Dissolving a Partnership Securely

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Abstract

We characterize security strategies and payoffs for three mechanisms for dissolving partnerships: the Texas Shoot-Out, the $K+1$ auction, and the compensation auction. A security strategy maximizes a participant's minimum payoff, and represents a natural starting point for analysis when a participant is either uncertain of the environment or uncertain of whether his rivals will play equilibrium. For the compensation auction, a dynamic dissolution mechanism, we introduce the notion of a perfect security strategy. Such a strategy maximizes a participant's minimum payoff along every path of play. We show that the compensation auction has a unique such strategy.

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1 Introduction

Fairly and efficiently resolving disputes often requires an arbitrator, lawyer, or parent to employ a system to help make allocation decisions. The prototypical example of such a system is the dissolution mechanism specified in partnership contracts. When a business partnership is formed, partners are advised to form a binding contract that specifies how assets will be divided in the event of death, disability, divorce, or departure. (These are sometimes called the 4 D's of business exit.) In such cases, an exit mechanism helps determine which partner receives the company and how the other partner(s) are compensated. There are many such mechanisms.

The literature on dissolving partnerships has focused on equilibrium analysis. However, a significant *practical obstacle* to the actual implementation of any dissolution mechanism is that the participants may be ignorant of their environment, e.g., the distribution of the values or the preferences of the other participants. Even if the environment is known, they may be uncertain of their own equilibrium strategy, uncertain of whether the other participants will play their part of an equilibrium, or uncertain of whether the other participants may collude. A participant is therefore uncertain of what payoff he is likely to obtain via the mechanism.

Here we step back from the usual equilibrium analysis of exit mechanisms and instead provide advice to the mechanism participants with the goal of guaranteeing that they do not do "too badly," regardless of the behavior of their rivals. We define a participant's security (or maximin) payoff for a mechanism to be the largest payoff that he can guarantee himself in the mechanism. Likewise, we define a security (or maximin) strategy as one that, if followed, always gives a participant at least his security payoff. We identify security payoffs and strategies for several different dissolution mechanisms: the Texas Shoot-Out, the $K + 1$ auction, and the compensation auction. From a practical point of view, the idea of finding a security strategy for a mechanism is natural and should be the *starting place* for any participant when determining how to behave in an exit mechanism.

We begin our analysis by briefly characterizing security strategies and payoffs for the Texas Shoot-Out, the exit mechanism most commonly used in practice. The Texas Shoot-Out is defined for two-person partnerships and is a direct application of the well-known "divide and choose" cake cutting procedure: the owner who wants to dissolve the partnership (the Divider) names a price p and the other owner (the Chooser) is compelled to either purchase his partner's share or sell his own share at the named price.

The mechanism has transparent security strategies and payoffs: A Divider who names a price that leaves him indifferent to whether his partner buys or sells is guaranteed to receive half of his value for the partnership. Likewise, a Chooser, by simply taking the best deal, either selling or buying at the proposed price, cannot leave with less than half of her value for the partnership. Since neither partner has a strategy which guarantees them more than half their value for the partnership, these strategies are security strategies.

The Texas Shoot-Out suffers from several shortcomings: equilibrium is not efficient, the mechanism treats the participants asymmetrically (favoring the chooser), and the mechanism does not scale to more than two participants.[1] By contrast, the $K+1$ auction, introduced by Cramton, Gibbons, and Klemperer (1987), and the compensation auction, introduced by Van Essen and Wooders (2016), are efficient, treat the partners symmetrically, and scale to accommodate any number of partners.

The second mechanism we analyze is the $K+1$ auction. In this auction the highest bidder obtains the partnership and pays each of the other bidders $1/N$-th of the weighted average $Kb_s+(1-K)b_f$ of the highest (b_f) and second

[1] McAfee (1992) characterizes equilibrium in the Texas Shoot-Out when partners have independent private values.

highest (b_s) bid, where $K \in [0,1]$. We show that the (unique) security strategy for a bidder is to bid his value and his security payoff is $1/N$-th of his value for the partnership. While Bayes Nash equilibrium bids depend on K, a participant's security strategy and payoff do not.

Finally, we identify security strategies in the compensation auction, a dynamic mechanism for dissolving partnerships introduced in Van Essen and Wooders (2016). The compensation auction is an ascending bid auction in which a bidder, in return for surrendering his claim on the item, obtains compensation equal to the difference in the price at which he drops out and the prior price at which a bidder dropped out. We show that it is a security strategy to drop out of the auction whenever compensation reaches $1/N$-th of a bidder's value, and this strategy yields a security payoff of $1/N$-th of the bidder's value. There is, however, a continuum of security strategies.

In dynamic mechanisms it is natural to be interested in strategies that maximize the payoff a participant can guarantee himself along the entire path of play. We call such a strategy a *perfect security strategy*, and we formally define this notion. We show that the compensation auction has a unique perfect security strategy. This strategy has the desirable property that the payoff a bidder can guarantee himself increases as the auction unfolds, so long as the bidder is never indifferent between remaining in the auction or dropping out. The perfect security strategy is also the limit of the Bayes Nash equilibrium bidding function, for CARA risk averse bidders, as bidders become infinitely risk averse.

For all three mechanisms the security payoff of a bidder is $1/N$-th of his value. Hence in any Bayes Nash equilibrium of these mechanisms, a bidder whose value is x_i obtains an equilibrium payoff of at least x_i/N. Thus, each bidder would be willing to participate in the mechanism if otherwise the partnership were allocated randomly to one of the bidders.

Related Literature

The origin of the notion of a security strategy is von Neumann's (1928) Minimax Theorem. Security strategies subsequently played a central role in the cake cutting literature, with contributions from mathematics, economics, political science and, more recently, computer science, which described cutting procedures and strategies which guaranteed a player a piece of cake of certain value – e.g., Steinhaus (1948) and Dubins and Spanier (1961).[2]

Security strategies were also prominent in the early mechanism design literature. Crawford (1980a) explains how a simple divide and choose mechanism can be used to implement Pareto-efficient allocations in an exchange economy, when agents act to maximize their minimum payoffs.[3] Demange (1984) studies a N player divide and choose game in which players bid to be the divider. In addition to characterizing the Nash equilibria of the game, it demonstrates that each player, by following a simple (non-equilibrium) maximin strategy, can guarantee themselves a share of resources with utility at least as great as would be obtained via equal division.

Moulin (1981) proposes and studies a voting scheme (he calls "voting by alternating veto") for choosing one of a finite number of public alternatives when utility is not transferable. He shows that both maximin play and Nash play select the same alternative. Finally, Moulin (1984) proposes an auction mechanism that implements the socially efficient decision, in a transferable utility setting, from among a finite number of possible decisions when players have maximin preferences. The same mechanism implements the socially efficient decision as the Nash equilibrium outcome when the players' preferences are commonly known. Thus the same mechanism "works" whether

[2]Brams and Taylor (1996) and Robertson and Webb (1998) both provide surveys of this literature. Chen, Lai, Parkes, and Procaccia (2013) is a recent example that incorporates both equity and strategic incentives into the design of cake cutting algorithms.

[3]The bulk of Crawford's work on divide and choose schemes is concerned with Nash rather than maximin play, e.g., Crawford (1977, 1979, 1980b).

the players are ignorant of their environment (and play maximin) or whether they are sophisticated (and play Nash equilibrium).

An emerging literature studies mechanism design when participants have maximin preferences. De Castro and Yannelis (2010) show that any efficient allocation is incentive compatible when the participants' beliefs about the types of others are unrestricted, and hence maximin preferences mitigate the fundamental conflict between efficiency and Pareto optimally. Wolitzky (2016) provides necessary conditions for an allocation rule to be maximin implementable when the participants' beliefs may be restricted. Mechanism design is not our focus, however, since equal-share partnerships are dissolved efficiently by the $K + 1$ auction and the compensation auction even when participants are expected utility maximizers. Our interest is in characterizing maximin strategies for several specific dissolution mechanisms, with the goal of providing practical advice to participants.

In the next section we define security strategies and payoffs. In Section 3 we characterize security strategies and payoffs for the Texas Shoot-Out, the $K+1$ auction, and the compensation auction. We also define the notion of a perfect security strategy and show that the compensation auction has a unique such strategy. In Section 4 we conclude with a discussion of dissolving unequal partnerships.

2 Preliminaries

A single indivisible item, e.g., a partnership, is to be allocated to one of $N \geq 2$ partners/players. Each partner $i \in \{1, \ldots, N\}$ has a private value $x_i \in [0, \bar{x}]$ for the partnership, where $\bar{x} < \infty$.

We study three mechanisms: the Texas Shoot-Out, the $K + 1$ auction, and the compensation auction. Write β^i for player i's strategy. Write $v_i(x_i, x_{-i}, \beta^i, \beta^{-i})$ for the payoff to a player whose value is x_i and who follows the strategy β^i, when $x_{-i} = (x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_N)$ and $\beta^{-i} = (\beta^1, \ldots, \beta^{i-1}, \beta^{i+1}, \ldots, \beta^N)$ are the values and strategies of the remaining players. The formal description of a strategy and the calculation of $v_i(x_i, x_{-i}, \beta^i, \beta^{-i})$ will depend on the mechanism at hand. Since the $K+1$ auction and the compensation auction are symmetric mechanisms, for these auctions we write $\bar{v}$ rather than $\bar{v}_i$ for security payoffs.

A player's security payoff for a particular mechanism is the largest payoff that he can guarantee himself, regardless of the values and strategies of the other players. A security strategy guarantees a player his security payoff. Formally:

Definition: *Player i's security payoff when his value is x_i* is the largest value $\bar{v}_i(x_i)$ for which he has a strategy $\bar{\beta}^i$ such that

$$v_i(x_i, x_{-i}, \bar{\beta}^i, \beta^{-i}) \geq \bar{v}_i(x_i) \ \forall x_{-i}, \beta^{-i}.$$

We say that $\bar{\beta}^i$ a *security strategy for player i* if for each $x_i \in [0, \bar{x}]$ the strategy guarantees him $\bar{v}_i(x_i)$.

3 Security Strategies and Payoffs for Three Dissolution Mechanisms

In this section, we identify the security strategies and security payoffs of the three dissolution mechanisms.

Security in the Texas Shoot-Out

The Texas Shoot-Out is defined for two-player partnerships. According to this mechanism, player 1 (the "Divider") names a price p. Player 2 (the "Chooser") observes p, and then chooses whether to buy or sell his share of the partnership at that price. If player i buys, then his payoff is $x_i - p$ and the seller's payoff is p. A strategy for player 1 is a mapping $\beta^1(x_1) : [0, \bar{x}] \to [0, \bar{x}]$ from values to price offers, while a strategy for player 2 is a mapping $\beta^2(x_2, p) : [0, \bar{x}] \times [0, \bar{x}] \to \{\text{Buy, Sell}\}$ from prices and values to a buy/sell decision.

Security strategies and payoff are well known for the Texas Shoot-Out. The following proposition is stated for completeness.[4]

Proposition 0: *In the Texas Shoot-Out, it is a security strategy for player 1 to chooses a price p equal to half of his value, i.e., $\bar{\beta}^1(x_1) = x_1/2$. It is a security strategy for player 2 to buy if the price is less than half his value and sell if it is greater than half his value, i.e.,*

$$\bar{\beta}^2(x_2, p) = \begin{cases} \text{Buy} & \text{if } p \le x_2/2 \\ \text{Sell} & \text{if } p > x_2/2. \end{cases}$$

The security payoff of player $i \in \{1, 2\}$ with value x_i is $x_i/2$. *Each player's security strategy is unique, up to the indifference of player 2 when $p = x_2$.*

Security in the K+1 Auction

In the $K + 1$ auction there are $N \ge 2$ partners who each submit a sealed bid for the whole partnership. The auction awards the partnership to the high bidder who then pays each of the others an amount of compensation equal to

$$\frac{1}{N}\left[K b_s + (1 - K) b_f\right],$$

where b_f and b_s are, respectively, the first and second highest bid and $K \in [0, 1]$.[5] A strategy for partner i is a function $\beta^i : [0, \bar{x}] \to [0, \bar{x}]$ mapping values to bids.

Proposition 1 shows that it is a security strategy for a bidder to bid his value, regardless of value of K.

[4]See, for instance, Raiffa (1982) pg. 297 where he works it out for a numerical example.

[5]In the event that $b_f = b_s$, the winner is selected randomly from among the high bids.

Proposition 1: In the $K+1$ auction, a bidder's unique security strategy is to bid his value, i.e., $\bar{\beta}^i(x_i) = x_i$, and his security payoff is x_i/N.

Security in the Compensation Auction

The compensation auction is a dynamic auction for dissolving a partnership, and it operates as follows: The price, starting from zero, rises continuously. Bidders may drop out at any point. A bidder who drops out surrenders his claim to the item and, in return, receives compensation from the (eventual) winner equal to the difference between the price at which he drops and the price at which the prior bidder dropped. The auction ends when exactly one bidder remains. That bidder wins the item and compensates the other bidders. Thus in an auction with N bidders, if $\{p_k\}_{k=1}^{N-1}$ is the sequence of dropout prices, then the compensation of the k-th bidder to drop is $p_k - p_{k-1}$, where $p_0 = 0$, and the winner's total payment is $p_{N-1} = \Sigma_{k=1}^{N-1}(p_k - p_{k-1})$.

A strategy for bidder i is a list of $N-1$ functions $\beta^i = (\beta_1^i, \ldots, \beta_{N-1}^i)$, where $\beta_k^i(x_i; p_1, \ldots, p_{k-1})$ gives bidder i's dropout price in round k, when $k-1$ bidders have previously dropped out at prices $p_1 \leq p_2 \leq \ldots \leq p_{k-1}$. Since a strategy must call for a feasible dropout price, we require that $\beta_k(x_i; p_1, \ldots, p_{k-1}) \geq p_{k-1}$ for each k and $p_1, \ldots, p_{k-1}$.

Proposition 2 identifies bidder i's security payoff and a simple security strategy which attains it.

Proposition 2: *In the compensation auction, the strategy which calls for bidder i to drop out when his compensation reaches x_i/N is a security strategy and realizes the security payoff of x_i/N. More formally, the strategy $\bar{\beta}_k^i(x_i; \mathbf{p}_{k-1}) = x_i/N + p_{k-1}$ for $k \in \{1, \ldots, N-1\}$, $x_i \in [0, \bar{x}]$, and $\mathbf{p}_{k-1}$ such that $0 \leq p_1 \leq \ldots \leq p_{k-1}$, is a security strategy.*

The strategy given in Proposition 2 is simple in the sense that the compensation a bidder demands does not depend on the prior history of dropout prices – he drops as soon as the current bid exceeds the prior dropout price by x_i/N. A bidder, however, has many security strategies. Of particular interest is the one which calls for bidder i to drop in stage k when the bid exceeds the prior dropout price by $(x_i - p_{k-1})/(N-k+1)$. Proposition 3 establishes that this strategy is also a security strategy.

Proposition 3: *Let $\bar{\beta}^i$ be any strategy such that $\bar{\beta}^i_k(x_i; \mathbf{p}_{k-1}) = (x_i - p_{k-1})/(N-k+1) + p_{k-1}$ for $k \in \{1, \ldots, N-1\}$, $x_i \in [0, \bar{x}]$, and $\mathbf{p}_{k-1}$ such that $0 \le p_1 \le \ldots \le p_{k-1} \le x_i$.*[6] *Then $\bar{\beta}^i$ is a security strategy.*

The next claim generalizes Proposition 3 by identifying a class of security strategies. It shows that any strategy in which the bidder demands compensation between $x_i/N + p_{k-1}$ and $(x_i - p_{k-1})/(N-k+1) + p_{k-1}$ is a security strategy.[7]

Proposition 4: *Let $\bar{\beta}^i$ be such that $\bar{\beta}^i_k(x_i; \mathbf{p}_{k-1}) \in [\frac{x_i}{N} + p_{k-1}, \frac{x_i - p_{k-1}}{N-k+1} + p_{k-1}]$ for $k \in \{1, \ldots, N-1\}$, $x_i \in [0, \bar{x}]$, and $\mathbf{p}_{k-1}$ such that $0 \le p_1 \le \ldots \le p_{k-1} \le x_i$. Then $\bar{\beta}^i$ is a security strategy.*

Perfect Security Strategies in the Compensation Auction

A bidder's security payoff is the maximum payoff he can guarantee himself at the start of the auction. In a dynamic mechanism it is natural to be interested in strategies that are responsive to the path of play and continue to maximize the payoff a bidder can guarantee himself as play unfolds. The main result in this subsection is to identify the unique strategy which maximizes the payoff a bidder guarantees himself following any sequence $p_1, \ldots, p_k$ of drop out prices. We show the security payoff of a bidder following this

[6] No restriction is placed on $\bar{\beta}^i_k(x_i; \mathbf{p}_{k-1})$ if $x_i < p_{k-1}$ since this contingency never arises if bidder i follows $\bar{\beta}^i$.

[7] We adopt the usual convention that $[a, b] = \{a\}$ if $a = b$, and $[a, b] = \emptyset$ if $a > b$. Observe that Proposition 4 places no restriction is on dropout prices for k, x_i, and $\mathbf{p}_{k-1}$ such that $[\frac{x_i}{N} + p_{k-1}, \frac{x_i - p_{k-1}}{N-k+1} + p_{k-1}]$ is empty.

strategy increases from one round to the next as long as he remains in the auction.

To proceed, it is useful to introduce the notation of a subauction of the compensation auction. The subauction $\Gamma(n, p_0)$ is a compensation auction in which there are $n \leq N$ bidders and the initial price ascends from $p_0 \geq 0$ rather than zero. If $p_1 \leq \ldots \leq p_{n-1}$ is the sequence of dropout prices in $\Gamma(n, p_0)$, then the winner pays the difference $p_k - p_{k-1}$ to the k-th bidder to drop for $k = \{1, \ldots, n-1\}$, and pays p_0 in addition.

Our results to this point concern the compensation auction $\Gamma(N, 0)$. However, after $k-1$ bidders have dropped out at prices $p_1, \ldots, p_{k-1}$, then the remaining bidders participate in $\Gamma(N-(k-1), p_{k-1})$, i.e., the subauction with $N-(k-1)$ bidders and the price ascending from p_{k-1}. If p_{N-1} is the final drop out price in the subaction, the winner of the subauction pays (total) compensation $p_{N-1} - p_{k-1}$ to the $N-k$ other bidders in the subauction and pays (total) compensation of p_{k-1} to the $k-1$ bidders who dropped out prior to the subauction.

Proposition 3 identified a security strategy for $\Gamma(N, 0)$. Proposition 5 is the analogue to Proposition 3 for $\Gamma(n, p_0)$. It identifies a bidder's security strategy and security payoff when the initial price p_0 need not be zero.

Proposition 5: *Let* $n \leq N$ and $p_0 \geq 0$. *In the subauction* $\Gamma(n, p_0)$ *the strategy* $\bar{\beta}^i$*, given by*

$$\bar{\beta}^i_k(x_i; \mathbf{p}_{k-1}) = \begin{cases} (x_i - p_{k-1})/(n-k+1) + p_{k-1} & \textit{if } x_i \geq p_{k-1} \\ \\ p_{k-1} & \textit{if } x_i < p_{k-1} \end{cases}$$

for $k \in \{1, \ldots, n-1\}$*,* $x_i \in [0, \bar{x}]$*, and* $\mathbf{p}_{k-1}$ *such that* $p_0 \leq p_1 \leq \ldots \leq p_{k-1}$*, is a security strategy. Furthermore, bidder* i*'s security payoff when his value is* x_i *is* $(x_i - p_0)/n$*.*

An implication of Proposition 5 is that a bidder's security payoff weakly increases from one round to the next when he follows the security strategy $\bar{\beta}^i$ identified in Proposition 3. To see this, consider a bidder whose value is x_i and who remains in the auction at round $k+1$ following drops at prices $p_1, \ldots, p_k$. By Proposition 5, his security payoff in the subauction $\Gamma(N-k, p_k)$ is

$$\frac{x_i - p_k}{N-k}.$$

Since the bidder did not drop in round k, then the bid at which a rival dropped must be less than his own round k bid, i.e.,

$$p_k \leq \bar{\beta}^i_k(x_i; \mathbf{p}_{k-1}) = \frac{x_i - p_{k-1}}{N-k+1} + p_{k-1}.$$

Hence

$$\frac{x_i - p_k}{N-k} \geq \frac{x_i - (\frac{x_i - p_{k-1}}{N-k+1} + p_{k-1})}{N-k} = \frac{x_i - p_{k-1}}{N-(k-1)},$$

where the right hand side was the bidder's security payoff in round k in $\Gamma(N-(k-1), p_{k-1})$. Indeed, so long as bidder i is never indifferent between dropping or continuing, the inequalities above are strict and bidder i's security payoff strictly increases from one round to the next.

A security strategy is *perfect* if it continues to be a security strategy in the auction that remains following any sequence of drops. Formalizing this idea requires introducing the notion of the restriction of a strategy (for $\Gamma(N,0)$) to a subauction. Let $\beta^i|_{\mathbf{p}_{k-1}}$ be the restriction of β^i to the auction $\Gamma(N-(k-1), p_{k-1})$ obtained after $k-1$ bidders in $\Gamma(N,0)$ drop at prices $(p_1, \ldots, p_{k-1})$, i.e., define

$$\begin{aligned}
\beta^i_1|_{\mathbf{p}_{k-1}}(x_i) &\equiv \beta^i_k(x_i; \mathbf{p}_{k-1}), \\
\beta^i_2|_{\mathbf{p}_{k-1}}(x_i; p_k) &\equiv \beta^i_{k+1}(x_i; \mathbf{p}_{k-1}, p_k), \\
&\vdots \\
\beta^i_{N-k}|_{\mathbf{p}_{k-1}}(x_i; p_k, \ldots, p_{N-2}) &\equiv \beta^i_{N-1}(x_i; \mathbf{p}_{k-1}, p_k, \ldots, p_{N-2}).
\end{aligned}$$

Formally, a perfect security strategy is defined as follows:

Definition: $\bar{\beta}^i$ is a **perfect security strategy** for bidder i if for $k \in \{1, \ldots, N-1\}$, $x_i \in [0, \bar{x}]$, and $\mathbf{p}_{k-1}$ such that $p_0 \leq p_1 \leq \ldots \leq p_{k-1}$, then $\bar{\beta}^i|_{\mathbf{p}_{k-1}}$ is a security strategy for bidder i in $\Gamma(N-(k-1), p_{k-1})$.

Proposition 6 shows that the security strategy identified in Proposition 3 is the unique perfect security strategy.

Proposition 6: *In the compensation auction* $\Gamma(N, 0)$ *the strategy* $\bar{\beta}^i$*, given by*

$$\bar{\beta}^i_k(x_i; \mathbf{p}_{k-1}) = \begin{cases} (x_i - p_{k-1})/(n-k+1) + p_{k-1} & \text{if } x_i \geq p_{k-1} \\ p_{k-1} & \text{if } x_i < p_{k-1} \end{cases}$$

for $k \in \{1, \ldots, N-1\}$*,* $x_i \in [0, \bar{x}]$*, and* $\mathbf{p}_{k-1}$ *such that* $0 \leq p_1 \leq \ldots \leq p_{k-1}$*, is the unique perfect security strategy.*

Proposition 4.2 of Van Essen and Wooders (2016) establishes that the security strategy given in Proposition 6 is obtained as the limit, as bidders become infinitely risk averse, of the Bayes Nash equilibrium when bidders have constant absolute risk aversion.

4 Discussion

We conclude by considering the players' incentives to participate in a dissolution mechanism. Cramton, Gibbons, and Klemperer (1987) take the disagreement payoff of bidder i with value x_i to be $r_i x_i$ if he refuses to participate in the mechanism, where r_i is the player's ownership share. We have shown that in the Texas Shoot-Out, the $K+1$ auction, and the compensation auction that each bidder's security payoff is $1/N$-th of his value, and hence participation is individually rational when ownership shares are equal.

The $K+1$ auction and compensation auction can dissolve partnerships with unequal shares as well, while still giving each player a security payoff which makes participation individually rational. To illustrate, suppose there are three owners with shares $r_1 = 1/4$, $r_2 = 1/3$, and $r_3 = 5/12$. Consider the compensation auction in which bidder 1 has 3 agents who participate on his behalf. Likewise, assign 4 agents to bidder 2 and 5 agents to bidder 3. The auction, therefore, will have 12 participating agents. If bidder i's agents each follow the security strategy of a bidder with value v_i in the $N = 12$ auction, then each of his agents secures a payoff of $1/N$-th of v_i and thus i's agents collectively secure $r_i v_i$. Participation is therefore individually rational for each bidder. This "trick" of assigning multiple agents to players is common in the cake cutting literature, see Robertson and Webb (1998).

5 Appendix

Proof of Proposition 0: The proof is well known and is only included for completeness. It is easy to verify that for the given strategies the players each obtain a payoff of at least $x_1/2$ and $x_2/2$, respectively.

We show there is no strategy which guarantees player 1 more than $x_1/2$. Consider a strategy β^1 such that $\beta^1(x_1) > x_1/2$ for some x_1. If player 1 with value x_1 sets a price $\beta^1(x_1) = p$ and player 2 chooses Sell, then player 1's payoff is $x_1 - p < x_1/2$. Likewise, if $\beta^1(x_1) = p < x_1/2$ for some x_1 then player 1 obtains a payoff less than $x_1/2$ if $\beta^2(x_2; p) =$"Buy".

Likewise, if $\beta^2(x_2, p) =$"Buy" for some $p > x_2/2$, then player 2 obtains a payoff $x_2 - p < x_2/2$ if player 1 offers price p. $\square$

Proof of Proposition 1: We need to establish two facts: (i) $\bar{\beta}^i(x_i) = x_i$ guarantees bidder i a payoff of at least x_i/N, and (ii) there is no strategy which guarantees bidder i a payoff above x_i/N. This establishes that $\bar{v}(x_i) = x_i/N$ is bidder i's security payoff and $\bar{\beta}^i$ is a security strategy.

Part (i): Suppose that $\bar{\beta}^i(x_i) = x_i$. If bidder i wins then he obtains a payoff of

$$x_i - \frac{N-1}{N}\left[Kb_s + (1-K)x_i\right] \geq x_i/N,$$

where the inequality holds since $x_i \geq b_s$ as x_i is the winning bid. If bidder i loses, then he obtains

$$\frac{1}{N}\left[Kb_s + (1-K)b_f\right] \geq x_i/N,$$

where the inequality holds since $b_f \geq b_s \geq x_i$. His payoff, therefore, is at least x_i/N.

Part (ii). Suppose to the contrary that for some $\hat{x}_i$, $\hat{\beta}^i$, and $\Delta > 0$ that

$$v(\hat{x}_i, x_{-i}, \hat{\beta}^i, \beta^{-i}) > \frac{\hat{x}_i}{N} + \Delta \ \forall x_{-i}, \beta^{-i}.$$

Since the inequality holds for all x_{-i} and β^{-i}, then it holds in particular for $\hat{x}_{-i} = (\hat{x}_i, \ldots, \hat{x}_i)$ and $\hat{\beta}^{-i} = (\hat{\beta}^i, \ldots, \hat{\beta}^i)$, i.e., $v(\hat{x}_i, \hat{x}_{-i}, \hat{\beta}^i, \hat{\beta}^{-i}) > \hat{x}_i/N + \Delta$. When every bidder has the same value $\hat{x}_i$ and follows the same strategy $\hat{\beta}^i$, then by symmetry every bidder has the same expected payoff, which is at least $\hat{x}_i/N + \Delta = \bar{v}(\hat{x}_i) + \Delta$. Summing across the N bidders, the total payoff is greater than $N\bar{v}(\hat{x}_i) = \hat{x}_i$. This is a contradiction since the total gain to allocating the item is $\hat{x}_i$ when every bidder's value is $\hat{x}_i$. □

Proof of Proposition 2: We prove that: (i) $\bar{\beta}^i$ guarantees bidder i a payoff of at least x_i/N, and (ii) there is no strategy which guarantees bidder i a payoff great than x_i/N.

Part (i). Suppose that bidder i has value x_i and follows $\bar{\beta}^i$ given in the proposition. Let x_{-i} and β^{-i} be arbitrary, and let $p_1, \ldots, p_{N-1}$ be the sequence of dropout prices that result. The sequence is uniquely determined unless there is a tie at some stage. If there is a tie then, depending on which bidder drops, one of several different prices sequences may result. In this case, let $(p_1, \ldots, p_{N-1})$ be an arbitrary such sequence.

Either bidder i drops out at some stage k, or all the other bidders drop out first. In the former case, i's payoff is $x_i/N + p_{k-1} - p_{k-1} = x_i/N$. Suppose that all the other bidders drop out before bidder i. Then it must be the case that $p_1 \leq x_i/N$, $p_2 - p_1 \leq x_i/N, \ldots, p_{N-1} - p_{N-2} \leq x_i/N$ since otherwise, if $p_k - p_{k-1} > x_i/N$ for some k, then bidder i would have dropped out at round k. Hence $p_1 + (p_2 - p_1) + \ldots + (p_{N-1} - p_{N-2}) \leq (N-1)x_i/N$ and thus bidder i's payoff is at least $x_i - (N-1)x_i/N = x_i/N$.

Part (ii). Suppose to the contrary that for some $\hat{x}_i \in [0, \bar{x}]$ that there is a strategy $\hat{\beta}^i$ for bidder i such that

$$v(\hat{x}_i, x_{-i}, \hat{\beta}^i, \beta^{-i}) > \bar{v}(\hat{x}_i) = \frac{\hat{x}_i}{N} \; \forall x_{-i}, \beta^{-i}.$$

Since the inequality holds for all x_{-i} and β^{-i}, then it holds in particular for $\hat{x}_{-i} = (\hat{x}_i, \ldots, \hat{x}_i)$ and $\hat{\beta}^{-i} = (\hat{\beta}^i, \ldots, \hat{\beta}^i)$, i.e., $v(\hat{x}_i, \hat{x}_{-i}, \hat{\beta}^i, \hat{\beta}^{-i}) > \hat{x}_i/N$. When every bidder has the same value $\hat{x}_i$ and follows the same strategy $\hat{\beta}^i$, then by symmetry every bidder has the same expected payoff, which exceeds $\bar{v}(\hat{x}_i)$. Summing across the N bidders, the total payoff exceeds $N\bar{v}(\hat{x}_i) = \hat{x}_i$. This is a contradiction since the total gain to allocating the item, i.e., the sum of the bidders' payoffs, must be $\hat{x}_i$ when every bidder's value is $\hat{x}_i$. $\square$

Proof of Proposition 3: Suppose that bidder i has value x_i and follows $\bar{\beta}^i$. Let x_{-i} and β^{-i} be arbitrary, and let $p_1, \ldots, p_{N-1}$ be the sequence of dropout prices that results. We show that bidder i's payoff is at least his security payoff of x_i/N. In the proof below, take $n = N$ and $p_0 = 0$.

Suppose that bidder i is not among the first $\hat{k} - 1$ bidders to drop. We show for $k \in \{1, \ldots, \hat{k} - 1\}$ that (i) $p_k - p_0 \leq k(x_i - p_0)/n$ and (ii) $p_k - p_{k-1} \leq (x_i - p_{k-1})/(n - k + 1)$. Assume $x_i > p_0$. If bidder i is not the first to drop, then

$$\bar{\beta}^i_1(x_i; p_0) = \frac{x_i - p_0}{n - 1 + 1} + p_0 \geq p_1,$$

i.e.,

$$p_1 - p_0 \leq \frac{x_i - p_0}{n}.$$

Hence (i) and (ii) hold for $k = 1$.

Assume that (i) and (ii) hold for some $k' < \hat{k} - 1$. We show they hold for $k' + 1$. By the induction hypothesis, $p_{k'} - p_0 \leq k'(x_i - p_0)/n$ and hence $k' < n$ and $x_i > p_0$ implies $p_{k'} - p_0 \leq x_i - p_0$, i.e., $p_{k'} \leq x_i$. Since bidder i did not drop at $k' + 1 \leq \hat{k} - 1$, then

$$\bar{\beta}^i_{k'+1}(x_i; \mathbf{p}_{k'}) = \frac{x_i - p_{k'}}{n - (k' + 1) + 1} + p_{k'} \geq p_{k'+1},$$

which establishes (ii) for $k = k' + 1$. Rearranging, we obtain

$$p_{k'+1} - p_0 \leq \frac{x_i + (n - k' - 1)p_{k'}}{n - k'} - p_0 \leq \frac{x_i + (n - k' - 1)(\frac{k'(x_i - p_0)}{n} + p_0)}{n - k'} - p_0 = \frac{k' + 1}{n}(x_i - p_0),$$

where the second inequality holds by the induction hypothesis. Hence (i) holds for $k = k' + 1$.

If bidder i drops in round $\hat{k}$, then his payoff is $(x_i - p_{\hat{k}-1})/(n - \hat{k} + 1)$. Since $p_{\hat{k}-1} \leq (\hat{k} - 1)(x_i - p_0)/n + p_0$ then

$$\frac{x_i - p_{\hat{k}-1}}{n - \hat{k} + 1} \geq \frac{x_i - (\frac{\hat{k}-1}{n}(x_i - p_0) + p_0)}{n - \hat{k} + 1} = \frac{x_i - p_0}{n}.$$

If bidder i is not among the first $n - 1$ bidders to drop, then $p_{n-1} - p_0 \leq (n - 1)(x_i - p_0)/n$. He wins the auction and his payoff is

$$x_i - p_{n-1} \geq x_i - (\frac{n - 1}{n}(x_i - p_0) + p_0) = \frac{x_i - p_0}{n}.$$

Hence $\bar{\beta}^i$ guarantee's bidder i his security payoff of $(x_i - p_0)/n$ and is therefore a security strategy. $\square$

Proof of Proposition 4: Suppose that bidder i has value x_i and follows $\bar{\beta}^i$. Let x_{-i} and β^{-i} be arbitrary, and let $p_1, \ldots, p_{N-1}$ be the sequence of dropout prices that results.

Suppose bidder i has not dropped at round $\hat{k} \geq 1$. We show that $p_k \leq kx_i/N$ for each $k \in \{1, \ldots, \hat{k}\}$. Since bidder i did not drop at round 1 then $p_1 \leq \bar{\beta}^i_1(x_i) = x_i/N$. Suppose that $p_k \leq kx_i/N$ for some $k' < \hat{k}$. We show that $p_{k'+1} \leq (k'+1)x_i/N$. Since $p_{k'} \leq k'x_i/N$, then $[\frac{x_i}{N} + p_{k'}, \frac{x_i - p_{k'}}{N-k'} + p_{k'}]$ is non-empty, and hence $\bar{\beta}^i_{k'+1}(x_i; \mathbf{p}_{k'}) \in [\frac{x_i}{N} + p_{k'}, \frac{x_i - p_{k'}}{N-k'} + p_{k'}]$. Since bidder i did not drop at round $k'+1$, then

$$p_{k'+1} \leq \bar{\beta}^i_{k'+1}(x_i; \mathbf{p}_{k'}) \leq \frac{x_i - p_{k'}}{N-k'} + p_{k'} = \frac{x_i + p_{k'}(N-k'-1)}{N-k'}.$$

Furthermore, $p_{k'} \leq k'x_i/N$ implies

$$p_{k'+1} \leq \frac{x_i + \frac{k'}{N}x_i(N-k'-1)}{N-k'} = \frac{(k'+1)x_i}{N}.$$

By induction, $p_k \leq kx_i/N$ for each $k \in \{1, \ldots, \hat{k}\}$.

Since $\bar{\beta}^i_1(x_i) = x_i/N$, if bidder i dropped at round 1 his payoff was x_i/N. If bidder i dropped at round $k > 1$ then $p_{k-1} \leq (k-1)x_i/N$ (since he did not drop at round $k-1$) and hence his payoff is

$$\bar{\beta}^i_k(x_i; \mathbf{p}_{k-1}) - p_{k-1} \geq \frac{x_i}{N} + p_{k-1} - p_{k-1} = \frac{x_i}{N}.$$

If bidder i wins the auction (i.e., he did not drop at round $N-1$) then $p_{N-1} \leq (N-1)x_i/N$ and his payoff is

$$x_i - p_{N-1} \geq x_i - \frac{N-1}{N}x_i = \frac{x_i}{N}.$$

Thus $\bar{\beta}^i$ is a security strategy for bidder i. $\square$

Proof of Proposition 5: If $x_i \geq p_0$, the proof of Proposition 3 goes through since it holds for general n and p_0.

If $x_i < p_0$, then bidder i's payoff is negative if he wins the auction. We first show that $\bar{\beta}^i$ guarantees bidder i a payoff of a least $(x_i - p_0)/n$. Since $\bar{\beta}^i_1$ calls for bidder i to drop immediately, his payoff is zero unless he wins the auction. The later occurs only if all $n-1$ other bidders drop immediately and ties are broken in bidder i's favor. In this case, bidder i's payoff is $x_i - p_0$.

Since this occurs with at most probability $1/n$, his expected payoff is at least $(x_i - p_0)/n$.

To see that there is no strategy which guarantees bidder i a payoff above $(x_i - p_0)/n$, simply note that for any strategy he follows, if all of his rivals follow the same strategy and have the same values, then by symmetry each bidder wins with probability $1/n$ and bidder i's payoff is $(x_i - p_0)/n$. $\square$

Proof of Proposition 6: Write $\bar{v}_{N-(k-1),p_{k-1}}(x_i)$ for the security payoff of a bidder with value x_i in the subauction $\Gamma(N-(k-1), p_{k-1})$. Suppose that $\bar{\beta}^i_k(x_i;\mathbf{p}_{k-1}) < (x_i - p_{k-1})/(N-(k-1)) + p_{k-1}$ for some k, x_i and $\mathbf{p}_{k-1}$ such that $p_0 \leq p_1 \leq \ldots \leq p_{k-1}$. We show that $\bar{\beta}^i$ is not a perfect security strategy. In particular, we show that $\bar{\beta}^i|_{\mathbf{p}_{k-1}}(x_i)$ yields a payoff less than $\bar{v}_{N-(k-1),p_{k-1}}(x_i)$ for some x_{-i} and β^{-i}.

From Proposition 5, the security payoff of bidder i in $\Gamma(N-(k-1), p_{k-1})$ is $\bar{v}_{N-(k-1),p_{k-1}}(x_i) = (x_i - p_{k-1})/(N-(k-1))$. Let x_{-i} and β^{-i} be such that the bids of the other $N-k$ bidders in round 1 of $\Gamma(N-(k-1), p_{k-1})$ are greater than $\bar{\beta}^i_1|_{\mathbf{p}_{k-1}}(x_i) = \bar{\beta}^i_k(x_i;\mathbf{p}_{k-1})$. Then bidder i drops in round 1 and his payoff is

$$\bar{\beta}^i_1|_{\mathbf{p}_{k-1}}(x_i) - p_{k-1} < \frac{x_i - p_{k-1}}{N-(k-1)} + p_{k-1} - p_{k-1} = \bar{v}_{N-(k-1),p_{k-1}}(x_i).$$

Hence $\bar{\beta}_i$ is not a perfect security strategy.

Suppose that $\bar{\beta}^i_k(x_i;\mathbf{p}_{k-1}) > (x_i - p_{k-1})/(N-k+1) + p_{k-1}$ for some k, x_i and $\mathbf{p}_{k-1}$ such that $p_0 \leq p_1 \leq \ldots \leq p_{k-1}$. Let x_{-i} and β^{-i} be such that (i) one of the other $N-k$ bidders in $\Gamma(N-(k-1), p_{k-1})$ has a dropout price $\hat{p}_k$ satisfying

$$\bar{\beta}^i_1|_{\mathbf{p}_{k-1}}(x_i) > \hat{p}_k > \frac{x_i - p_{k-1}}{N-(k-1)} + p_{k-1},$$

and (ii) the remaining bidders' dropout prices are above $\bar{\beta}^i_1|_{\mathbf{p}_{k-1}}(x_i) = \bar{\beta}^i_k(x_i;\mathbf{p}_{k-1})$. Then bidder 1 does not drop out in round 1 of $\Gamma(N-(k-1), p_{k-1})$, but enters the subauction $\Gamma(N-k, \hat{p}_k)$. From Proposition 3 the largest payoff he can guarantee himself in this subauction is $\bar{v}_{N-k,\hat{p}_k}(x_i) = (x_i - \hat{p}_k)/(N-k)$. We have that

$$\frac{x_i - \hat{p}_k}{N-k} < \frac{x_i - \left[\frac{x_i - p_{k-1}}{N-(k-1)} + p_{k-1}\right]}{N-k} = \frac{x_i - p_{k-1}}{N-(k-1)} < \bar{v}_{N-(k-1),p_{k-1}}(x_i).$$

Hence $\bar{\beta}_i$ is not a perfect security strategy. $\square$

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