

Ascending Bid Auctions with a Buy-Now Price

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Abstract

Internet auctions run by eBay and Yahoo! allow sellers to set up their auctions with a *Buy-Now* option. Sellers set a price, termed a *buy price*, at which a bidder may purchase the item and end the auction. This option is now widely used on internet auctions. This might seem surprising at first, since setting a buy price is tantamount to imposing a price ceiling in the auction, and this seems contrary to a seller's interest. We formulate and analyze a model of ascending bid auctions with a buy price. We assume two bidders with symmetric, independent private values drawn from a uniform distribution. The details of the model differ for the eBay and Yahoo! versions, since these two sites implement the buy-now option differently. We characterize unique symmetric equilibrium strategies for risk neutral bidders in both versions of the buy-now mechanism. Expected seller revenue in equilibrium in the eBay buy-now auction is less than in the ascending bid auction if any bidder types accept the buy price. For the Yahoo! buy-now auction there exists an interval of buy prices at the upper end of the support of values such that expected revenue for the seller in equilibrium is equal to revenue in an English auction

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- a revenue equivalence result. Finally, we show that seller revenue may be higher in a buy-now auction than in an English auction when bidders are risk averse.

1 Introduction

The expansion of commerce conducted over the Internet has sparked a surge of interest in auctions and new auction forms. Many new online auction sites appeared, and quite a few subsequently disappeared. These online auctions use a variety of formats and rules. In many cases online auctions have adapted procedures that had been used for running auctions long before the Internet came into being. In a few cases online auctions have introduced features which appear to be new and unique to the online environment. Lucking-Reiley (2000) describes the wide variety of online auction formats that were being used as of 1999.

An example of a new twist in online auction formats appears in Yahoo! and eBay auctions. In 1999 Yahoo! introduced the *Buy-Now!* feature into its ascending bid auctions. The Buy-Now! feature allows the seller to set a price, termed a buy price, at which any bidder may purchase the item at any time during the auction (for more on this format see, <http://auctions.yahoo.com/phtml/auc/us/promo/buynow.html>). This feature, in effect, allows the seller to post a maximum price for the item, since no rational bidder would permit the bid price to rise above the buy price. In 2000 eBay introduced its own version of a maximum price feature into its online auctions via the *Buy It Now* option. In contrast to the Yahoo! format, eBay permits bidders to select the buy price only at the opening of the auction, before any bids are submitted, or in the case of an auction with a reservation price, before bids reach the reservation price (see, <http://pages.ebay.com/services/buyandsell/buyitnow.html>). We use the expression "buy-now auction" as the generic term for an ascending bid auction with a buy price.

The buy-now auction has become quite popular in both Yahoo! and eBay auctions. Table One lists the total number of current auctions and the number of buy-now

auctions on eBay and Yahoo! in selected categories on a recent day (the categories are similar, but not identical across the two auction sites). Overall, about 40% of these eBay auctions and 66% of these Yahoo! auctions utilize the buy-now feature. A recent article also cites a 40% figure for the fraction of eBay auctions that use the buy-now feature (see Hof (2001)). Note that the total number of auctions on eBay dwarfs the number of auctions on Yahoo!

Table One

Data on Buy-Now and All Auctions for Yahoo! and eBay
 (data is for current auctions on March 27, 2002)

Category	Yahoo!Auctions		eBay Auctions	
	# buy-now	total #	# buy-now	total #
Automobiles	157	186	662	2,378
Clothing	148	223	2,678	10,078
DVD Players	86	115	981	2,251
VCR's	91	136	384	1,078
Digital Cameras	337	583	7,179	13,284
TV's	23	39	596	2,073

Setting a maximum price in an auction may appear to be irrational for a seller. After all, an ascending bid auction is intended to illicit high bids from potential buyers, and putting a cap on these bids would seem to limit the seller's expected revenue. But there are at least two reasons why a buy-now auction might yield greater revenue for the seller. First, a buy price may help reduce the extent of uncertainty for some bidders. Consider a bidder whose value exceeds the buy price. If the bidder waits then they may be able to acquire the item at a price less than the buy price, but the bidder runs the risk of losing the auction (this would happen if another bidder accepts the buy price, or if they are outbid in the ascending auction). By accepting the buy price the bidder can ensure acquisition of the item at the buy

price. A risk-averse bidder may opt to accept the buy price rather than face the uncertainty regarding acquisition at a lower price. A seller may be able to exploit this kind of bidder behavior by appropriately setting the buy price. A second reason why a buy price might increase seller revenue involves a reduction in waiting time for the winning bidder. Online ascending bid auctions often have a fixed closing date; Yahoo! and eBay auctions typically run for 7 days. Acceptance of a buy price may permit a bidder to gain possession of the item several days earlier than would be possible by waiting for the auction to close. The reduction in waiting time associated with the buy now option may raise bidders' willingness to pay under this option relative to the wtp when a winner gains possession after the auction close. A seller may be able to exploit this increase in willingness to pay under the buy-now option by appropriately setting the buy price.

In this paper we formulate and analyze a model of an ascending bid auction with a buy-now feature. We utilize a symmetric independent private values framework with a continuous distribution of values for bidders. We derive equilibrium results for a version of the model with two bidders and uniformly distributed values. We compare and contrast equilibrium results under the Yahoo! and eBay versions of the buy now mechanism. We do not consider time discounting or any other cost of waiting in our analysis. Instead, we focus on how introduction of the buy price alters the incentives of bidders in the absence of costs of waiting, and on how equilibrium bidding behavior with a buy price affects seller revenue.

We demonstrate several results. First, we characterize unique symmetric equilibrium strategies for bidders in the eBay version of the buy now mechanism. Equilibrium strategies involve a value cut-off, above which a bidder will accept the buy price, and below which a bidder will reject the buy price and participate in the ascending bid auction. If both bidders accept the buy price, then the winning bidder is selected randomly. If bidders are risk neutral then in equilibrium (1) no bidder types accept the buy price if it is in the upper half of the values support; all bidder types wait for the ascending bid portion of the auction and the allocation is efficient, and (2) some bidder types accept the buy price if it is in the lower half of the values support; the

allocation is (ex ante) inefficient. Expected seller revenue in the buy now auction is less than in the ascending bid auction if any bidder types accept the buy price. Next, we characterize unique symmetric equilibrium strategies for risk neutral bidders in the Yahoo! version of the buy now mechanism. The critical part of deriving equilibrium strategies involves characterizing a threshold function. This function specifies the bid price (or, clock time) at which a buyer will accept the buy price, as a function of the buyer's value, for values above the buy price. This time may be either at the opening of the auction, or after the opening. Second, we show a conditional revenue equivalence result. There exists an interval of buy prices at the upper end of the support of values such that the auction with a buy price yields the same expected revenue for the seller as the ascending bid auction. This is an application of the well-known revenue equivalence result for auction formats with risk neutral bidders that yield efficient allocations. If buy prices are in the lower half of the values support then the allocation is (ex ante) inefficient and seller revenue is less than revenue in an ascending bid auction. Finally, we demonstrate how seller revenue is impacted by bidder risk aversion for eBay buy-now auctions. We derive equilibrium results for constant absolute risk averse (CARA) bidders and we show that a seller can earn greater expected revenue with a buy-now auction than with an English auction.

We are aware of one prior analysis of an auction with a buy price. Budish and Takeyama (2001) analyze an ascending bid auction with a buy price using a model with two bidders, symmetric, independent private values, and with two possible valuations for each bidder. They demonstrate that (1) if bidders are risk neutral and the seller sets the buy price optimally then the seller earns the same expected revenue in an auction with a buy price as in the conventional ascending bid auction, and (2) if bidders are risk averse and the seller sets the buy price optimally then the seller can earn greater expected revenue in an auction with a buy price than in a conventional ascending bid auction. Risk aversion can drive a high-value bidder to accept a relatively high buy price rather than take the chance that the bid price will rise above the buy price. The Budish and Takeyama analysis applies to the eBay

buy-now mechanism, but not to the Yahoo! mechanism.

2 The Model

There are two bidders for a single item, whose values are independently and identically distributed, according to a uniform distribution with support $[\underline{v}, \bar{v}]$ and *c.d.f.* F . Denote by v_i the value of bidder i . Let B denote the buy-now price set by the seller. We assume that $\underline{v} < B < \bar{v}$, since a seller would never wish to set $B \leq \underline{v}$, whereas if $B \geq \bar{v}$ then no bidder will ever take it. Bidders are assumed to have a common utility function u that depends on the monetary payoff received. Utility is assumed to be strictly increasing, concave, and twice-differentiable, with $u(0) = 0$. This section deals with the case when bidders are risk neutral; the next section considers risk averse bidders.

Both the eBay and the Yahoo! buy-now auctions are modelled as clock auctions, with a continuously rising bid price. The precise rules under which the buy now option is exercised differ for the two auctions:

eBay: At the opening instant of the auction the bidders simultaneously decide whether to exercise the buy now option. If both bidders reject the buy price then the buy-now option disappears, and bidding proceeds according to an ascending clock auction.¹ If either bidder exercises the buy-now option, then the auction ends, the bidder exercising the option wins the item and pays the seller B . If both bidders choose the buy-now option, then the winner is randomly assigned.

Yahoo!: Bidding proceeds as in an ascending clock auction, with the buy-now option remaining in effect so long as the auction is open. If either bidder exercises the buy-now option at any point, then the auction ends, the bidder exercising the option wins the item and pays the seller B .

¹Here we deal with auctions in which there is no reserve, in which case the buy-now option disappears as soon as a bid is placed. In eBay auctions with a reserve, the buy-now option disappears only once there is a bid placed about the reserve price.

2.1 Buy It Now (eBay)

In the eBay auction a bidder's strategy tells him, for each of his possible values, whether to exercise the buy-now option or whether to wait. (We will assume that if neither bidder chooses to buy now, then in the ascending clock auction that follows, both bidders follow their dominant strategy of remaining in the auction as long as their value is above the bid price.) We focus on equilibria in "cutoff strategies." A **cutoff strategy** for bidder is a value $c \in [\underline{v}, \bar{v}]$ such that he accepts the buy price if his value exceeds c but waits if his value is below c . Suppose that bidder i 's value is v_i and his rival's cutoff strategy is c . If bidder i accepts the buy price, he wins the auction if either his rival waits (which occurs with probability $F(c)$) or his rival accepts the buy price but bidder i wins the coin toss (which occurs with probability $\frac{1}{2}[1 - F(c)]$). Bidder i 's expected payoff is

$$U^b(v_i, c) = \left[\frac{1}{2}(1 - F(c)) + F(c) \right] u(v_i - B).$$

If bidder i waits, then he wins the auction only if his rival also waits (i.e., $v_j < c$) and his value is higher than his rival's (i.e., $v_j < v_i$). His expected payoff is

$$U^w(v_i, c) = \int_{\underline{v}}^{\min\{v_i, c\}} u(v_i - v_j) dF(v_j).$$

A cutoff value c^* is a **symmetric equilibrium** if $U^w(v_i, c^*) > U^b(v_i, c^*)$ for all $v_i \in [\underline{v}, c^*)$ and $U^w(v_i, c^*) < U^b(v_i, c^*)$ for all $v_i \in (c^*, \bar{v}]$, i.e., given that bidder j uses the cutoff value c^* then it's optimal for bidder i to use the same cutoff.

Proposition 1 indicates that equilibrium and efficiency results can be divided into two cases, depending on whether the buy price is relatively high or not.

Proposition 1: Suppose the bidders are risk neutral.

- (i) If $B \in [\frac{1}{2}\underline{v} + \frac{1}{2}\bar{v}, \bar{v})$ then the buy price is never accepted by a bidder (i.e., the unique equilibrium cutoff value is $c^* = \bar{v}$).
- (ii) If $B \in (\underline{v}, \frac{1}{2}\underline{v} + \frac{1}{2}\bar{v})$ then there is a unique symmetric equilibrium cutoff value of

$$c^* = \frac{B(\bar{v} - 2\underline{v}) + \underline{v}^2}{\bar{v} - B}, \quad (1)$$

and the cutoff value is increasing in the buy price. Equilibrium is inefficient since the item is awarded to the low-value bidder with positive probability.

Proposition 1 indicates that the buy-now auction is inefficient in equilibrium as long as the buy price is set low enough so that some bidder types would be willing to accept. The inefficiency is similar to the inefficiency that results when a single item is offered for sale at a fixed price to multiple buyers. If there is no mechanism to put the high-value buyer at the head of the queue of buyers then there is a positive probability that the high-value buyer will not receive the item. The only case in which the buy-now auction is (ex ante) efficient is when the buy price is set high enough so that no bidder types will accept it (case (i) in Proposition 1).

Proposition 2: Suppose that bidders are risk neutral. If the buy price is set so that there is a positive probability that B will be accepted in equilibrium (i.e., $B \in (\underline{v}, \frac{1}{2}\underline{v} + \frac{1}{2}\bar{v})$) then expected seller revenue in equilibrium is less than expected seller revenue in an English auction.

Note that results for a model with a continuous distribution of values differ from those of the Budish and Takeyama (2001) model with two possible values. In BT the seller can set a buy price so that a high-value buyer is willing to accept and expected seller revenue equals expected revenue in an English auction. With a continuous distribution of values, any buy price that is accepted with positive probability yields lower expected seller revenue than an English auction.

2.2 Buy-Now! (Yahoo!)

The Yahoo! buy-now auction involves more complex strategies for bidders than the eBay buy-now auction. For the eBay version we were able to characterize the decision to accept the buy price based on a simple cut-off strategy. In contrast, for the Yahoo! version a bidder's strategy must indicate at what point during the auction the bidder will accept the buy price, conditional upon their value. We simplify the analysis of the Yahoo! version by supposing that if a bidder's value is less than B , then he

never chooses to “buy it now” and he exits as soon as the price reaches his value; recall that this is a dominant strategy in the English auction. We focus on how bidders with values above B choose the bid price (correspondingly, the time on the auction clock) at which to accept the buy price. We say that $t : [B, \bar{v}] \rightarrow [\underline{v}, B]$ is a **threshold strategy** if either (i) t is continuous and decreasing on $[B, \bar{v}]$, or (ii) there is a $z \in (B, \bar{v})$ such that t is continuous and strictly decreasing on $[B, z)$, but t jumps down to $t(v) = \underline{v}$ for $v \in (z, \bar{v}]$.

A threshold strategy has the following interpretation: if $t(v) > \underline{v}$ then a bidder with value v remains in the auction until the bid price rises to $t(v)$, at which point he elects to buy-it-now; if $t(v) = \underline{v}$ then the bidder elects to buy-it-now immediately at the auction open. Let $[\underline{t}, \bar{t}]$ denote the range of threshold values for which $t(v)$ is strictly decreasing.

Suppose bidder 2 follows the threshold strategy $t(v_2)$ shown in Figure 1 and bidder 1 chooses the threshold t_1 . If bidder 2’s value v_2 is below t_1 , then bidder 2 drops out when the price reaches his value. Bidder 1 wins and pays v_2 . If bidder 2’s value is above t_1 but below B , then bidder 1 exercises the buy now option when the price reaches t_1 (and before bidder 2 drops out). If bidder 2’s value is between B and $t^{-1}(t_1)$, then bidder 2’s threshold is greater than t_1 , and bidder 1 exercises the buy now option when the price reaches t_1 (and before bidder 2 exercises the option). Finally, if bidder 2’s value is above $t^{-1}(t_1)$, he chooses a threshold less than t_1 , and bidder 2 exercises the buy-now option at a bid price below t_1 .

Figure 1 goes here.

We now formally define the bidder’s payoff functions: If a bidder’s value is v , he chooses the threshold \tilde{t} , and the other bidder follows the threshold strategy t (one without a jump down) then the bidder’s expected utility is

$$U(\tilde{t}, v; t) = \begin{cases} \int_{\underline{v}}^{\tilde{t}} u(v-x)dF(x) + [F(t^{-1}(\tilde{t})) - F(\tilde{t})]u(v-B) & \text{if } \tilde{t} \in [\underline{t}, \bar{t}] \\ \int_{\underline{v}}^{\tilde{t}} u(v-x)dF(x) + [1 - F(\tilde{t})]u(v-B) & \text{if } \tilde{t} < \underline{t}. \end{cases}$$

Note that if $\tilde{t} < \underline{t}$ then the bidder wins for sure, paying the other bidder’s value when the value is less than \tilde{t} and paying B otherwise.

We say that t is a **(symmetric) equilibrium in threshold strategies** if for each $v \in [B, \bar{v}]$ we have

$$U(t(v), v; t) \geq U(\tilde{t}, v; t) \quad \forall \tilde{t} \in [\underline{v}, B].$$

In other words, for each value v a bidder's optimal threshold is $t(v)$ when all the other bidders follow strategy t .

Proposition 3: If bidders are risk neutral then there is a unique symmetric equilibrium in threshold strategies that are continuous and differentiable (except possibly at one point where the threshold strategy jumps down). If $B \geq (\bar{v} + \underline{v})/2$ then the equilibrium threshold strategy satisfies $t(v) = 2B - v$. If $B < (\bar{v} + \underline{v})/2$, then

$$t(v) = \begin{cases} 2B - v & \text{if } v \leq \frac{\underline{v}^2 + B(\bar{v} - 2\underline{v})}{\bar{v} - B} \\ \underline{v} & \text{otherwise.} \end{cases}$$

Figure 2(a) below shows the equilibrium threshold function without a jump down, and is useful for comparing the seller's expected revenue in the Yahoo! buy-now auction to an English auction. Let bidder 1's value v_1 be fixed and assume that $v_2 \leq v_1$. If $v_1 \leq B$, then seller revenue is v_2 in both the Yahoo! buy-now and the English auction. If $v_1 > B$ then revenue is still v_2 in both auctions if $v_2 \leq 2B - v_1$. Hence, if revenue is different between the two auctions, then it must be because revenues differ when $v_2 \in [2B - v_1, v_1]$, i.e., revenues must differ in the heavily shaded region. However, the seller's expected revenue in the English auction, conditional on $v_2 \in [2B - v_1, v_1]$, is²

$$\int_{2B-v_1}^{v_1} v_2 \frac{1}{v_1 - (2B - v_1)} dv_2 = B.$$

Thus, conditional on $v_2 \in [2B - v_1, v_1]$ the seller gets B for sure in the Yahoo! buy-now auction, while he gets the same amount in expectation in the English auction. We have show that for each fixed v_1 , given $v_2 \leq v_1$, the seller's expected revenue is

²Given $v_2 \in [2B - v_1, v_1]$, then v_2 is conditionally distributed $U[2B - v_1, v_1]$. Since the mean of the $U[a, b]$ density is $\frac{a+b}{2}$, the conditional expectation of v_2 is B .

the same in both auctions. Since this is true for each v_1 , the seller's unconditional expected revenue must be the same for both auctions. This establishes Proposition 4(i).³

Figure 2 goes here.

The revenue properties of the Yahoo! buy-now auction when the threshold function has a jump down can be understood by considering Figure 2(b). The key observation is that the value at which a bidder's threshold function jumps down in the Yahoo! buy-now auction is the same as the bidder's cutoff value in the eBay buy-now auction. Again, fix v_1 and assume that $v_2 \leq v_1$. If $v_1 \leq \frac{v^2 + B(\bar{v} - 2\underline{v})}{\bar{v} - B}$, then the eBay and Yahoo auctions both generate the same expected revenue to the seller; in particular, both yield the same expected revenue as in the English auction. (If $v_1 > B$ the equality of revenues follows from the argument in above.) If $v_1 > \frac{v^2 + B(\bar{v} - 2\underline{v})}{\bar{v} - B}$ then the seller's revenue is B in both auctions. Since expected revenue is the same for both auctions for each fixed v_1 , the seller's unconditional expected revenue is also the same in both auctions. This establishes Proposition 4(iii). Proposition 4(ii) follows immediately from Propositions 2 and 4(iii).

Proposition 4: Assume that the bidders are risk neutral.

- (i) If $B \geq (\bar{v} + \underline{v})/2$ then the Yahoo! buy-now auction yields the same expected revenue as the English auction.
- (ii) If $B < (\bar{v} + \underline{v})/2$ then the Yahoo! buy-now auction yields less revenue than the English auction.
- (iii) The Yahoo! buy-now auction yields the same revenue as the eBay buy-now auction.

While the Yahoo! and eBay buy-now auctions both yield the same expected revenue, the ex post outcomes may be quite different. If $B \geq (\bar{v} + \underline{v})/2$ then the buy-now price is accepted for some value combinations in the Yahoo! auction while

³Alternatively, Proposition 4(i) can be seen as an application of the well-known revenue equivalence property of efficient auctions.

it is never accepted in the eBay buy-now auction. If $B < (\bar{v} + \underline{v})/2$ then the set of value combinations for which the buy-now price is accepted immediately is the same in both auctions; in the Yahoo! auction, however, the buy-now price may also be accepted after some delay.

3 Bidder Risk Aversion

Bidder risk aversion has no effect on behavior in English ascending bid auctions. It is a dominant strategy for a risk averse bidder to remain in the auction until the bid price equals the bidder's value, just as it is for a risk neutral bidder. Therefore, expected seller revenue in the ascending bid auction is the same regardless of bidder risk preferences. But risk aversion does affect behavior in a buy-now auction. In this section we provide several results on the effects of bidder risk aversion on bidder behavior and on seller revenue. We focus on the case of constant absolute risk aversion (CARA), since preferences of this type yield tractable equilibrium conditions.⁴ We suppose that both bidders have CARA utility of the form, $u(x) = 1 - e^{-\alpha x}$, where $\alpha > 0$ is the index of bidder risk aversion.

3.1 Buy It Now (eBay)

We first examine bidder risk aversion in the eBay buy-now auction. The following proposition describes equilibrium bidder behavior.

Proposition 5: Assume the bidders are risk averse with index of risk aversion $\alpha > 0$. For each $B \in [\underline{v}, \bar{v}]$ there is a unique symmetric equilibrium in cutoff strategies. Furthermore, there is a critical value $\tilde{v}_\alpha \in (\underline{v}, \bar{v})$ such that:

(i) if $B \in (\underline{v}, \tilde{v}_\alpha)$ then the symmetric equilibrium cutoff value satisfies $c^* \in (B, \bar{v})$ and is strictly increasing in B .

⁴We explored the properties of constant relative risk averse (CRRA) preferences in this model. We found that for some buy prices, a symmetric equilibrium in cutoff strategies does not exist for the eBay buy-now auction. If an equilibrium exists, then it must be of some alternative form.

(ii) if $B \in [\tilde{v}_\alpha, \bar{v}]$ then the symmetric equilibrium cutoff value is $c^* = \bar{v}$, i.e., the buy-now price is never accepted by a bidder.

The following proposition indicates how risk aversion changes the bidders' willingness to accept the buy price. In equilibrium, risk averse bidders choose lower cutoff values than risk neutral bidders, i.e., they accept the buy price, compared to the case of risk neutral bidders.

Proposition 6: Let \tilde{v}_α be the critical value given in Proposition 5. If $B \in (\underline{v}, \tilde{v}_\alpha)$ then the symmetric equilibrium cutoff value is strictly lower when bidders are risk averse with index of risk aversion α than if bidders are risk neutral. If $B \in [\tilde{v}_\alpha, \bar{v})$ then the symmetric equilibrium cutoff is $c = \bar{v}$ whether bidders are risk averse or risk neutral.

Proposition 7: If bidders have CARA utility with index of risk aversion $\alpha > 0$ then there exists a buy price in the eBay buy-now auction such that expected seller revenue exceeds expected revenue in the English ascending bid auction.

Table Two provides numerical calculations of the optimal (expected revenue maximizing) buy price for a seller, for increasing indices of bidder risk aversion. Note that the optimal buy price rises as bidder risk aversion increases; the equilibrium cutoff value becomes closer to the buy price as bidder risk aversion increases.

Table Two
Optimal Buy Price for Seller

(CARA Bidders)*

Index of Bidder	Buy Price	Cutoff Value	Seller Revenue	% Gain in Revenue
Risk Aversion (α)				
0	—	—	133.3	—
0.05	154.8	174.8	138.1	3.5%
0.1	159.9	172.9	141.0	5.7%
0.5	167.8	171.1	145.5	9.1%
2.0	169.9	170.8	146.7	10.0%

* Two bidders with *iid* values uniformly distributed on [100,200].

3.2 Buy-Now! (Yahoo!)

INCOMPLETE

4 Appendix

Proof of Proposition 1: If bidder i accepts the buy price, then his expected payoff when his rival uses the cutoff strategy c , is

$$U^b(v_i, c) = \left[\frac{1}{2}(1 - F(c)) + F(c) \right] u(v_i - B) = \frac{(\frac{1}{2}\bar{v} + \frac{1}{2}c - \underline{v})}{\bar{v} - \underline{v}} u(v_i - B),$$

whereas if he waits then his expected payoff is

$$U^w(v_i, c) = \int_{\underline{v}}^{\min\{v_i, c\}} u(v_i - v_j) dF(v_j) = \frac{1}{\bar{v} - \underline{v}} \int_{\underline{v}}^{\min\{v_i, c\}} u(v_i - v_j) dv_j.$$

Clearly, a necessary condition for c to be an interior equilibrium cutoff value is that $U^b(c, c) = U^w(c, c)$.⁵ Since bidders are risk neutral, this equality is equivalent to,

$$\frac{1}{2}(\bar{v} + c - 2\underline{v})(c - B) = \int_{\underline{v}}^c (c - v_j) dv_j = \frac{1}{2}(c - \underline{v})^2.$$

⁵If $U^B(c, c) > U^W(c, c)$ then by the continuity of U^B and U^W we have $U^B(v', c) > U^W(v', c)$ for some $v' \in (B, c)$, which contradicts that c is a cutoff equilibrium. If $U^B(c, c) < U^W(c, c)$ then by the continuity of U^B and U^W we have $U^B(v', c) < U^W(v', c)$ for some $v' \in (c, \bar{v})$, again a contradiction.

Solving for c yields

$$c^* = \frac{B(\bar{v} - 2\underline{v}) + \underline{v}^2}{\bar{v} - B},$$

which is the value cut-off indicated in part (ii) of the proposition. The condition, $\underline{v} < B < \frac{1}{2}\underline{v} + \frac{1}{2}\bar{v}$ implies $\underline{v} < c^* < \bar{v}$, and hence $F(c^*) < 1$.

We now show for $B \in [\underline{v}, \frac{1}{2}\underline{v} + \frac{1}{2}\bar{v})$ that c^* as given above is an equilibrium cutoff value. It's easy to verify that $U^b(v_i, c^*)$ is linear in v_i , $U^w(v_i, c^*)$ is convex in v_i for $v_i < c^*$, and $U^w(v_i, c^*)$ is linear in v_i for $v_i > c^*$. Furthermore,

$$\left. \frac{\partial U^b(v_i, c^*)}{\partial v_i} \right|_{v_i=c^*} = \frac{1}{2}(1 + F(c^*)) > F(c^*) = \left. \frac{\partial U^w(v_i, c^*)}{\partial v_i} \right|_{v_i=c^*},$$

which means that the slope of the expected payoff function from accepting the buy now price is steeper at $v_i = c^*$ than the expected payoff function from waiting. These facts imply that $U^w(v_i, c^*) > U^b(v_i, c^*)$ for $v_i < c^*$ and $U^w(v_i, c^*) < U^b(v_i, c^*)$ for $v_i > c^*$, i.e., c^* is a symmetric equilibrium. This establishes Proposition 1(ii).

If $B \in [\frac{1}{2}\underline{v} + \frac{1}{2}\bar{v}, \bar{v}]$ then $c^* \geq \bar{v}$ (for c^* given above) and hence there is no interior equilibrium cutoff. Clearly \underline{v} is not an equilibrium cutoff since for $v_i < B$ we have $U^w(v_i, \underline{v}) = 0 > U^b(v_i, \underline{v}) = \frac{1}{2}(v_i - B)$ for $v_i < B$. We now show that \bar{v} is an equilibrium. We have $U^b(v_i, \bar{v}) = v_i - B$ and

$$U^w(v_i, \bar{v}) = \int_{\underline{v}}^{v_i} (v_i - v_j) dF(v_j) = \frac{(v_i - \underline{v})^2}{2(\bar{v} - \underline{v})}.$$

It is straightforward to show that $U^w(v_i, \bar{v}) > U^b(v_i, \bar{v})$ for all $v_i < \bar{v}$. This establishes (i). \square

Proof of Proposition 2: Let Δ be the difference between expected revenue in the buy-now auction and a standard English auction. Let c^* be the symmetric equilibrium cutoff value. This is in the interior of the values support, given the hypothesis regarding B in Proposition 2. If both bidders have a value less than or equal to c^* then seller revenue is the same in both types of auction; revenue is equal to the lower of the values of the two bidders. If one or both bidders has a value above c^* then the

seller receives B in the buy now auction. The difference in revenue between the two types of auction is B minus the lower of the two values of the bidders. So,

$$\Delta = 2 \int_{c^*}^{\bar{v}} \left[\int_{\underline{v}}^{v_1} (B - v_2) dF(v_2) \right] dF(v_1)$$

After integrating the right-hand side

$$\Delta = \frac{\bar{v} - c^*}{3(\bar{v} - \underline{v})^2} (3 [B(\bar{v} - 2\underline{v}) + \underline{v}^2] - \bar{v}^2 - \bar{v}c^* + 3Bc^* - c^{*2}).$$

The expression in square brackets above is equal to $(\bar{v} - B)c^*$, according to the definition of c^* in equation (1). Substituting for the expression in curly brackets and simplifying yields

$$\Delta = -\frac{(\bar{v} - c^*)^3}{3(\bar{v} - \underline{v})^2} < 0,$$

since $c^* < \bar{v}$ for $B \in (\underline{v}, \frac{1}{2}\underline{v} + \frac{1}{2}\bar{v})$. \square

Proof of Proposition 3: We begin by writing the payoff function when the threshold function has a jump down. If a bidder's value is v , he chooses the threshold \tilde{t} , and every other bidder follows the threshold strategy t (which jumps down to \underline{v} at z), then the bidder's expected utility is

$$U(\tilde{t}, v; t) = \begin{cases} \int_{\underline{v}}^{\tilde{t}} u(v - x) dF(x) + [F(B) - F(\tilde{t})]u(v - B) & \text{if } \tilde{t} > \bar{t} \\ \int_{\underline{v}}^{\tilde{t}} u(v - x) dF(x) + [F(t^{-1}(\tilde{t})) - F(\tilde{t})]u(v - B) & \text{if } \tilde{t} \in [\underline{t}, \bar{t}] \\ \int_{\underline{v}}^{\tilde{t}} u(v - x) dF(x) + [F(t^{-1}(\underline{t})) - F(\tilde{t})]u(v - B) & \text{if } \tilde{t} \in (\underline{v}, \underline{t}) \\ u(v - B)(F(z) + \frac{1}{2}(1 - F(z))) & \text{if } \tilde{t} = \underline{v}. \end{cases}$$

The proof proceeds in several steps.

STEP I: We first show that a (symmetric) equilibrium threshold function satisfies $t(B) = B$. Suppose to the contrary that a bidder with value $v = B$ chooses a threshold $t(B) < B$. Then

$$U(t(B), B; t) = \int_{\underline{v}}^{t(B)} u(B - x) dF(x) < \int_{\underline{v}}^B u(B - x) dF(x) = U(B, B; t),$$

where the inequality is strict since $u(B-x) > 0$ for $x \in [t(B), B)$ and since $dF(x) > 0$. In other words, a bidder with value $v = B$ obtains a higher payoff choosing a threshold of B than of $t(B)$ when the other bidder follows t . This contradicts that t is a symmetric equilibrium.

STEP II: Let t be an equilibrium threshold strategy which is differentiable (except at one point \hat{v} , if t jumps down at \hat{v}). We show that $t(v) = 2B - v$ for $v \in [B, \bar{v}]$ if t has no jump and $t(v) = 2B - v$ for $v \in [B, \hat{v})$ if t jumps down at \hat{v} .

Consider the case where t jumps down at \hat{v} . For $\tilde{v} \leq \hat{v}$ define

$$U(\tilde{v}, v) = \int_{\underline{v}}^{t(\tilde{v})} (v-x) \frac{1}{\bar{v}-\underline{v}} dx + (v-B) \frac{\tilde{v}-t(\tilde{v})}{\bar{v}-\underline{v}}$$

to be a bidder's expected payoff when his value is v , he chooses a threshold as though his value was \tilde{v} , and the other bidder follows t . For $v \in (B, \hat{v})$ we have $\bar{v} < t(v) < B$; hence a necessary condition for t to be an equilibrium is that $\frac{dU(\tilde{v}, v)}{d\tilde{v}}|_{\tilde{v}=v} = 0$ for $v \in (B, \hat{v})$, i.e.,

$$Bt' - tt' + v - B = 0. \tag{2}$$

Integrating (2) yields

$$-t^2 + 2Bt + v^2 - 2Bv = C,$$

where C is a constant. The boundary condition that $t(B) = B$ implies that $C = 0$, and thus

$$(v-t)(t-(2B-v)) = 0.$$

There are two solutions for the threshold strategy: (i) $t(v) = 2B - v$ and (ii) $t(v) = v$. The second solution, however, can be discarded since a threshold strategy must satisfy $t(v) \leq B$ but we have $v > B$. The proof for the case where t has no jump is straightforward and therefore omitted.

STEP III: Let t be an equilibrium threshold strategy which is differentiable (except at one point \hat{v} , if t jumps down at \hat{v}). We show that if $B \geq (\bar{v} + \underline{v})/2$ then $t(v) = 2B - v$

for $v \in [B, \bar{v}]$ and if $B < (\bar{v} + \underline{v})/2$ then t jumps down at $\hat{v} = \frac{v^2 + B(\bar{v} - 2\underline{v})}{\bar{v} - B}$, i.e.,

$$t(v) = \begin{cases} 2B - v & \text{if } v \leq \frac{v^2 + B(\bar{v} - 2\underline{v})}{\bar{v} - B} \\ \underline{v} & \text{otherwise.} \end{cases}$$

Suppose that t jumps down at z . Define the function

$$L(v) = \int_{\underline{v}}^{2B-v} (v-x) \frac{1}{\bar{v}-\underline{v}} dx + (v-B) \int_{2B-v}^{\min\{v,z\}} \frac{1}{\bar{v}-\underline{v}} dx$$

to be the payoff to a bidder when his value is v , his threshold is $2B - v$, and his rival follows t . Simplifying, we obtain

$$L(v) = \frac{1}{\bar{v}-\underline{v}} \left[(v-B)(\min\{v,z\} - v) + \frac{1}{2}(v-\underline{v})^2 \right].$$

Define

$$R(v) = \frac{1}{2} \frac{\bar{v}-z}{\bar{v}-\underline{v}} (v-B) + \frac{z-\underline{v}}{\bar{v}-\underline{v}} (v-B)$$

to be the payoff to a bidder when his value is v , his threshold is \underline{v} , and his rival follows t .

A necessary condition for t to jump down at z is that when a bidder's value is z then he is indifferent to a threshold of $2B - z$ and a threshold of \underline{v} , i.e., that $L(v) = R(v)$ for $v = z$. If $L(z) > R(z)$, for example, then, since L and R are both continuous in v , there is an $\varepsilon > 0$ such that $L(z + \varepsilon) > R(z + \varepsilon)$, and hence a bidder whose value is $z + \varepsilon$ obtains a higher payoff by deviating from t and setting a threshold of $2B - (z + \varepsilon)$, rather than following t and choosing a threshold of \underline{v} . $L(z) = R(z)$ implies $z = \frac{v^2 + B(\bar{v} - 2\underline{v})}{\bar{v} - B}$. It's easy to see that $z > B$ for $B > \underline{v}$. If $B > (\bar{v} + \underline{v})/2$ then $z > \bar{v}$, which contradicts that t jumps down for some $z \in (B, \bar{v})$. If $B < (\bar{v} + \underline{v})/2$ then $z \in (B, \bar{v})$.

STEP IV: We have shown that if t is an equilibrium in threshold strategies which are continuous and differentiable (except at possibly one point), then it is as given in Proposition 3. We show that the threshold strategy given in Proposition 3 is indeed an equilibrium, and therefore an equilibrium exists.

Case (i): Suppose $B < (\bar{v} + \underline{v})/2$, and hence t jumps down at \hat{v} , and suppose the bidder's value is $v \in [B, \hat{v}]$. The bidder's payoff, if he chooses a threshold as though his true value were $\tilde{v} \in [B, \hat{v}]$, is

$$U(\tilde{v}, v) = \int_{\underline{v}}^{2B-\tilde{v}} (v-x) \frac{1}{\bar{v}-\underline{v}} dx + (v-B) \int_{2B-\tilde{v}}^{\tilde{v}} \frac{1}{\bar{v}-\underline{v}} dx.$$

Note that

$$\frac{dU(\tilde{v}, v)}{d\tilde{v}} = \frac{1}{\bar{v}-\underline{v}}(v-\tilde{v}),$$

and

$$\frac{d^2U(\tilde{v}, v)}{d\tilde{v}^2} = \frac{-1}{\bar{v}-\underline{v}} < 0.$$

Hence $U(\tilde{v}, v)$ is concave in \tilde{v} (for fixed v), and $\tilde{v} = v$ maximizes $U(\tilde{v}, v)$ on the interior of $[B, \hat{v}]$. Equivalently, a threshold of $t = 2B - v$ maximizes the bidder's payoff for thresholds in $[2B - \hat{v}, B]$. We now show that $t = 2B - v$ maximize the bidder's payoff for all thresholds in $[0, B]$.

Clearly a threshold of $t \in (0, 2B - \hat{v})$ is never optimal. The bidder's expected payoff with a threshold of $\tilde{t} \in (0, 2B - \hat{v})$ is

$$\int_{\underline{v}}^{\tilde{t}} (v-x) \frac{1}{\bar{v}-\underline{v}} dx + (v-B) \int_{\tilde{t}}^{\hat{v}} \frac{1}{\bar{v}-\underline{v}} dx,$$

which is strictly less than

$$\int_{\underline{v}}^{2B-\hat{v}} (v-x) \frac{1}{\bar{v}-\underline{v}} dx + (v-B) \int_{2B-\hat{v}}^{\hat{v}} \frac{1}{\bar{v}-\underline{v}} dx.$$

In particular, if the bidder raises his threshold to from \tilde{t} to $2B - \hat{v}$ this leaves the probability of winning unchanged, but reduces the expected price.

Finally, we show that when the bidder's value is $v \leq \hat{v}$, then a threshold of $t = \underline{v}$ is not optimal. Choosing a threshold of $t = 2B - v$ the bidder obtains

$$L(v) = \frac{1}{\bar{v}-\underline{v}} \left[(v-B)(\hat{v}-v) + \frac{1}{2}(v-\underline{v})^2 \right],$$

whereas choosing a threshold of $t = \underline{v}$ he obtains

$$R(v) = \frac{1}{2} \frac{\bar{v}-\hat{v}}{\bar{v}-\underline{v}} (v-B) + \frac{\hat{v}-\underline{v}}{\bar{v}-\underline{v}} (v-B).$$

Clearly, $L(v)$ is concave in v and $R(v)$ is linear in v . Furthermore, $L(B) = \frac{1}{2}(B - \underline{v})^2 > R(B) = 0$ and $L(\hat{v}) = R(\hat{v})$, and therefore $L(v) > R(v)$ for $v < \hat{v}$ and $L(v) < R(v)$ for $v > \hat{v}$. Hence, the threshold $t = 2B - v$ is optimal for a bidder with a value $v \in [B, \hat{v}]$, and a threshold of $t = \underline{v}$ is optimal for a bidder with a value $v \in [\hat{v}, \bar{v}]$. \square

Proof of Proposition 5: If bidder i accepts the buy price then his expected utility, when his rival uses the cutoff strategy c , is

$$U^b(v_i, c) = \left[\frac{1}{2}(1 - F(c)) + F(c) \right] u(v_i - B) = \frac{\bar{v} + c - 2\underline{v}}{2(\bar{v} - \underline{v})} [1 - e^{-\alpha(v_i - B)}],$$

whereas if he waits then his expected utility is

$$\begin{aligned} U^w(v_i, c) &= \int_{\underline{v}}^{\min\{v_i, c\}} u(v_i - v_j) dF(v_j) \\ &= \frac{\min\{v_i, c\} - \underline{v} - \frac{e^{-\alpha v_i}}{\alpha} (e^{\alpha \min\{v_i, c\}} - e^{\alpha \underline{v}})}{\bar{v} - \underline{v}}. \end{aligned}$$

Note that $u(v_i - v_j) \geq 0$ for $v_j \in [\underline{v}, \min\{v_i, c\}]$ and hence $U^w(v_i, c) \geq 0$.

We first show that $c = \underline{v}$ is a symmetric (boundary) equilibrium if and only if $B = \underline{v}$. In particular, $c = \underline{v}$ is a symmetric boundary equilibrium if $U^b(v_i, \underline{v}) > U^w(v_i, \underline{v}) \forall v_i \in (\underline{v}, \bar{v}]$. But $U^b(v_i, \underline{v}) > 0 \forall v_i \in (\underline{v}, \bar{v}]$ implies $B = \underline{v}$.

A necessary condition for c to be a symmetric interior equilibrium cutoff value is that $U^b(c, c) = U^w(c, c)$. This equality is equivalent to

$$\frac{1}{2}(\bar{v} + c - 2\underline{v})[1 - e^{-\alpha(c - B)}] = c - \underline{v} - \frac{1}{\alpha} + \frac{1}{\alpha}e^{-\alpha(c - \underline{v})},$$

or

$$e^{\alpha B} = \frac{(\alpha \bar{v} - \alpha c + 2)e^{\alpha c} - 2e^{\alpha \underline{v}}}{\alpha(\bar{v} + c - 2\underline{v})}.$$

It's convenient to solve for the buy-now price B as a function of the cutoff value c .

We obtain

$$B = \frac{1}{\alpha} \ln \left[\frac{(\alpha \bar{v} - \alpha c + 2)e^{\alpha c} - 2e^{\alpha \underline{v}}}{\alpha(\bar{v} + c - 2\underline{v})} \right].$$

Let $\lambda(c)$ be the function inside the square brackets and define $\mu(c) = \frac{1}{\alpha} \ln \lambda(c)$. Then $B = \mu(c)$.

Proof of (i): Let $\tilde{\nu}_\alpha = \mu(\bar{\nu})$ be the critical value given in the Proposition. We begin by establishing some properties of $\mu(c)$. It's easy to verify that $\mu(\underline{\nu}) = \underline{\nu}$. To see that μ is strictly increasing in c observe that

$$\mu'(c) = \frac{1}{\alpha} \frac{\lambda'(c)}{\lambda(c)},$$

where

$$\lambda'(c) = \frac{e^{\alpha c} \left\{ \alpha(\bar{\nu}^2 - c^2) + 2c - 2\underline{\nu}(1 + \alpha\bar{\nu} - \alpha c) - \frac{2}{\alpha} \right\} + \frac{2}{\alpha} e^{\alpha \underline{\nu}}}{(\bar{\nu} + c - 2\underline{\nu})^2}.$$

Clearly $\lambda(c) > 0$ for $c \in [\underline{\nu}, \bar{\nu}]$. To see that $\lambda'(c) > 0$ also, define $g(c)$ as

$$g(c) = e^{\alpha c} \left\{ \alpha(\bar{\nu}^2 - c^2) + 2c - 2\underline{\nu}(1 + \alpha\bar{\nu} - \alpha c) - \frac{2}{\alpha} \right\},$$

so that

$$\lambda'(c) = \frac{g(c) + \frac{2}{\alpha} e^{\alpha \underline{\nu}}}{(\bar{\nu} + c - 2\underline{\nu})^2}.$$

Since $g'(c) = e^{\alpha c} \alpha^2 (\bar{\nu} - c) (\bar{\nu} + c - 2\underline{\nu}) \geq 0$, then

$$g(c) \geq g(\underline{\nu}) = e^{\alpha \underline{\nu}} \left\{ \alpha(\bar{\nu}^2 - \underline{\nu}^2) - 2\underline{\nu}\alpha(\bar{\nu} - \underline{\nu}) - \frac{2}{\alpha} \right\},$$

and hence

$$\lambda'(c) \geq \frac{e^{\alpha \underline{\nu}} \left\{ \alpha(\bar{\nu}^2 - \underline{\nu}^2) - 2\underline{\nu}\alpha(\bar{\nu} - \underline{\nu}) - \frac{2}{\alpha} \right\} + \frac{2}{\alpha} e^{\alpha \underline{\nu}}}{(\bar{\nu} + c - 2\underline{\nu})^2} = \frac{e^{\alpha \underline{\nu}} \alpha (\bar{\nu} - \underline{\nu})^2}{(\bar{\nu} + c - 2\underline{\nu})^2} > 0,$$

for $c \in [\underline{\nu}, \bar{\nu}]$. Thus μ is strictly increasing in c .

Finally, we show that $\mu(c) < c$ for $c > \underline{\nu}$, i.e.,

$$\mu(c) = \frac{1}{\alpha} \ln \left[\frac{(\alpha\bar{\nu} - \alpha c + 2)e^{\alpha c} - 2e^{\alpha \underline{\nu}}}{\alpha(\bar{\nu} + c - 2\underline{\nu})} \right] < c.$$

Equivalently, we need to establish that

$$\frac{(\alpha\bar{\nu} - \alpha c + 2)e^{\alpha c} - 2e^{\alpha \underline{\nu}}}{\alpha(\bar{\nu} + c - 2\underline{\nu})} < e^{\alpha c}.$$

Rearranging this expression yields

$$-\alpha(c - \bar{v}) + 1 < e^{-\alpha(c - \bar{v})}.$$

However, this inequality holds for $c > \underline{v}$ since $e^x > x + 1$ for $x \neq 0$.⁶ In particular, choosing $x = -\alpha(c - \underline{v})$ yields the result.

In sum, we have shown that $\mu(\underline{v}) = \underline{v}$, $\mu(c)$ is strictly increasing in c on $[\underline{v}, \bar{v}]$, and $\mu(c) < c \forall c > \underline{v}$. In particular, $\tilde{v}_\alpha = \mu(\bar{v}) < \bar{v}$. These facts imply $c = \mu^{-1}(B)$ is well-defined and strictly increasing in B on $[\underline{v}, \tilde{v}_\alpha]$. This establishes that for each $B \in [\underline{v}, \tilde{v}_\alpha]$ there is a unique $c \in (\underline{v}, \bar{v})$ satisfying the necessary condition $U^b(c, c) = U^w(c, c)$.

Let $B \in (\underline{v}, \tilde{v}_\alpha)$ and let $c = \mu^{-1}(B)$. Then $c \in (\underline{v}, \bar{v})$. We show that c is a symmetric equilibrium in cutoff strategies, i.e., if $v_i \in [\underline{v}, c)$ then $U^w(v_i, c) > U^b(v_i, c)$ and if $v_i \in (c, \bar{v}]$ then $U^b(v_i, c) > U^w(v_i, c)$. We establish this by showing that $U^b(v_i, c)$ is everywhere steeper than $U^w(v_i, c)$. Hence they cross at $v_i = c$, with U^b below U^w for $v_i \in [\underline{v}, c)$ and U^b above U^w for $v_i \in (c, \bar{v}]$. We have

$$\frac{\partial U^b(v_i, c)}{\partial v_i} = \frac{\bar{v} + c - 2\underline{v}}{2(\bar{v} - \underline{v})} \alpha e^{-\alpha(v_i - B)},$$

and

$$\frac{\partial U^w(v_i, c)}{\partial v_i} = \begin{cases} \frac{1}{\bar{v} - \underline{v}} [1 - e^{-\alpha(v_i - \underline{v})}] & \text{if } v_i \leq c \\ \frac{1}{\bar{v} - \underline{v}} [e^{-\alpha(v_i - c)} - e^{-\alpha(v_i - \underline{v})}] & \text{if } v_i > c. \end{cases}$$

Note that $U^w(v_i, c)$ is differentiable at $v_i = c$. Note also that for $v_i \in [c, \bar{v})$ that

$$e^{\alpha B} = \lambda(c) > \frac{2e^{\alpha c} - 2e^{\alpha \underline{v}}}{\alpha(\bar{v} + c - 2\underline{v})}.$$

Rearranging yields

$$\frac{\alpha(\bar{v} + c - 2\underline{v})}{2(\bar{v} - \underline{v})} e^{\alpha B} > \frac{e^{\alpha c} - e^{\alpha \underline{v}}}{\bar{v} - \underline{v}},$$

⁶Let $f(x) = e^x - (x + 1)$. Since $f'(x) = e^x - 1$ and $f''(x) = e^x > 0$, this means that f has a global minimum at $x = 0$ where $f(0) = 0$.

which implies for $v_i \in [c, \bar{v})$ that

$$\frac{\partial U^b(v_i, c)}{\partial v_i} = \frac{\bar{v} + c - 2\underline{v}}{2(\bar{v} - \underline{v})} \alpha e^{-\alpha(v_i - B)} > \frac{1}{\bar{v} - \underline{v}} [e^{-\alpha(v_i - c)} - e^{-\alpha(v_i - \underline{v})}] = \frac{\partial U^w(v_i, c)}{\partial v_i}.$$

We have, in particular for $v_i = c$ that

$$\left. \frac{\partial U^b(v_i, c)}{\partial v_i} \right|_{v_i=c} > \left. \frac{\partial U^w(v_i, c)}{\partial v_i} \right|_{v_i=c}.$$

For $v_i < c$, then $U^b(v_i, c)$ is strictly concave in v_i and $U^w(v_i, c)$ is strictly convex in v_i . Hence this same inequality holds for $v_i < c$. This establishes that $c = \mu(B)$ is the only interior symmetric equilibrium.

We now show that if $B \in (\underline{v}, \tilde{v}_\alpha)$ then there is no symmetric boundary equilibrium. Since $B > \underline{v}$ then $c = \underline{v}$ is not a symmetric equilibrium (see above). To see that $c = \underline{v}$ is not a symmetric equilibrium, note that by construction $U^b(\bar{v}, \bar{v}) = U^w(\bar{v}, \bar{v})$ when $B = \tilde{v}_\alpha$. Since $\frac{dU^b(\bar{v}, \bar{v})}{dB} < 0$ and $\frac{dU^w(\bar{v}, \bar{v})}{dB} = 0$, then for $B < \tilde{v}_\alpha$ we have $U^b(\bar{v}, \bar{v}) > U^w(\bar{v}, \bar{v})$. The continuity of U^b and U^w implies there is an $\varepsilon > 0$ such that $U^b(v_i, \bar{v}) > U^w(v_i, \bar{v})$ for $\forall v_i \in (\bar{v} - \varepsilon, \bar{v}]$. Thus, \bar{v} is not a symmetric equilibrium.

Proof of (ii): Suppose that $B \geq \tilde{v}_\alpha$. Then

$$e^{\alpha B} \geq \lambda(\bar{v}) = \frac{e^{\alpha \bar{v}} - e^{\alpha \underline{v}}}{\alpha(\bar{v} - \underline{v})}. \quad (4)$$

Let $x = \alpha(\bar{v} - v_i) \geq 0$; this inequality is strict if $v_i < \bar{v}$. Using the definition of x ,

$$e^{\alpha \bar{v}} = e^{\alpha v_i} e^x \geq e^{\alpha v_i} (x + 1) = e^{\alpha v_i} (\alpha(\bar{v} - v_i) + 1) \quad (5)$$

where the inequality follows from $e^x \geq x + 1$, for $x \geq 0$. Combining inequalities (4) and (5) yields

$$e^{\alpha B} \geq \frac{e^{\alpha v_i} (\alpha(\bar{v} - v_i) + 1) - e^{\alpha \underline{v}}}{\alpha(\bar{v} - \underline{v})},$$

or

$$e^{-\alpha(v_i - B)} \geq \frac{\alpha(\bar{v} - v_i) + 1 - e^{-\alpha(v_i - \underline{v})}}{\alpha(\bar{v} - \underline{v})}.$$

This implies

$$U^b(v_i, \bar{v}) = 1 - e^{-\alpha(v_i - B)} \leq \frac{1}{(\bar{v} - \underline{v})} \left[v_i - \underline{v} - \frac{1}{\alpha} + \frac{1}{\alpha} e^{-\alpha(v_i - \underline{v})} \right] = U^w(v_i, \bar{v}).$$

The inequality is strict for every $v_i \in [\underline{v}, \bar{v}]$ if $B > \tilde{v}_\alpha$; the inequality is strict for $v_i < \bar{v}$ if $B = \tilde{v}_\alpha$. \square

Proof of Proposition 6: By Proposition 1, when bidders are risk neutral and $B \in (\underline{v}, \frac{1}{2}\underline{v} + \frac{1}{2}\bar{v})$, then the unique symmetric equilibrium cutoff value is

$$c = \frac{B(\bar{v} - 2\underline{v}) + \underline{v}^2}{\bar{v} - B}.$$

It will be convenient to express this in inverse form:

$$B = \frac{c\bar{v} - \underline{v}^2}{\bar{v} + c - 2\underline{v}}$$

or equivalently,

$$e^{\alpha B} = n(c) = e^{\frac{\alpha(c\bar{v} - \underline{v}^2)}{\bar{v} + c - 2\underline{v}}}.$$

>From the proof of Proposition 5, when bidders are risk averse and $B \in (\underline{v}, \tilde{v}_{\alpha\alpha})$, the symmetric equilibrium cutoff value is

$$e^{\alpha B} = \lambda(c) = \frac{(\alpha\bar{v} - \alpha c + 2)e^{\alpha c} - 2e^{\alpha\underline{v}}}{\alpha(\bar{v} + c - 2\underline{v})}.$$

To establish the proposition it is sufficient to show that $\lambda(c) > n(c)$ for $c \in (\underline{v}, \bar{v}]$.

Let $\hat{\lambda}(c) = \alpha(\bar{v} + c - 2\underline{v})\lambda(c)$ and $\hat{n}(c) = \alpha(\bar{v} + c - 2\underline{v})n(c)$. The inequality $\lambda(c) > n(c)$ is equivalent to $\hat{\lambda}(c) > \hat{n}(c)$. Note that $\hat{\lambda}(\underline{v}) = \hat{n}(\underline{v}) = \alpha(\bar{v} - \underline{v})e^{\alpha\underline{v}}$. The derivatives of $\hat{\lambda}(c)$ and $\hat{n}(c)$ are,

$$\hat{\lambda}'(c) = e^{\alpha c}[\alpha^2(\bar{v} - c) + \alpha]$$

and

$$\hat{n}'(c) = e^{\frac{\alpha(c\bar{v} - \underline{v}^2)}{\bar{v} + c - 2\underline{v}}} \left[\alpha + \frac{\alpha^2(\bar{v} - \underline{v})^2}{\bar{v} + c - 2\underline{v}} \right].$$

We will show that $\hat{\lambda}'(c) > \hat{n}'(c)$ by establishing several claims that lead to this result.

Let $z = \bar{v} - \underline{v} > 0$ and let $y = c - \underline{v} \in [0, z]$.

Claim #1:

$$\frac{\alpha y^2}{y+z} + 1 = \frac{y+z+\alpha y^2}{y+z} \geq \frac{y+z+\alpha z^2}{(y+z)(1+\alpha(z-y))}$$

with strict inequality for $y < z$. This claim follows from algebraic manipulation.

Claim #2:

$$e^{\frac{\alpha y^2}{y+z}} \geq \frac{\alpha y^2}{y+z} + 1$$

with strict inequality for $y > 0$. This claim follows from the properties of the exponential function.

Claim #3:

$$e^{\frac{\alpha y^2}{y+z}} > \frac{y+z+\alpha z^2}{(y+z)(1+\alpha(z-y))}$$

for $y = c - \underline{v} \in [0, z]$. This claim follows from claims 1 and 2.

Using the definitions of y and z , Claim #3 may be expressed as

$$e^{\frac{\alpha(c-\underline{v})^2}{\bar{v}+c-2\underline{v}}} > \frac{\bar{v}+c-2\underline{v}+\alpha(\bar{v}-\underline{v})^2}{(\bar{v}+c-2\underline{v})(1+\alpha(\bar{v}-c))}.$$

This inequality implies that

$$\hat{\lambda}'(c) = e^{\alpha c} [\alpha^2(\bar{v}-c) + \alpha] > e^{\frac{\alpha(c\bar{v}-\underline{v}^2)}{\bar{v}+c-2\underline{v}}} \left[\alpha + \frac{\alpha^2(\bar{v}-\underline{v})^2}{\bar{v}+c-2\underline{v}} \right] = \hat{n}'(c),$$

which is the desired result. \square

Proof of Proposition 7: It will be convenient to express the buy price as a function of the equilibrium cutoff value, as in the proof of Proposition 5. Let $B = \mu(c)$, where the function μ is defined in (3). The function μ is differentiable and increasing in c . If $c \in (\underline{v}, \bar{v})$ then $B = \mu(c)$ is such that some bidder types will accept the buy price in equilibrium.

Seller expected revenue in the buy-now auction with risk averse bidders may be expressed as a function of the equilibrium cutoff value, c :

$$R(c) = 2 \left[\int_{\underline{v}}^c \left(\int_{\underline{v}}^{v_1} v_2 dF(v_2) dF(v_1) \right) + \int_c^{\bar{v}} \left(\int_{\underline{v}}^{v_1} \mu(c) dF(v_2) dF(v_1) \right) \right]$$

This simplifies to,

$$R(c) = \frac{2}{(\bar{v} - \underline{v})^2} \left[\int_{\underline{v}}^c \left(\frac{1}{2}v_1^2 - \frac{1}{2}\underline{v}^2 \right) dv_1 + \mu(c) \int_c^{\bar{v}} (v_1 - \underline{v}) dv_1 \right]$$

or,

$$R(c) = \frac{2}{(\bar{v} - \underline{v})^2} \left[\frac{1}{6}c^3 - \frac{1}{2}\underline{v}^2c + \frac{1}{3}\underline{v}^3 + \mu(c) \left(\frac{1}{2}\bar{v}^2 - \underline{v}(\bar{v} - c) - \frac{1}{2}c^2 \right) \right].$$

If $c = \bar{v}$ then no bidder types accept the buy price, $B = \mu(\bar{v})$. Seller expected revenue is equal to

$$R(\bar{v}) = \frac{2}{(\bar{v} - \underline{v})^2} \left[\frac{1}{6}\bar{v}^3 - \frac{1}{2}\underline{v}^2\bar{v} + \frac{1}{3}\underline{v}^3 \right] = \underline{v} + \frac{1}{3}(\bar{v} - \underline{v}),$$

which is the expected seller revenue for the English ascending bid auction. The revenue function $R(c)$ defined above is differentiable in c , since $\mu(c)$ is differentiable. In particular,

$$R'(\bar{v}) = \frac{2}{\bar{v} - \underline{v}} \left[\frac{1}{2}(\bar{v} + \underline{v}) - \mu(\bar{v}) \right] < 0.$$

The inequality follows since, in the proof of Proposition 6, it is established that

$$\frac{(\alpha\bar{v} - \alpha c + 2)e^{\alpha c} - 2e^{\alpha\bar{v}}}{\alpha(\bar{v} + c - 2\underline{v})} > e^{\frac{\alpha(c\bar{v} - \underline{v}^2)}{\bar{v} + c - 2\underline{v}}}$$

for $c \in (\underline{v}, \bar{v}]$, and hence

$$\mu(c) > \frac{c\bar{v} - \underline{v}^2}{\bar{v} + c - 2\underline{v}}$$

for $c \in (\underline{v}, \bar{v}]$. For $c = \bar{v}$ we obtain $\mu(c) > \frac{1}{2}(\bar{v} + \underline{v})$.

The result that $R'(\bar{v}) < 0$ implies that there exists a cutoff value c' less than \bar{v} and corresponding buy price, $B' = \mu(c')$, such that the seller's expected revenue in the buy-now auction exceeds $R(\bar{v})$. \square

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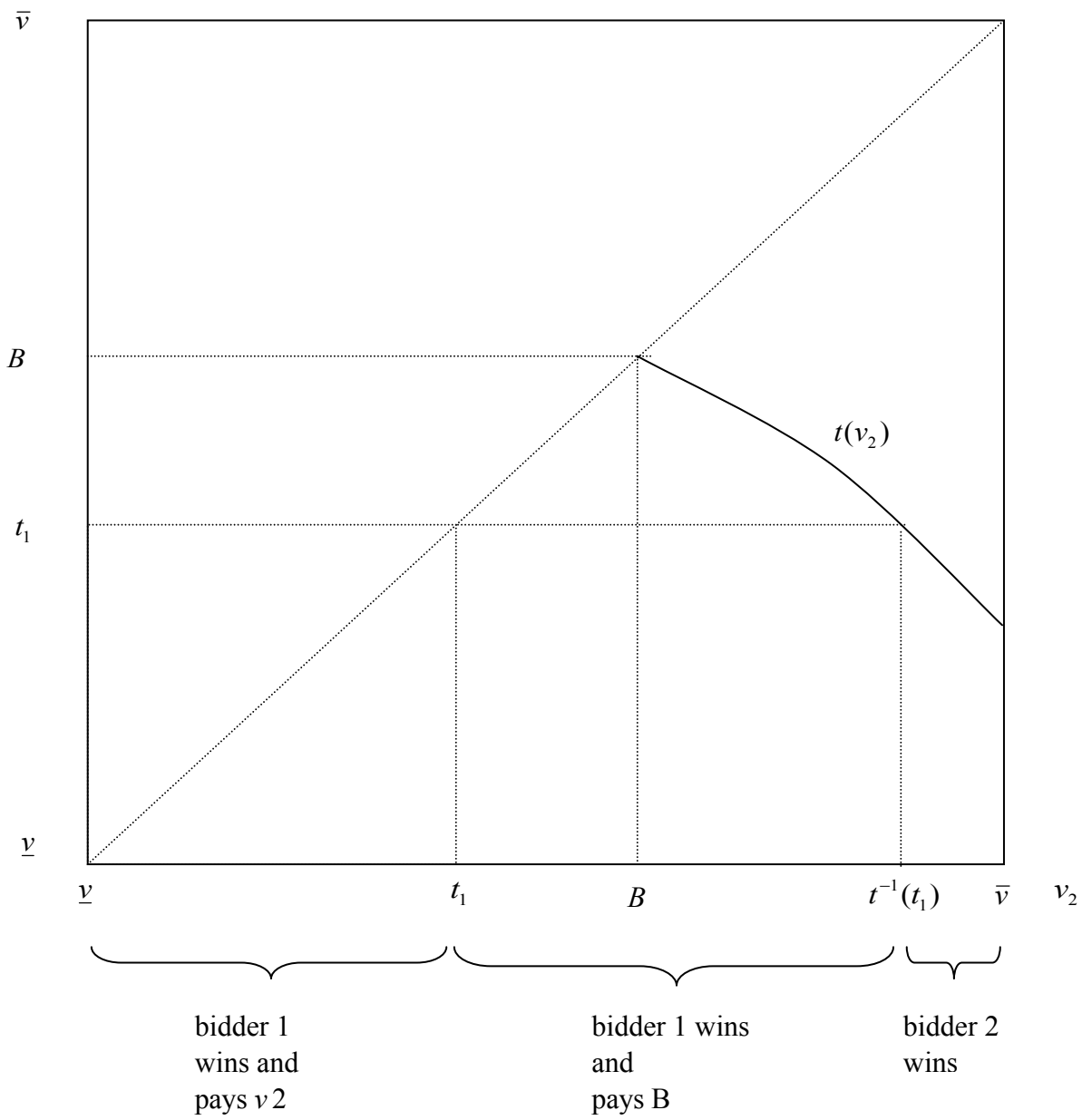


Figure 1

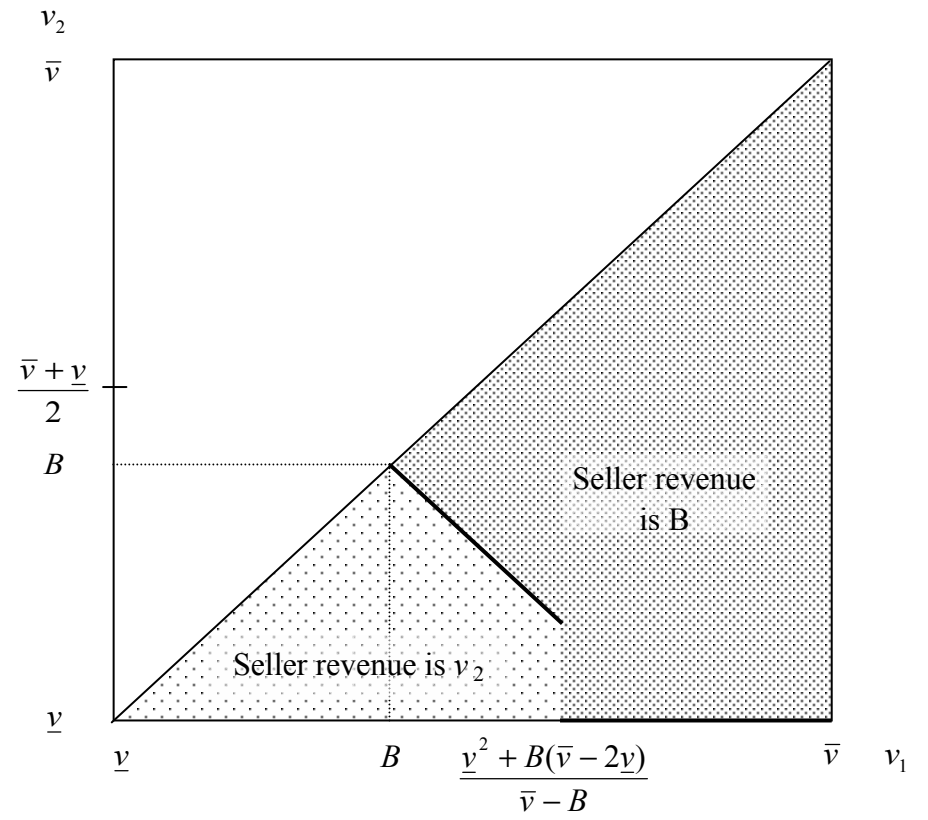
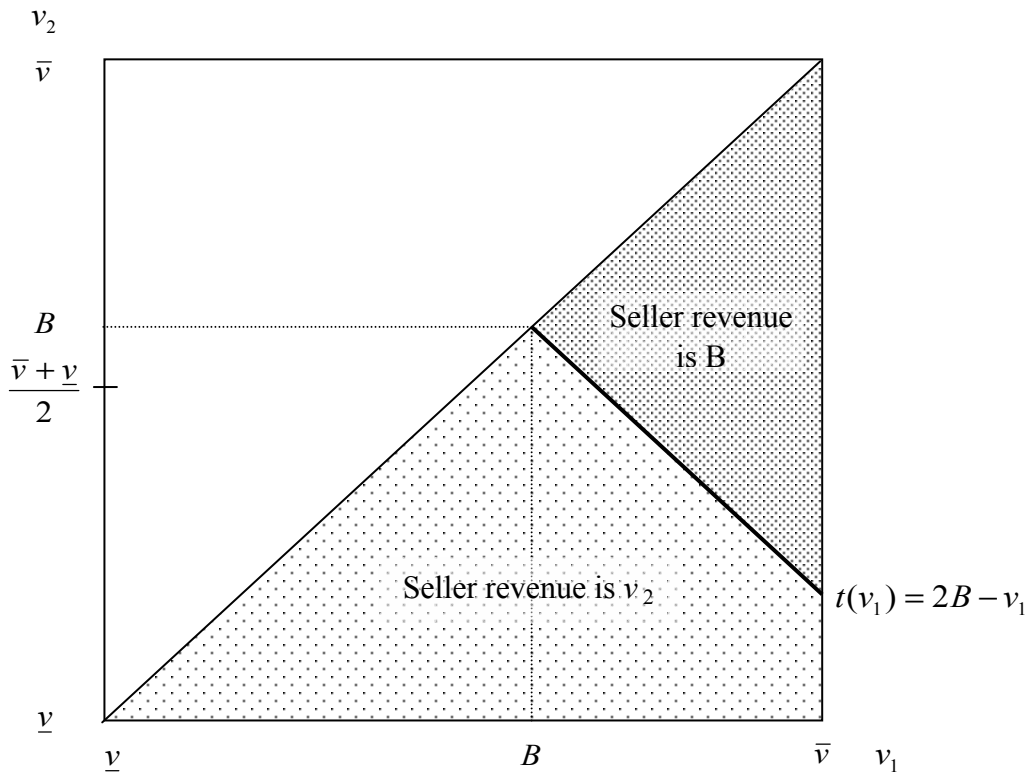


Figure 2: Seller Revenue in the Yahoo! Buy-Now Auction