eBay buy-now auction raises strictly less revenue than the eBay auction without a buy price and the same reserve.

Bidder risk aversion has no effect on seller revenue in an eBay auction without a buy price. However, if bidders are risk averse then setting a buy price can be advantageous for the seller. Consider any buy price B which is accepted with positive probability by bidders with index of risk aversion $\alpha > 0$, but which would be rejected if bidders were risk-neutral (that is, consider any B satisfying $\delta_0(\bar{v}) < B < \delta_\alpha(\bar{v})$). Let c_α^* denote the equilibrium cutoff. (Since $B < \delta_\alpha(\bar{v})$ then $c_\alpha^* < \bar{v}$.) An auction with such a buy price and with reserve r raises more revenue than an eBay auction with the same reserve and no buy price. To see this, let bidder 1's value v_1 be fixed and suppose, without loss of generality, that y (the maximum of his rivals' values) is less than v_1. If $v_1 < c_\alpha^*$ then the buy price is not accepted by any bidder and seller revenue is $\max\{r, y\}$, the same as in the auction without a buy price. If $v_1 > c_\alpha^*$ then seller revenue is B in the buy-now auction and is $\max\{r, y\}$ in the auction without a buy price. Now, B may be either more or less than $\max\{r, y\}$. However, B is greater than the expectation of $\max\{r, y\}$ since

$$E[\max\{r, y\} | \underline{v} \leq y \leq v_1] = \delta_0(v_1) \leq \delta_0(\bar{v}) < B,$$

where the equality holds by the definition of $\delta_0(v_1)$, the weak inequality holds since $\delta_0(v)$ is increasing in v, and the strict inequality holds by assumption. We have shown that the seller's expected revenue conditional on bidder 1 winning is (i) the same whether or not the seller sets a buy price if $v_1 < c_\alpha^*$, and is (ii) higher in the auction with the buy price if $v_1 > c_\alpha^*$. Since $c_\alpha^* < \bar{v}$, then $v_1 > c^*$ with positive probability, and hence the seller's *ex-ante* expected revenue is higher in the auction with the buy price.[14]

The following corollary summarizes these results.

Corollary 1 *Suppose bidders are CARA risk averse with index of risk aversion* $\alpha > 0$*. Consider an eBay buy-now auction with reserve price r and buy price B. If* $\delta_0(\bar{v}) < B < \delta_\alpha(\bar{v})$ *then expected seller revenue in the buy-now auction exceeds expected revenue in the eBay auction with the same reserve and no buy price.*

Corollary 1 suggests why eBay introduced the buy-now auction format, and why it has proven to be so popular with sellers—this format has the potential for raising seller revenue and eBay's own auction revenue (which is a percentage of seller revenue) relative to standard eBay auctions.

Corollary 1, while it doesn't identify the optimal buy price, does show the seller how to set a buy price that raises revenue. Given the uncertainty a seller is likely to face regarding the degree of bidder risk aversion and the distribution of bidders' values, providing a range of revenue-improving buy prices may be of more practical use than providing conditions characterizing an optimal buy price.

[14] Note that CARA is not necessary for this argument. Let u be a concave utility function and define $\delta_u(v)$ such that $u(v - \delta_u(v)) = E[u(v - \max\{r, y\}) | \underline{v} \leq y \leq v]$. Provided that a symmetric equilibrium in cutoff strategies exists and the cutoff is less than $\bar{v}$, the same argument shows that any buy price B satisfying $\delta_0(\bar{v}) < B < \delta_u(\bar{v})$ raises seller revenue.

One can see that for any *given* buy price, seller revenue increases as bidders become more risk averse. Suppose bidders have index of risk aversion α', but the index of risk aversion increases to α''. Consider a buy-now auction with reserve r and buy price B, where $B < \delta_{\alpha''}(\bar{v})$. Let $c^*_{\alpha'}$ and $c^*_{\alpha''}$ denote the equilibrium cutoff for $\alpha = \alpha'$ and $\alpha = \alpha''$, respectively. By Proposition 1(ii) we have $c^*_{\alpha''} < c^*_{\alpha'}$. Let bidder 1's value v_1 be fixed and suppose $v_1 > y$. If $v_1 < c^*_{\alpha''}$ then seller revenue is $\max\{r, y\}$, whether bidders have risk aversion index α' or α''; if $v_1 > c^*_{\alpha'}$ then seller revenue is B whether $\alpha = \alpha'$ or $\alpha = \alpha''$. If $c^*_{\alpha''} < v_1 < c^*_{\alpha'}$ then revenue is B if $\alpha = \alpha''$, whereas if $\alpha = \alpha'$ then expected revenue is

$$E[\max\{r, y\}|\underline{v} \le y \le v_1] = \delta_0(v_1) < \delta_{\alpha'}(c^*_{\alpha'}) \le B,$$

where the first inequality holds since $\delta_\alpha(v)$ is increasing in α and in v. If $c^*_{\alpha'} = \bar{v}$ then by Proposition 1(ii) we have $B \ge \delta_{\alpha'}(\bar{v})$ and the second equality holds immediately. If $c^*_{\alpha'} < \bar{v}$ then $B < \delta_{\alpha'}(\bar{v})$ and

$$u(c^*_{\alpha'} - B)Q(F(c^*_{\alpha'})) = u(c^*_{\alpha'} - \delta_{\alpha'}(c^*_{\alpha'}))G(c^*_{\alpha'}).$$

Since $Q(F(c^*_{\alpha'})) > G(c^*_{\alpha'})$ then $\delta_{\alpha'}(c^*_{\alpha'}) < B$ and, again, the second inequality holds. Hence, seller revenue increases as bidders become more risk averse.

Corollary 2 *Consider an eBay buy-now auction with reserve price r and buy price B. If the index of bidder risk aversion increases from α' to α'' then seller revenue strictly increases unless $B \ge \delta_{\alpha''}(\bar{v})$, i.e., unless the buy price is always rejected even when bidders have the higher index of risk aversion α''.*

The key difference between the eBay and Yahoo buy-now auctions is the temporary nature of the buy price in the eBay auction. To understand incentives in the Yahoo auction it is useful to consider the incentives of a bidder in the eBay auction if he had the (hypothetical) option to accept the buy price once the bid begins to ascend. A bidder who waits and observes the bid price begin to rise above the reserve learns (i) $y < c^*$, i.e., the "good" news that no rival has a value above c^*, and (ii) $y > r$, i.e., the "bad" news that at least one rival has a value above r. Hence, a bidder with value $v < c^*$ who waits has, once the ascending bid phase of the eBay auction begins, an expected utility of

$$E[u(v - y)|r < y \le v]\frac{G(v) - G(r)}{G(c^*) - G(r)},$$

whereas if he could accept the buy price his utility would be $u(v - B)$. We say that there is **no regret** in the buy-now auction if

$$E[u(c^* - y)|r < y \le c^*] \ge u(c^* - B).$$

This condition states that an eBay bidder with value c^* has an expected utility, once the bid begins to ascend, at least as great as he would obtain were he able to accept the buy price. In other words, a bidder does not regret forgoing the buy now option. He would not accept the buy price even if it were available. The no regret condition implies that a bidder with a value $v < c^*$ would also not accept the buy price (if it were available) once the ascending bid phase of the auction begins.

Intuitively, no regret will be satisfied if (i) either the buy price is high, or (ii) not too much bad news is revealed when a bidder doesn't win at the reserve price r; in other words, the probability that $y \in [\underline{v}, r]$ is small. In particular, no regret always holds when there is no reserve. Remark 1 formalizes this idea. No regret will tend to fail as the buy-now auction comes to resemble a posted price mechanism, with the reserve and the buy price close.

Remark 1 The no regret condition holds if either (i) the buy price is sufficiently high, or (ii) the reserve is sufficiently small or there is no reserve, i.e., if $r = \underline{v}$.

Proof Appendix.

As we shall see, the satisfaction of the no regret condition plays an important role in the existence of an equilibrium in the Yahoo buy-now auction.

4 Yahoo buy-now auctions

In this section we compare the Yahoo buy-now auction to the Yahoo auction without a buy price. We begin by establishing that the eBay and Yahoo buy-now auctions are utility equivalent for bidders. This result will be exploited in order to characterize equilibrium in the Yahoo buy-now auction.

Proposition 2 *Assume bidders are risk neutral* ($\alpha = 0$) *or CARA risk averse with index of risk aversion* $\alpha > 0$*. Consider an eBay auction and a Yahoo auction where the reserve price is* r *and the buy price is* B *for both auctions. Let* t *be an equilibrium threshold function of the Yahoo auction* (*which is differentiable except possibly at one point* z *where it jumps down*)*, and let* c^* *be the equilibrium cutoff in the eBay auction* (*see Proposition 1*)*.*

- (i) *The Yahoo and eBay auctions are utility equivalent for the bidders, i.e., a bidder whose value is* v *obtains the same expected utility in the Yahoo auction as in the eBay auction.*
- (ii) *If* $B \geq \delta_\alpha(\bar{v})$ *then the equilibrium threshold strategy* t *has no jump down. If* $B < \delta_\alpha(\bar{v})$ *then* t *jumps down at* c^**, with* $t(v) > r$ *if* $v \leq c^*$ *and* $t(v) = r$ *if* $v > c^*$.

Proof Appendix.

Proposition 2(i) shows that bidders are indifferent between the eBay and Yahoo buy-now auctions when the reserve and buy prices are the same in both auctions. For sufficiently high buy prices (i.e. $B \geq \delta_\alpha(\bar{v})$), bidders are indifferent between the eBay buy-now auction and the English ascending bid auction.[15] This implies that bidder utility is constant in the Yahoo buy-now auction even as the buy price decreases, so long as it remains above $\delta_\alpha(\bar{v})$. Matthews (1987) shows that CARA bidders are indifferent between the English ascending bid auction and the first-price sealed-bid auction. These results imply that bidders are indifferent between all four auction formats—eBay buy-now, Yahoo buy-now, English ascending bid,

[15] If $B > \delta_\alpha(\bar{v})$ then by Proposition 1(i) the buy price is not accepted by any bidder in the eBay buy-now auction, and hence the auction is trivially equivalent to the English ascending bid auction.

and first-price sealed-bid—when in each case the reserve is the same and the buy price is high.

Propositions 1 and 2(ii) shows that the set of values for which a bidder accepts the buy price immediately is the same for the Yahoo and eBay auctions. If $B < \delta_\alpha(\bar{v})$ then bidders in both types of auctions will accept the buy price immediately if their value is above c^*, but wait otherwise. If $B \geq \delta_\alpha(\bar{v})$ then the buy price is not *immediately* accepted in either auction. In the Yahoo auction, however, the buy price is accepted with higher total probability since it is accepted with positive probability in the ascending bid phase of the auction.

Even though the two auction formats are utility equivalent, the *ex-post* outcomes are generally different. In the eBay auction if all the bidders wait, then the final price is the second highest value (which may be more or less than B). In the Yahoo auction the final price never exceeds B.

Proposition 3 *Assume bidders are CARA risk averse with index of risk aversion $\alpha \geq 0$. Consider a Yahoo auction with reserve price r and buy price B such that "no regret" holds. There is a unique symmetric equilibrium $t(v)$ in threshold strategies that are differentiable (except possibly at one point where the threshold strategy jumps down).*

(i) *If $B \geq \delta_\alpha(\bar{v})$ then $t(v)$ is defined implicitly by*

$$E[u(v-y)|t(v) \leq y \leq v] = u(v-B), \tag{2}$$

for $v \in [B, \bar{v}]$, or equivalently

$$\int_{t(v)}^{v} \left(e^{\alpha y} - e^{\alpha B}\right) dG(y) = 0.$$

(ii) *If $B < \delta_\alpha(\bar{v})$ then $t(v)$ is as given above for $v \in [B, c^*]$ and $t(v) = r$ for $v \in (c^*, \bar{v}]$.*

Proof Appendix.

Proposition 3 is established by exploiting the utility equivalence for bidders of the eBay and Yahoo auction. According to Eq. (2), the equilibrium threshold $t(v)$ has a natural economic interpretation: $t(v)$ makes the buy price B equal to the certainty equivalent of the random payment a bidder would make in an English ascending auction in which the maximum of his rivals' values is known to be between $t(v)$ and v. Equation (2) can be used to easily numerically calculate the equilibrium threshold function.

4.1 Seller revenue

If $B \geq \delta_0(\bar{v})$ then in the Yahoo buy-now auction the bidder with the highest value wins the auction, and hence by the Revenue Equivalence Theorem the Yahoo buy-now auction is revenue for risk-neutral bidders equivalent to the Yahoo auction without a buy price. If $B < \delta_0(\bar{v})$, then by Proposition 3(ii) the buy price is accepted immediately with positive probability. For the reasons discussed for the eBay auction, in this case the introduction of a buy price lowers seller revenue.

Intuition would suggest that bidders are quicker to accept the buy price as they are more risk averse. Corollary 3 shows this is indeed the case; bidders in a Yahoo buy-now auction choose lower thresholds when they are more risk averse.

Corollary 3 *The equilibrium threshold function shifts down as bidders become more risk averse. In particular, let $\alpha'' > \alpha' \geq 0$ and let t_α be the equilibrium threshold function of a Yahoo auction with buy price B and reserve r when bidders are CARA risk averse with index of risk aversion α. Then for $v > B$ we have $t_{\alpha''}(v) < t_{\alpha'}(v)$ if $t_{\alpha'}(v) > r$ and $t_{\alpha''}(v) = r$ if $t_{\alpha'}(v) = r$.*

Proof Appendix.

An immediate consequence of Corollary 3 is that for any given buy price, seller revenue increases as bidders become more risk averse. Figure 2 shows a shift down in the equilibrium threshold function (from $t_{\alpha'}$ to $t_{\alpha''}$) when the index of risk aversion increases from α' to α''. Let bidder 1's value v_1 be fixed and suppose that v_1 exceeds the maximum value y of his rival. For combinations (v_1, y) below $t_{\alpha''}(v_1)$, seller revenue is y whether bidders have index of risk aversion α' or α''. For (v_1, y) above $t_{\alpha'}(v_1)$, seller revenue is B in each case. For (v_1, y) that lie between the two threshold functions, seller revenue is B when bidders are more risk averse (i.e., $\alpha = \alpha''$), and seller revenue is y, where $y < B$, when bidders are less risk averse (i.e., $\alpha = \alpha'$).

Hence we have the following Corollary.

Corollary 4 *Consider a Yahoo buy-now auction with reserve r and buy price B. If the index of risk aversion increases from α' to α'' then seller revenue strictly increases.*

As noted earlier, when bidders are risk neutral and $B \geq \delta_0(\bar{v})$ then the Yahoo buy-now auction yields the same revenue as the auction without a buy price. By

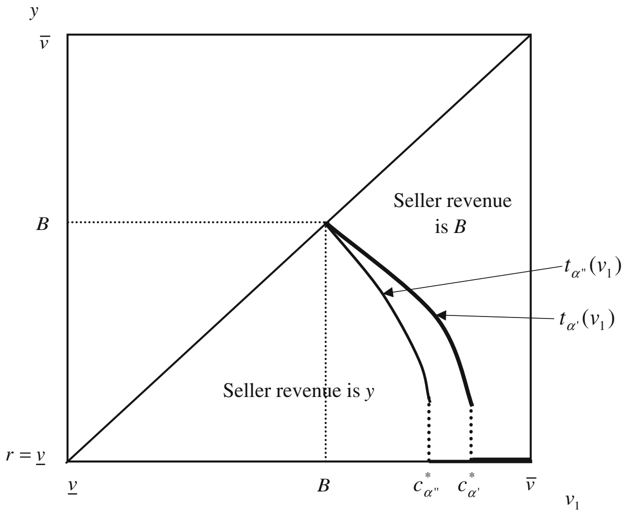

Fig. 2 Equilibrium threshold strategies with $\alpha = \alpha'$ and $\alpha = \alpha''$

Corollary 4 revenue in the buy-now auction increases as bidder become more risk averse. Since revenue in the Yahoo auction without a buy price doesn't depend on risk attitudes, we have the following result.

Corollary 5 *Suppose bidders are CARA risk averse with index of risk aversion $\alpha > 0$. Consider a Yahoo buy-now auction with reserve price r and buy price B, where $B \geq \delta_0(\bar{v})$. Then expected seller revenue in the buy-now auction exceeds expected revenue in the Yahoo auction with the same reserve and no buy price.*

Corollary 5 shows that when bidders are risk averse, introducing a buy price into a Yahoo auction raises revenue for a wide range of buy prices. Corollaries 2 and 4 point out another difference between the eBay and Yahoo buy-now auction. In the eBay auction seller revenue is constant as bidders become more risk averse as long as the buy price is so high that it is not accepted; in the Yahoo buy-now auction seller revenue increases as bidders become more risk averse, for any B less than $\bar{v}$.

5 Comparing eBay and Yahoo

We now turn to a comparison of eBay and Yahoo buy-now auctions assuming that bidders are risk averse. In addition to the case of CARA bidders considered in prior sections, we will consider bidders with decreasing absolute risk aversion (DARA) and increasing absolute risk aversion (IARA).[16] Denote the equilibrium cutoff in the eBay auction by $c_e \in (B, \bar{v}]$. Denote by $c_y \in (B, \bar{v}]$ the point at which the Yahoo equilibrium threshold function jumps down with $t(v) = r$ for $v \in (c_y, \bar{v}]$. Proposition 2 established that $c_e = c_y$ for the case of CARA utility.

Proposition 4 *Consider an eBay auction and a Yahoo auction where the reserve price is r and the buy price is B for both auctions. Let t be an equilibrium threshold function of the Yahoo auction with a jump down at c_y. Let c_e be the equilibrium cutoff in the eBay auction.*

(i) *If bidders have DARA then either $c_y = c_e = \bar{v}$, i.e., the Yahoo threshold function has no jump down, or $c_y < c_e$, i.e., the Yahoo threshold function jumps down at a value below the eBay equilibrium cutoff. If bidders have IARA, then either $c_y = c_e = \bar{v}$ or $c_y > c_e$.*
(ii) *If bidders have DARA then bidders prefer the eBay auction to the Yahoo auction. In particular, for each value $v \in (B, \bar{v}]$ the equilibrium payoff of a bidder whose value is v is higher in the eBay auction than the Yahoo auction. If bidders have IARA then bidders prefer the Yahoo auction to the eBay auction.*

Proof Appendix.

Under CARA we have $c_y = c_e$ by Proposition 2, while under DARA $c_y < c_e$ by Proposition 4. Hence an implication of these propositions is that the probability the buy price is accepted immediately is at least as high in the Yahoo auction as

[16] Sections 3 and 4 provide equilibrium existence results for eBay and Yahoo buy-now auctions for CARA bidders. The comparisons we make for DARA and IARA bidders are contingent on existence of equilibria for eBay and Yahoo buy-now auctions.

in the eBay auction when bidders have CARA or DARA. Since the buy price is accepted with positive probability in the ascending bid phase of the Yahoo auction, another testable implication of the model is that the buy price is accepted with higher overall probability in the Yahoo auction.

The following Corollary establishes that seller revenue is higher in the Yahoo auction if bidders have CARA or DARA.

Corollary 6 *Consider an eBay auction and a Yahoo auction where the reserve price is r and the buy price is B for both auctions. If bidders are CARA risk averse with index of risk aversion $\alpha > 0$ or if bidders have DARA, then the Yahoo auction raises more revenue than the eBay auction. If bidders are risk neutral then the seller's expected revenue is the same for both types of auctions.*

Proof Appendix.

The intuition for the revenue superiority of the Yahoo to the eBay auction is as follows: When a bidder accepts the buy price he pays a premium to avoid the uncertainty of the random payment he would make if he were to continue in the auction. In particular, suppose that bidder 1 has the highest value v_1. If $c_y \leq c_e$ then the seller extracts this risk premium in both types of auctions when $v_1 \geq c_e$ since in this case the buy price is accepted immediately in both auctions. If $v_1 < c_e$ then the buy price is rejected in the eBay auction and the seller captures no risk premium. In contrast, in the Yahoo auction the continues to extract a risk premium, either at the auction open (if $c_y < v_1 < c_e$) or in the ascending bid phase of the auction if ($v_1 < c_y$ and $t(v_1) < y$). Hence, ex-ante, the Yahoo auction raises more revenue.

Under IARA, where $c_e < c_y$, the revenue comparison is not clear. If $v_1 \in (c_e, c_y)$ then seller revenue is higher in the eBay auction. In particular, in the eBay auction revenue is B, while in the Yahoo auction it is $\max\{r, y\}$ if $y < t(v_1)$ and it is B if $t(v_1) < y$. If $v_1 \in (B, c_e]$ then revenue is higher in the Yahoo auction since the buy price may be accepted in the ascending bid phase of the auction, while it is rejected in the eBay auction. The overall comparison of revenue depends on which of these effects dominates. We have computed equilibria for several examples with IARA and have found higher revenue in the Yahoo auction.

6 Conclusion

We have formulated models that capture key features of auctions with buy prices, as implemented on the eBay and Yahoo auction sites. The eBay and Yahoo versions of the buy-now auction differ in the timing of the buy price option; in eBay the buy price is available to bidders only at the beginning of the auction, whereas in Yahoo the buy price is available throughout the auction. We characterized equilibrium strategies for risk neutral and risk averse bidders in buy-now auctions, using an independent private values framework. When bidders are risk neutral, the eBay and Yahoo buy-now auctions yield the same expected seller revenue, given that the buy price and the reserve are the same in the two auctions. These two buy-now auctions also yield the same expected seller revenue as the ascending bid auction, given the same reserve, if the buy price is high enough that it is not accepted at the beginning of the auction (for the eBay buy-now auction, this means that the buy price is never accepted). If the buy price is accepted with positive probability at the beginning of the auction, then the buy-now auctions yield lower expected seller revenue than the ascending bid auction with the same reserve.

When bidders are risk averse an auction with a buy price is a simple mechanism that permits a seller to earn more expected revenue than an ascending bid auction. We focus on the case of bidders with constant absolute risk aversion. Given CARA bidders and a particular reserve, we show that there are wide ranges of buy prices for eBay and Yahoo buy-now auctions that yield higher expected revenue for the seller than the ascending bid auction. CARA bidders are indifferent between an eBay and Yahoo buy-now auction, if the auctions have the same buy price and the same reserve. However, the seller is not indifferent. Given CARA bidders, a seller earns higher expected revenue in the Yahoo buy-now auction than in the eBay buy-now auction. We show that this seller revenue ranking of Yahoo and eBay buy-now auctions is preserved if bidders have decreasing absolute risk aversion.

There are a number of ways in which this analysis might be extended. More general preferences for risk averse bidders could be introduced into the model. One could introduce sequential (possibly random) entry of bidders into the auction or bidder impatience. (Mathews (2003a,b) addresses these issues in eBay buy-now auctions with risk-neutral bidders.) Within the independent private values framework one could consider an uncertain numbers of bidders. It would also be interesting to consider a common or affiliated values setting, since some goods auctioned on the Internet surely have a common value component (see Bajari and Hortacsu (2003)).

Appendix

Lemma 1 *The certainty equivalent* $\delta_\alpha(v)$ *in* (1) *satisfies*

$$\delta_0(v) = E[\max\{r, y\}|y \le v],$$

if bidders are risk neutral (i.e., $\alpha = 0$*), and it satisfies*

$$e^{\alpha\delta_\alpha(v)} = \frac{1}{G(v)}\left[G(r)e^{\alpha r} + \int_r^v e^{\alpha y}\mathrm{d}G(y)\right], \tag{3}$$

if bidders are CARA risk averse with index of risk aversion $\alpha > 0$.

Proof We prove the lemma only for the case where bidders are risk averse. By the definition of $\delta_\alpha(v)$ we have

$$\frac{1 - e^{-\alpha(v-\delta_\alpha(v))}}{\alpha} = \frac{1}{G(v)}\left[\frac{1 - e^{-\alpha(v-r)}}{\alpha}G(r) + \int_r^v \frac{1 - e^{-\alpha(v-y)}}{\alpha}\mathrm{d}G(y)\right],$$

or

$$e^{-\alpha(v-\delta_\alpha(v))} = \frac{1}{G(v)}\left[e^{-\alpha(v-r)}G(r) + \int_r^v e^{-\alpha(v-y)}\mathrm{d}G(y)\right].$$

Multiplying both sides by $e^{\alpha v}$ yields the result. □

Lemma 2 *Whether bidders are risk neutral or risk averse, for $v > r$ we have*

$$\delta'_\alpha(v) = \frac{G'(v)}{G(v)} \frac{u(v - \delta_\alpha(v))}{u'(v - \delta_\alpha(v))} > 0. \tag{4}$$

Proof When $\alpha > 0$, then differentiating (3) with respect to v yields

$$\alpha\delta'_\alpha(v)\mathrm{e}^{\alpha\delta_\alpha(v)} = \frac{1}{G(v)}\mathrm{e}^{\alpha v}G'(v) - \frac{G'(v)}{G(v)^2}\left[G(r)\mathrm{e}^{\alpha r} + \int_r^v \mathrm{e}^{\alpha y}\mathrm{d}G(y)\right].$$

Using (3) again and simplifying yields

$$\delta'_\alpha(v) = \frac{G'(v)}{G(v)} \frac{1 - \mathrm{e}^{-\alpha(v-\delta_\alpha(v))}}{\alpha\mathrm{e}^{-\alpha(v-\delta_\alpha(v))}},$$

which is (4). Since $v > \delta_\alpha(v)$ for $v > r$, then $\delta'_\alpha(v) > 0$. A symmetric argument establishes the result when $\alpha = 0$. □

Proof of Proposition 1 Consider a bidder with value $v \geq B$, and suppose all his rivals use the cutoff strategy c. (If $c = \bar{v}$ then all rival bidders reject the buy price for all of their values.) If the bidder accepts the buy price, then his expected utility is

$$\begin{aligned} U^b(v, c) &= u(v - B)\sum_{k=0}^{n-1}\binom{n-1}{k}\frac{1}{k+1}(1 - F(c))^k F(c)^{n-1-k} \\ &= u(v - B)Q(F(c)), \end{aligned} \tag{5}$$

where for $x \in [0, 1)$ we define

$$Q(x) = \frac{1 - x^n}{n(1 - x)},$$

and where we define $Q(1) = \lim_{x\to 1} Q(x) = 1$. Note that Q is continuously differentiable on $[0, 1]$ with $Q'(x) > 0$, and $Q(0) = \frac{1}{n}$. Furthermore, $Q(x) > x^{n-1}$ for $x \in [0, 1)$ implies $Q(F(c)) > G(c)$ for $c < \bar{v}$. If the bidder waits, then his expected utility is

$$U^w(v, c) = G(r)u(v - r) + \int_r^{\min\{v,c\}} u(v - y)\mathrm{d}G(y). \tag{6}$$

Using the certainty equivalent payment $\delta_\alpha(v)$, we can rewrite $U^w(v, c)$ as

$$U^w(v, c) = u(v - \delta_\alpha(\min\{v, c\}))G(\min\{v, c\}).$$

We prove Proposition 1(ii) first. Assume $B \in (r, \delta_\alpha(\bar{v}))$. A necessary condition for c to be an equilibrium cutoff is

$$U^b(c, c) = U^w(c, c).$$

Rewriting, we obtain

$$u(c-B)Q(F(c)) = u(c-\delta_\alpha(c))G(c), \tag{7}$$

which is the condition given in Proposition 1(ii). We now show there is a unique c satisfying (7). Define $\hat{U}^b(c)=u(c-B)Q(F(c))$ and $\hat{U}^w(c)=u(c-\delta_\alpha(c))F(c)^{n-1}$. We have $\hat{U}^b(B)=0$, and $B>r$ implies $\hat{U}^w(B)>0$. Also, $B<\delta_\alpha(\bar{v})$ implies $\hat{U}^b(\bar{v})=u(\bar{v}-B)>u(\bar{v}-\delta_\alpha(\bar{v}))=\hat{U}^w(\bar{v})$. Therefore, since $\hat{U}^b(c)$ and $\hat{U}^w(c)$ are both continuous, there is some $c\in(B,\bar{v})$ such that $\hat{U}^b(c)=\hat{U}^w(c)$.

We now show that there is a unique such c. We have

$$\frac{\mathrm{d}\hat{U}^b(c)}{\mathrm{d}c} = Q'(F(c))F'(c)u(c-B)+Q(F(c))u'(c-B)$$

and

$$\begin{aligned}\frac{\mathrm{d}\hat{U}^w(c)}{\mathrm{d}c} &= G(c)u'(c-\delta_\alpha(c))(1-\delta'_\alpha(c))+G'(c)u(c-\delta_\alpha(c))\\ &= G(c)u'(c-\delta_\alpha(c)),\end{aligned}$$

where the second equality follows from Lemma 2. If $\hat{U}^b(c)=\hat{U}^w(c)$ for some $c\in(B,\bar{v})$, i.e., c satisfies (7), then $Q(F(c))>G(c)$ implies $u(c-B)<u(c-\delta_\alpha(c))$, and hence $u'(c-B)\geq u'(c-\delta_\alpha(c))$ since u is concave. Thus, $Q(F(c))u'(c-B)>G(c)u'(c-\delta_\alpha(c))$ which, together with $Q'(F(c))F'(c)u(c-B)>0$, implies $\frac{\mathrm{d}\hat{U}^b(c)}{\mathrm{d}c}>\frac{\mathrm{d}\hat{U}^w(c)}{\mathrm{d}c}$. We have shown that if $\hat{U}^b(c)$ and $\hat{U}^w(c)$ cross at c, then $\hat{U}^b$ is steeper than $\hat{U}^w$ at c. This establishes there is a unique c at which $\hat{U}^b$ and $\hat{U}^w$ cross.

Next we show that the solution to $\hat{U}^b(c)=\hat{U}^w(c)$ is (a) increasing in B, (b) decreasing in r, and (c) decreasing in α. Since $\hat{U}^b(c)$ is steeper than $\hat{U}^w(c)$ where they cross and since $\hat{U}^b(c)$ shifts down as B increases while $\hat{U}^w(c)$ remains fixed, the solution is increasing in B. As r increases $\hat{U}^b(c)$ remains fixed, while $\hat{U}^w(c)$ shifts down since $\delta_\alpha(c)$ is increasing in r. Hence the solution to $\hat{U}^b(c)=\hat{U}^w(c)$ is decreasing in r.

To establish (c) it is useful to explicitly express the dependence of $\hat{U}^b(c)$ and $\hat{U}^w(c)$ on α, writing $\hat{U}^b_\alpha(c)$ and $\hat{U}^w_\alpha(c)$. Suppose α increases from α' to α''. Denote by c' the solution to $\hat{U}^b_{\alpha'}(c)=\hat{U}^w_{\alpha'}(c)$ and denote by c'' the solution to $\hat{U}^b_{\alpha''}(c)=\hat{U}^w_{\alpha''}(c)$. We have $\hat{U}^b_{\alpha''}(B)=0<\hat{U}^w_{\alpha''}(B)$. To establish $c'>c''$ we need to show that $\hat{U}^b_{\alpha''}(c')>\hat{U}^w_{\alpha''}(c')$, which then implies $c''\in(B,c')$, which is (c). We have $\hat{U}^b_{\alpha'}(c')=\hat{U}^w_{\alpha'}(c')$, i.e.,

$$\frac{1-\mathrm{e}^{-\alpha'(c'-B)}}{\alpha'}Q(F(c')) = \frac{1-\mathrm{e}^{-\alpha'(c'-\delta_{\alpha'}(c'))}}{\alpha'}G(c'). \tag{8}$$

Since $Q(F(c'))>G(c')$ then $c'-B<c'-\delta_{\alpha'}(c')$. One can show, for x and y fixed and $x<y$, that

$$\frac{1-\mathrm{e}^{-\alpha x}}{1-\mathrm{e}^{-\alpha y}}$$

is increasing in α. Hence, choosing $x = c' - B$ and $y = c' - \delta_{\alpha'}(c')$ this implies

$$\frac{1 - \mathrm{e}^{-\alpha''(c'-B)}}{1 - \mathrm{e}^{-\alpha''(c'-\delta_{\alpha'}(c'))}} > \frac{1 - \mathrm{e}^{-\alpha'(c'-B)}}{1 - \mathrm{e}^{-\alpha'(c'-\delta_{\alpha'}(c'))}} = \frac{G(c')}{Q(F(c'))},$$

where the equality holds by (8). Hence

$$\frac{1 - \mathrm{e}^{-\alpha''(c'-B)}}{\alpha''} Q(F(c')) > \frac{1 - \mathrm{e}^{-\alpha''(c'-\delta_{\alpha'}(c'))}}{\alpha''} Q(F(c')).$$

Since $\delta_\alpha(c)$ is increasing in α, then

$$\hat{U}^b_{\alpha''}(c') = \frac{1 - \mathrm{e}^{-\alpha''(c'-B)}}{\alpha''} Q(F(c')) > \frac{1 - \mathrm{e}^{-\alpha''(c'-\delta_{\alpha''}(c'))}}{\alpha''} Q(F(c')) = \hat{U}^w_{\alpha''}(c'),$$

which establishes (c).

We now show that the cutoff c satisfying (7) is an equilibrium cutoff. We establish that $U^b(v, c) < U^w(v, c)$ for $v \in [B, c)$ and $U^b(v, c) > U^w(v, c)$ for $v \in (c, \bar{v}]$ by showing that $U^b(v, c)$ has a greater slope than $U^w(v, c)$. We have

$$\frac{\partial U^b(v, c)}{\partial v} \equiv U^b_v(v, c) = Q(F(c))u'(v - B). \tag{9}$$

For $v < c$ we have

$$\begin{aligned}\frac{\partial U^w(v, c)}{\partial v} &\equiv U^w_v(v, c) = G'(v)u(v - \delta_\alpha(v)) + G(v)u'(v - \delta_\alpha(v))(1 - \delta'_\alpha(v)) \\ &= G(v)u'(v - \delta_\alpha(v)),\end{aligned}$$

where the second equality follows from Lemma 2. For $v > c$ we have

$$U^w_v(v, c) = G(c)u'(v - \delta_\alpha(c)).$$

Case (i) $v < c$. Since $Q(F(c)) > G(c)$ then $B > \delta_\alpha(c)$ by (7). Further, since $\delta_\alpha(v)$ is increasing in v, we have $B > \delta_\alpha(v)$ for all $v < c$. Therefore $u'(v - B) \geq u'(v - \delta_\alpha(v))$ for all $v \in [B, c]$. Thus,

$$U^b_v(v, c) = u'(v - B)Q(F(c)) > u'(v - \delta_\alpha(v))G(v) = U^w_v(v, c) \tag{10}$$

since $Q(F(c)) > G(c) \geq G(v)$. Equation (10) and $U^b(c, c) = U^w(c, c)$ imply that $U^b(v, c) < U^w(v, c)$ for $v < c$.

Case (ii) $v > c$. Since $Q(F(c)) > G(c)$ then $B > \delta_\alpha(c)$. The concavity of u implies $u'(v - B) \geq u'(v - \delta_\alpha(c))$ for all $v > c$. Thus

$$U^b_v(v, c) = Q(F(c))u'(v - B) > G(c)u'(v - \delta_\alpha(c)) = U^w_v(v, c).$$

Therefore, $U^b(v, c) > U^w(v, c)$ for $v > c$. This establishes that c satisfying (7) is an equilibrium cutoff.

We now prove Proposition 1(i). Assume that $B \geq \delta_\alpha(\bar{v})$. We first show that there is no equilibrium cutoff with $c < \bar{v}$. An equilibrium cutoff of $c = B$ implies that $U^w(v, B) < U^b(v, B)$ for all $v \in (B, \bar{v}]$. As v approaches B, however, $U^b(v, B) = (v - B)Q(F(B))$ approaches zero, while $U^w(v, B) = u(v - \delta_\alpha(B))G(B)$ is strictly positive, which contradicts that $c = B$ is an equilibrium cutoff. Suppose there is an equilibrium cutoff c, with $B < c < \bar{v}$; let c be the largest such cutoff. Earlier it was shown that $\hat{U}^b(v)$ is steeper than $\hat{U}^w(v)$ at $v = c$, where the two functions cross. Hence $\hat{U}^b(v) > \hat{U}^w(v)\ \forall v \in (c, \bar{v})$ and for $v = \bar{v}$ either (i) $\hat{U}^b(\bar{v}) > \hat{U}^w(\bar{v})$, or (ii) $\hat{U}^b(\bar{v}) = \hat{U}^w(\bar{v})$. Since $B \geq \delta_\alpha(\bar{v})$ then

$$\hat{U}^w(\bar{v}) = u(\bar{v} - \delta_\alpha(\bar{v})) \geq u(\bar{v} - B) = \hat{U}^b(\bar{v}),$$

which contradicts (i). If $\hat{U}^b(\bar{v}) = \hat{U}^w(\bar{v})$ then $\hat{U}^b(v)$ is steeper than $\hat{U}^w(v)$ at $v = \bar{v}$, which contradicts $\hat{U}^b(v) > \hat{U}^w(v)\ \forall v \in (c, \bar{v})$.

We now show that $c = \bar{v}$ is an equilibrium cutoff. Since $B \geq \delta_\alpha(\bar{v})$ then

$$U^w(\bar{v}, \bar{v}) = u(\bar{v} - \delta_\alpha(\bar{v})) \geq u(\bar{v} - B) = U^b(\bar{v}, \bar{v}).$$

Further, $F(\bar{v}) = 1$ implies $U_v^w(v, \bar{v}) = G(v)u'(v - \delta_\alpha(v))$ and $U_v^b(v, \bar{v}) = u'(v - B)$. For $v < \bar{v}$ we have

$$u'(v - B) \geq u'(v - \delta_\alpha(v)) > G(v)u'(v - \delta_\alpha(v)),$$

where the weak inequality follows from $B \geq \delta_\alpha(\bar{v}) > \delta_\alpha(v)$ and the strict inequality follows from $G(v) < 1$. This establishes that $U_v^b(v, \bar{v})$ is steeper than $U_v^w(v, \bar{v})$ for $v < \bar{v}$. Since $U^w(\bar{v}, \bar{v}) \geq U^b(\bar{v}, \bar{v})$ then $U^w(v, \bar{v}) \geq U^b(v, \bar{v})\ \forall v < \bar{v}$, which establishes the result. □

Proof of Remark 1 We first show that no regret holds if $r = \underline{v}$. Consider the cases $B > \delta_\alpha(\bar{v})$ and $B < \delta_\alpha(\bar{v})$. Suppose $B > \delta_\alpha(\bar{v})$. Then $c^* = \bar{v}$ by Proposition 1. The no regret condition holds since

$$\begin{aligned} E[u(\bar{v} - y)|r < y \leq \bar{v}] &= E\left[u(\bar{v} - \max\{r, y\})|\underline{v} < y \leq \bar{v}\right] \\ &= u(\bar{v} - \delta_\alpha(\bar{v})) > u(\bar{v} - B). \end{aligned}$$

If $B < \delta_\alpha(\bar{v})$, then by Proposition 1(ii) c^* satisfies

$$u(c^* - B)Q(F(c^*)) = u(c^* - \delta_\alpha(c^*))G(c^*).$$

Since $Q(F(c^*)) > G(c^*)$ then $u(c^* - B) < u(c^* - \delta_\alpha(c^*))$. Hence no regret holds as

$$\begin{aligned} E[u(c^* - y)|r < y \leq c^*] &= E\left[u(c^* - \max\{r, y\})|\underline{v} < y \leq c^*\right] \\ &= u(c^* - \delta_\alpha(c^*)) > u(c^* - B). \end{aligned}$$

In either case, if $r > \underline{v}$ then

$$E\left[u(c^* - \max\{r, y\})|r < y \leq c^*\right] < E\left[u(c^* - \max\{r, y\})|\underline{v} \leq y \leq c^*\right].$$

However, no regret will continue to hold if r is sufficiently close to $\underline{v}$.

No regret will also hold if B is sufficiently large, for given r. Suppose $B > \delta_\alpha(\bar{v})$. Then $c^* = \bar{v}$ and no regret is

$$E[u(\bar{v} - y)|r < y \leq \bar{v}] > u(\bar{v} - B).$$

Since $E[u(\bar{v} - y)|r < y \leq \bar{v}] > 0$, this inequality clearly holds for B sufficiently close to $\bar{v}$. □

Proof of Proposition 2 Let $t(v)$ be an equilibrium in threshold strategies, where t is differentiable except, possibly, at one point $z \in (B, \bar{v})$ where t jumps down. Denote by c^* the equilibrium cutoff of the eBay auction. Utility equivalence of the eBay and Yahoo buy-now auction clearly holds for a bidder whose value is less than the buy price. To establish utility equivalence we need to show that (a) $U(t(v), v; t) = U^w(v, z)\ \forall\, v \in [B, z]$, (b) $U(r, v; t) = U^b(v, z)\ \forall\, v \in [z, \bar{v}]$, and (c) if $t(v)$ jumps down at z, then $z = c^*$. Proving (a)–(c) establishes Proposition 2(i).

Proof of (a) We show that for a bidder with value $v \in [B, z]$, his expected payoff in an eBay and Yahoo buy-now auction (with a reserve r and a buy price B in both) is the same as in a second-price sealed-bid auction with the same reserve. In the second-price sealed-bid auction it is a dominant strategy for a bidder with value $v \in [B, z)$ to bid his value, i.e.,

$$v \in \arg\max_{x \geq r} \{G(x)E[u(v - \max\{r, y\})|\underline{v} \leq y \leq x]\}. \tag{11}$$

By (1) we have

$$u(x - \delta_\alpha(x)) = E[u(x - \max\{r, y\})|\underline{v} \leq y \leq x].$$

Since u is CARA this equality implies

$$u(v - \delta_\alpha(x)) = E[u(v - \max\{r, y\})|\underline{v} \leq y \leq x]. \tag{12}$$

Combining (11) and (12) yields

$$v \in \arg\max_{x \geq r} G(x)u(v - \delta_\alpha(x)). \tag{13}$$

The bidder's payoff in the second price auction is $G(v)u(v - \delta_\alpha(v))$. For $v \leq z$ this is equal to $U^w(v, z)$ by (6).

In the Yahoo auction, a bidder with value $v \in [B, z]$ who chooses his threshold as though his true value were $x \in [B, z]$ obtains an expected utility of

$$U(t(x), v; t) = \int_{\underline{v}}^{t(x)} u(v - \max\{r, y\})\mathrm{d}G(y) + \int_{t(x)}^{x} u(v - B)\mathrm{d}G(y).$$

Denote the bidder's payment as a function of the maximum of his rivals' values by $p(y)$, where $p(y) = r$ if $y \in [\underline{v}, r)$, $p(y) = y$ if $y \in [r, t(x))$, $p(y) = B$ if $y \in [t(x), x)$, and $p(y) = 0$ if $y > x$. We can rewrite the expression above as

$$U(t(x), v; t) = G(x)E[u(v - p(y))|\underline{v} \leq y \leq x].$$

Let $\gamma_\alpha(x)$ be the certainty equivalent of a buyer with value x for the price he would pay conditional on winning in the Yahoo auction, i.e.,

$$u(x - \gamma_\alpha(x)) = E[u(x - p(y))|\underline{v} \leq y \leq x].$$

By CARA

$$u(v - \gamma_\alpha(x)) = E[u(v - p(y))|\underline{v} \leq y \leq x]. \tag{14}$$

Since t is an equilibrium threshold strategy then

$$v \in \arg\max_{x\in[B,z]} G(x)E[u(v-p(y))|\underline{v} \leq y \leq x]. \tag{15}$$

Substituting (14) into (15) yields

$$v \in \arg\max_{x\in[B,z]} G(x)u(v-\gamma_\alpha(x)). \tag{16}$$

Hence in the Yahoo auction $U(t(v), v; t) = G(v)u(v-\gamma_\alpha(v))$.

Equations (13) and (16) provide conditions on the certainty equivalents of payments made in equilibrium in a second-price auction and in a Yahoo auction. The two equations are identical except for the certainty equivalent functions $\delta_\alpha(x)$ and $\gamma_\alpha(x)$. We now show that $\delta_\alpha(v) = \gamma_\alpha(v)$ for all $v \in [B, z]$. A bidder with value $v = B$ has the same equilibrium expected utility in both the second-price auction and the Yahoo auction, i.e., $G(B)u(B-\delta_\alpha(B)) = G(B)u(B-\gamma_\alpha(B))$. Hence $\delta_\alpha(B) = \gamma_\alpha(B)$. Differentiating (13) with respect to x yields the first-order condition

$$G'(v)u(v-\delta_\alpha(v)) - G(v)u'(v-\delta_\alpha(v))\delta'_\alpha(v) = 0,$$

for the second-price auction, or

$$\delta'_\alpha(v) = \frac{G'(v)u(v-\delta_\alpha(v))}{G(v)u'(v-\delta_\alpha(v))}.$$

This is an ordinary differential equation in δ_α. Differentiating (16) shows the γ_α function satisfies exactly the same ordinary differential equation. Both equations have the same initial condition at $v = B$ since $\delta_\alpha(B) = \gamma_\alpha(B)$. Hence for $v \in [B, z]$, we have $\delta_\alpha(v) = \gamma_\alpha(v)$ and so

$$U^w(v, z) = G(v)u(v-\delta_\alpha(v)) = G(v)u(v-\gamma_\alpha(v)) = U(t(v), v; t).$$

This proves (a).

Proof of (b) For $v \in [z, \bar{v}]$ it is trivial to see that $U(r, v; t) = U^b(v, z)$.

Proof of (c) Suppose that t jumps down at $z \in (B, \bar{v})$. A necessary condition for t to jump down at z is that

$$U(t(z), z; t) = U(r, z; t),$$

i.e., a bidder with value z is indifferent between the threshold $t(z)$ and r. By parts (a) and (b) this is equivalent to

$$U^w(z, z) = U^b(z, z),$$

which is (7) with c replaced by z. In other words, the value at which t jumps down is defined by the same condition as the eBay equilibrium cutoff. This proves Proposition 2(i).

If $B \geq \delta_\alpha(\bar{v})$ then the equilibrium cutoff is $c^* = \bar{v}$ by Proposition 1, and hence the threshold function t has no jump down. If $B \in (r, \delta_\alpha(\bar{v}))$ then there is a unique equilibrium cutoff $c^* \in (B, \bar{v})$, and hence t jumps down at $z = c^*$. The proves Proposition 2(ii). $\square$

Proof of Proposition 3 We have shown that if $t(v)$ is an equilibrium threshold function, then bidder equilibrium expected utilities are the same in the eBay and the Yahoo buy-now auctions. We now show that an equilibrium threshold function exists and is unique.

Consider a bidder with value $v \in [B, c^*]$. In the Yahoo buy-now auction the bidder's equilibrium expected utility is

$$G(r)u(v-r) + \int_r^{t(v)} u(v-y)\mathrm{d}G(y) + [G(v) - G(t(v))]u(v-B).$$

In the eBay buy-now auction his equilibrium expected utility is

$$G(r)u(v-r) + \int_r^{v} u(v-y)\mathrm{d}G(y).$$

By Proposition 2(i) the difference of these two utilities is zero, i.e.,

$$\int_{t(v)}^{v} [u(v-y) - u(v-B)]\mathrm{d}G(y) = 0.$$

If $v = B$ this equation implies $t(v) = B$. If $v > B$ there is the trivial solution $t(v) = v$, which we dismiss since a threshold cannot exceed B. For $t(v) < v$ this equality can be re-written as

$$E[u(v-y)|t(v) \leq y \leq v] = u(v-B). \tag{17}$$

We show that (17) defines $t(v)$ uniquely, and $t(v)$ is decreasing in v. Clearly,

$$E[u(v-y)|B \leq y \leq v] < u(v-B). \tag{18}$$

The "no regret" assumption means

$$E[u(c^*-y)|r \leq y \leq c^*] \geq u(c^*-B).$$

Since $v \leq c^*$ this implies

$$E[u(c^*-y)|r \leq y \leq v] \geq u(c^*-B).$$

Since bidders have CARA preferences, this is equivalent to

$$E[u(v-y)|r \leq y \leq v] \geq u(v-B). \tag{19}$$

Since $E[u(v-y)|t \leq y \leq v]$ is continuous and strictly decreasing in t, Eqs. (18) and (19) imply there is a unique $t(v) \in [r, B)$ satisfying (17). To see that $t(v)$ is decreasing, note that (17) can be re-written as

$$\int_{t(v)}^{v} \left(\mathrm{e}^{\alpha y} - \mathrm{e}^{\alpha B}\right) \mathrm{d}G(y) = 0.$$

For $t(v)$ fixed, the LHS is increasing in v since $v > B$. For v fixed, the LHS is also increasing in $t(v)$ since $t(v) < B$. Hence $t(v)$ must be decreasing in v for the equality to hold as v increases. Let $t(v) = r$ for $v > c^*$.

We now show that $t(v)$, as constructed above, is an equilibrium. Consider a bidder with value $v \in [B, c^*]$. In the proof of Proposition 2 it is established for $v \in [B, c^*]$ that

$$G(x)u(v - \delta_\alpha(x)) = G(x)u(v - \gamma_\alpha(x)) \; \forall x \in [B, c^*],$$

where $\delta_\alpha(x)$ is the certainty equivalent of the payment made by the winning bidder with value x in an eBay auction without a buy price (and also a second-price auction), and $\gamma_\alpha(x)$ is the certainty equivalent in the Yahoo buy-now auction. Value-bidding is optimal in the second-price auction, i.e.,

$$v \in \arg \max_{x \in [B, c^*]} G(x)u(v - \delta_\alpha(x)).$$

Hence $v \in \arg\max_{x \in [B,c^*]} G(x)u(v - \gamma_\alpha(x))$, i.e., a bidder with value $v \le c^*$ in the Yahoo buy-now auction obtains a higher payoff with a threshold $t(v)$ than with any other threshold $t(x) \in [t(c^*), B]$. Clearly a threshold $t \in (r, t(c^*))$ is not optimal, since a threshold of $t(c^*)$ yields a higher expected payoff.

A bidder with value $v \in [B, c^*]$ obtains a higher payoff with a threshold of $t(v)$ than with a threshold of r, as we now show. If the bidder chooses a threshold of r his payoff is

$$U(r, v; t) = u(v - B)Q(F(c^*)) = U^b(v, c^*).$$

Since c^* is an equilibrium cutoff and $v \le c^*$, then $U^b(v, c^*) \le U^w(v, c^*)$. For $v \le c^*$, by payoff equivalence of the eBay and Yahoo buy-now auctions we have

$$U^w(v, c^*) = U(t(v), v; t).$$

These equalities and inequalities yield the result $U(r, v; t) \le U(t(v), v; t)$. We have established that $t(v)$ is an optimal threshold for a bidder with value $v \in [B, c^*]$.

We complete the proof by showing that $t(v) = r$ is an optimal threshold for a bidder with value $v > c^*$. Clearly, any threshold $\tilde{t} \in (r, t(c^*))$ is dominated by the threshold $t(c^*)$. Consider a threshold $\tilde{t} \in [t(c^*), B]$. We will show that $U(r, v; t) > U(\tilde{t}, v; t)$. For $v > c^*$ by payoff equivalence of the Yahoo and eBay buy-now auction we have $U(r, v; t) = U^b(v, c^*)$. Since c^* is an equilibrium cutoff and since $v > c^*$ then $U^b(v, c^*) > U^w(v, c^*)$. Let $\tilde{v}$ be such that $t(\tilde{v}) = \tilde{t}$; note that $\tilde{v} \in [B, c^*]$. We also have

$$\begin{aligned} U^w(v, c^*) &= G(r)u(v - r) + \int_r^{c^*} u(v - y)\mathrm{d}G(y) \\ &\ge G(r)u(v - r) + \int_r^{\tilde{v}} u(v - y)\mathrm{d}G(y), \end{aligned}$$

where the equality is by the definition of $U^w(v, c^*)$ and where the inequality follows from $v > c^* \ge \tilde{v}$. Since $\tilde{v} \le c^*$, by payoff equivalence we have $U^w(\tilde{v}, c^*) =$

$U(t(\tilde{v}), \tilde{v}; t)$. Since bidders are CARA risk averse, this equality can be rewritten as

$$G(r)u(v-r) + \int_r^{\tilde{v}} u(v-y)\mathrm{d}G(y) = U(t(\tilde{v}), v; t).$$

These inequalities yield $U(r, v; t) > U(t(\tilde{v}), v; t)$, which completes the proof. □

Proof of Corollary 3 Suppose that $\alpha'' > \alpha' \geq 0$. Let t_α be the equilibrium threshold function of the Yahoo auction when bidders are CARA risk averse with index of risk aversion α. By Proposition 3, if $t_\alpha(v) > r$ then

$$E[u(v-y)|t_\alpha(v) \leq y \leq v] = u(v-B).$$

If $t_{\alpha'}(v) > r$ and $\alpha' > 0$ then this equality can be re-written as

$$E[\mathrm{e}^{\alpha' y}|t_{\alpha'}(v) \leq y \leq v] = \mathrm{e}^{\alpha' B}.$$

Raising both sides to the power α''/α' yields

$$\left(E[\mathrm{e}^{\alpha' y}|t_{\alpha'}(v) \leq y \leq v]\right)^{\frac{\alpha''}{\alpha'}} = \mathrm{e}^{\alpha'' B}.$$

Since $x^{\alpha''/\alpha'}$ is convex, we have

$$\begin{aligned}\left(E[\mathrm{e}^{\alpha' y}|t_{\alpha'}(v) \leq y \leq v]\right)^{\frac{\alpha''}{\alpha'}} &< E\left[\left(\mathrm{e}^{\alpha' y}\right)^{\frac{\alpha''}{\alpha'}} |t_{\alpha'}(v) \leq y \leq v\right] \\ &= E\left[\mathrm{e}^{\alpha'' y}|t_{\alpha'}(v) \leq y \leq v\right].\end{aligned}$$

Hence,

$$\mathrm{e}^{\alpha'' B} = E[\mathrm{e}^{\alpha'' y}|t_{\alpha''}(v) \leq y \leq v] < E[\mathrm{e}^{\alpha'' y}|t_{\alpha'}(v) \leq y \leq v],$$

which implies $t_{\alpha''}(v) < t_{\alpha'}(v)$, i.e., more risk averse bidders choose lower thresholds.

If $t_{\alpha'}(v) > r$ and $\alpha' = 0$ then

$$B = E[y|t_{\alpha'}(v) \leq y \leq v].$$

Also,

$$\mathrm{e}^{\alpha'' B} = E[\mathrm{e}^{\alpha'' y}|t_{\alpha''}(v) \leq y \leq B] > \mathrm{e}^{\alpha'' E[y|t_{\alpha''}(v) \leq y \leq B]},$$

where the equality holds by (2) and the inequality holds since $\mathrm{e}^{\alpha'' y}$ is convex. Hence $B > E[y|t_{\alpha''}(v) \leq y \leq B]$. Thus

$$E[y|t_{\alpha''}(v) \leq y \leq B] < E[y|t_{\alpha'}(v) \leq y \leq B],$$

and so $t_{\alpha''}(v) < t_{\alpha'}(v)$.

If $t_{\alpha'}(v) = r$ then $t_{\alpha''}(v) = r$ since the c^*, the value at which the threshold jumps down, is decreasing in α by Proposition 1. □

Proof of Proposition 4 The equilibrium expected payoff of an eBay bidder with value $v \in [B, c_e]$ is

$$U^e(v, c_e) = G(r)u(v - r) + \int_r^v u(v - y)\mathrm{d}G(y), \tag{20}$$

and, since $u(0) = 0$, then

$$U_v^e(v, c_e) = G(r)u'(v - r) + \int_r^v u'(v - y)\mathrm{d}G(y).$$

The expected payoff of a Yahoo bidder with value $v \in [B, c_y]$ is

$$U^y(v, c_y) = G(r)u(v - r) + \int_r^{t(v)} u(v - y)\mathrm{d}G(y) + \int_{t(v)}^v u(v - B)\mathrm{d}G(y). \tag{21}$$

Since t is an equilibrium threshold function we have

$$U_v^y(v, c_y) = G(r)u'(v - r) + \int_r^{t(v)} u'(v - y)\mathrm{d}G(y) + \int_{t(v)}^v u'(v - B)\mathrm{d}G(y).$$

Since $t(B) = B$ then $U^e(B, c_e) = U^y(B, c_y)$.

Assume bidders exhibit DARA. Let $c_m = \min\{c_e, c_y\}$. We now show that if $v \in (B, c_m]$ and $U^e(v, c_e) \leq U^y(v, c_y)$ then $U_v^e(v, c_e) > U_v^y(v, c_y)$. Let $v \in (B, c_m]$. If $U^e(v, c_e) \leq U^y(v, c_y)$ then by (20) and (21) we have

$$\int_{t(v)}^v u(v - y)\mathrm{d}G(y) \leq \int_{t(v)}^v u(v - B)\mathrm{d}G(y),$$

i.e.,

$$E[u(v - y)|t(v) \leq y \leq v] \leq u(v - B).$$

Hence there is a $B' > B$ such that

$$E[u(v - y)|t(v) \leq y \leq v] = u(v - B').$$

Since u has DARA, this implies (see p. 638 of Matthews (1987)) that

$$E[u'(v - y)|t(v) \leq y \leq v] > u'(v - B').$$

Since u is concave then $u'(v - B') > u'(v - B)$ and hence

$$E[u'(v - y)|t(v) \leq y \leq v] > u'(v - B)$$

i.e.,

$$\int_r^v u'(v-y)\mathrm{d}G(y) > \int_r^{t(v)} u'(v-y)\mathrm{d}G(y) + \int_{t(v)}^v u'(v-B)\mathrm{d}G(y).$$

Adding $G(r)u'(v-r)$ to both sides of this inequality yields $U_v^e(v, c_e) > U_v^y(v, c_y)$.

We have shown that (i) $U^e(B, c_e) = U^y(B, c_y)$ and (ii) $U^e(v, c_e) \le U^y(v, c_y)$ implies $U_v^e(v, c_e) > U_v^y(v, c_y)$. Consider the following lemma, which is a variation of Lemma 2 in Milgrom and Weber (1982):

Lemma *Let g and h be differentiable functions for which* (i) $g(\underline{x}) = h(\underline{x})$ *and* (ii) $x > \underline{x}$ *and* $g(x) \le h(x)$ *implies* $g'(x) > h'(x)$. *Then* $g(x) > h(x)$ *for all* $x > \underline{x}$.

Applying the Lemma yields $U^e(v, c_e) > U^y(v, c_y)$ for $v \in (B, c_m]$.

Suppose contrary to Proposition 4(i) that bidders have DARA, $c_e \le c_y$ and $c_e < \bar{v}$. Then $c_m = c_e$. Consider a bidder whose value is c_m. Since $c_e < \bar{v}$, then by the definition of c_e the expected payoff in the eBay auction of a bidder with value c_m is $U^e(c_m, c_e) = u(c_m - B)Q(F(c_e))$. Since $c_m \le c_y$ the expected payoff in the Yahoo auction of a bidder with value c_m satisfies

$$U^y(c_m, c_y) \ge u(c_m - B)Q(F(c_y)),$$

i.e., the bidder obtains a greater payoff rejected the buy price rather than accepting it. Since $Q(F(x))$ is increasing in x then $U^y(c_m, c_y) \ge U^e(c_m, c_e)$, a contraction.

The symmetric argument establishes that if bidders have IARA then (i) $U^e(v,c_e) < U^y(v, c_y)$ for $v \in (B, c_m]$ and (ii) either $c_y = c_e = \bar{v}$ or $c_y > c_e$.

We now establish (ii) when bidders have DARA. By Proposition 4(i) we have $c_m = c_y$ and, as just shown $U^e(v, c_e) > U^y(v, c_y)$ for $v \in (B, c_y]$. If $c_e = c_y = \bar{v}$ then the result is established. Otherwise, if $c_y < c_e$, consider a bidder whose value is $v \in (c_y, c_e)$. Such a bidder rejects the buy price in the eBay auction but accepts it in the Yahoo auction. We have

$$U^e(v, c_e) > u(v-B)Q(F(c_e)) > u(v-B)Q(F(c_y)) = U^y(v, c_e),$$

where the middle inequality holds since $c_e > c_y$ and $Q(F(x))$ is increasing in x. A bidder whose value is $v \in (c_e, \bar{v})$ accepts the buy price in both the eBay and the Yahoo auction and hence

$$U^e(v, c_e) = u(v-B)Q(F(c_e)) > u(v-B)Q(F(c_y)) = U^y(v, c_e).$$

Hence for each value $v \in (B, \bar{v}]$ a bidder obtains a higher payoff in the eBay auction. The symmetric argument bidder prefer the Yahoo buy-now auction to the eBay buy now auction when bidders have IARA. □

Proof of Corollary 6 It is useful to first consider a second-price sealed-bid auction without a buy price. Suppose that bidder 1 has the highest value v_1. He wins the second-price auction and makes a random payment of $\max\{r, y\}$, where y is the second highest value. The certainty equivalent payment, denote by $\delta_u(v_1)$, is defined by

$$u(v_1 - \delta_u(v_1)) = E[u(v_1 - \max\{r, y\})|\underline{v} \le y \le v_1].$$

Note that $\delta_u(v_1)$ is increasing in v_1. As before, denote by $\delta_0(v_1)$ the certainty equivalent payment of a risk-neutral bidder.

By Proposition 2(i), if bidders are CARA risk averse then $c_y = c_e$, while bidders have DARA then $c_y < c_e$. If $v_1 \leq c_y$ and $y < t(v_1)$ then seller revenue is $\max\{r, y\}$ in both the eBay and the Yahoo auction. If $v_1 \leq c_y$ and $y \geq t(v_1)$ then revenue in the Yahoo auction is B and expected revenue in the eBay auction is $E[\max\{r, y\}|t(v_1) \leq y \leq v_1]$. By Proposition 4(ii) we have $U^e(v_1, c_e) \geq U^y(v_1, c_y)$ and using (20) and (21) we obtain

$$E[u(v_1 - y)|t(v_1) \leq y \leq v_1] \geq u(v_1 - B).$$

Since u is concave then

$$u(E[v_1 - y|t(v_1) \leq y \leq v_1]) > E[u(v_1 - y)|t(v_1) \leq y \leq v_1].$$

Hence $u(E[v_1 - y|t(v_1) \leq y \leq v_1]) > u(v_1 - B)$, which implies $E[y|t(v_1) \leq y \leq v_1]) < B$ since u is increasing. Thus if $v_1 \leq c_y$ then revenue is higher in the Yahoo auction than in the eBay auction.

If $v_1 \in [c_y, c_e]$ then revenue in the Yahoo auction is B and expected revenue in the eBay auction is $E[\max\{r, y\}|r \leq y \leq v_1]$. There are two cases to consider, $c_e < \bar{v}$ and $c_e = \bar{v}$. Suppose $c_e < \bar{v}$. Then a bidder whose value is equal to the equilibrium cutoff c_e is indifferent between accepting or rejecting the buy price in the eBay auction, i.e.,

$$u(c_e - B)Q(F(c_e)) = u(c_e - \delta_u(c_e))G(c_e).$$

Since $Q(F(x)) > G(x)$ for $x > 0$, then $u(c_e - B) < u(c_e - \delta_u(c_e))$ and hence $B > \delta_u(c_e)$. Note that $\delta_u(c_e) > \delta_0(c_e) > \delta_0(v_1)$ where the first inequality holds since the certainty equivalent payment is higher when bidders are risk averse than when they are risk neutral and where the second holds since $\delta_0(v)$ is increasing in v. Hence we have established that

$$B > \delta_u(c_e) > \delta_0(v_1) = E[\max\{r, y\}|r \leq y \leq v_1],$$

i.e., revenue is higher in the Yahoo auction when $v_1 \in [c_y, c_e]$ and $c_e < \bar{v}$. If $c_e = \bar{v}$ then $B \geq \delta_u(\bar{v})$ since otherwise a bidder with value $v = \bar{v}$ optimally accepts the buy price, contradicting that $c_e = \bar{v}$ is an equilibrium cutoff. Hence

$$B \geq \delta_u(\bar{v}) \geq \delta_0(v_1) = E[\max\{r, y\}|r \leq y \leq v_1],$$

i.e., revenue is higher in the Yahoo auction. If $v_1 > c_e$ then seller revenue is B in both the eBay and the Yahoo buy-now auctions.

The proof that seller revenue is the same for both types of auctions when bidders are risk neutral is straightforward and is omitted. □

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