

On the Irrelevance of Risk Attitudes in Repeated Two-Outcome Games¹

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We study equilibrium and maximin play in supergames consisting of the sequential play of a finite collection of stage games, where each stage game has two outcomes for each player. We show that for two-player supergames in which each stage game is strictly competitive, in any Nash equilibrium of the supergame, play at each stage is a Nash equilibrium of the stage game provided preferences over *certain* supergame outcomes satisfy a natural monotonicity condition. In particular, equilibrium play does not depend on risk attitudes. We establish an invariance result for games with more than two players when the solution concept is subgame perfection. *Journal of Economic Literature* Classification Numbers: C72, C9. © 2001 Academic Press

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1. INTRODUCTION

It is well known that the Nash equilibria of games in strategic form generally depend upon players' risk attitudes, and that the Nash equilibria of repeated games depend upon the players' time preferences as well. A well-known exception to the former is the class of games in which each player's

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only possible outcomes are “win” and “lose.” In this paper we extend this class substantially, to all finite repetitions of “win–loss” games, and establish that the Nash equilibria of such games are invariant to the players’ time preferences as well.

For experimental tests of solution concepts such as minimax, Nash equilibrium, and subgame perfection, it is also desirable to identify games whose solution does not depend upon detailed knowledge of the players’ intertemporal preferences or their preferences over lotteries. In an ingenious design, O’Neill (1987) conducted experiments using a simple card game in which two players each choose one of the four cards Ace, Two, Three, and Joker. The row player wins if both players choose Joker or if the players choose number cards that differ. Otherwise the column player wins. (In O’Neill’s experiment the outcome \mathcal{W} , i.e., “winning,” means receiving a transfer of 5 cents from the other player.) The strategic form of this game is presented below.

	A	2	3	J
A	\mathcal{L}	\mathcal{W}	\mathcal{W}	\mathcal{L}
2	\mathcal{W}	\mathcal{L}	\mathcal{W}	\mathcal{L}
3	\mathcal{W}	\mathcal{W}	\mathcal{L}	\mathcal{L}
J	\mathcal{L}	\mathcal{L}	\mathcal{L}	\mathcal{W}

Since this game has only two outcomes for each player, the game’s unique Nash (and minimax) solution has each player choose each number card with probability 0.2 and choose Joker with probability 0.4, provided that each player’s preference over lotteries is increasing in his own probability of winning.⁴

In O’Neill’s experiment, fixed pairs of subjects played the game 105 times. Clearly, the supergame consisting of the repeated play of this game has more than two outcomes. In the twice-repeated game, for example, a player can win or lose at each stage, and therefore the supergame has the four possible outcomes \mathcal{WW} , \mathcal{WL} , \mathcal{LW} , and \mathcal{LL} .

In the present paper we analyze finite supergames which consist of the sequential play of stage games (not necessarily the same game at each stage), with each stage game having only “win” and “loss” as possible outcomes for each player. It is commonly believed that since such games, like the repeated O’Neill game, have more than two outcomes, the issue of risk

⁴Another means for controlling for risk attitudes is the lottery risk neutralization procedure introduced in Roth and Malouf (1979) and extended by Berg *et al.* (1986).

attitudes is brought squarely back into the analysis of equilibrium.⁵ Our results show that this belief is not correct.

Our first result is for two-player supergames. Theorem 1 establishes that if each stage game in a sequence of stage games is strictly competitive (i.e., a loss for one player is a win for the other player), then in every Nash equilibrium of the supergame, play at each stage will be a Nash equilibrium of the stage game, provided that each player's von Neumann-Morgenstern expected utility function over supergame outcomes is "monotone." An expected utility function u_i on supergame outcomes (i.e., on sequences of stage game outcomes) is monotone if whenever two outcomes differ only at a single stage then u_i assigns a higher utility to the outcome in which player i wins at that stage. In the twice-repeated O'Neill game, for example, u_i is monotone if the utility of WW is greater than the utility of either WL or LW , and if each of these utilities is greater than the utility of LL .

Monotonicity of preferences is a natural condition on the players' ordering of *certain* supergame outcomes. Our first result requires no further knowledge of the players' preferences over lotteries on supergame outcomes, such as their intertemporal preferences or their risk attitudes, requiring only that their preferences be representable by von Neumann-Morgenstern expected utility functions. Applied to the repeated O'Neill game, for example, our result establishes that in the supergame's Nash equilibrium, play at each stage is the Nash equilibrium of the one-shot game, regardless of players' risk attitudes or time preferences.

Our next results concern the subgame perfect equilibria of supergames with any finite number of players, consisting of the sequential play of win-loss stage games. Theorem 2 establishes that if each player's utility function over supergame outcomes is monotone, then playing at each stage the Nash equilibrium of the stage game is a *subgame perfect* equilibrium of the supergame. Theorem 3 establishes that if each stage game in the sequence has a unique Nash equilibrium and each player's utility function over supergame outcomes is monotone, then the supergame has a unique subgame perfect equilibrium in which, at each stage, play is the Nash equilibrium of the stage game. These results establish that a stronger solution concept (i.e., subgame perfection rather than Nash equilibrium) enlarges

⁵Few studies of win-loss games acknowledge the repeated game resulting from fixed pairings, with the notable exceptions of Brown and Rosenthal (1990), who state that the Nash equilibrium of the repeated O'Neill game consists of play at each stage of the stage game equilibrium when each player maximizes the sum of his payoffs (and who conjecture that equilibrium has this character for more general preferences), and Binmore *et al.* (1996) who state that since the repeated game has more than two outcomes, equilibrium play depends on the subjects' risk attitudes.

the class of supergames for which equilibrium play is invariant to risk attitudes and time preferences. Together our results establish that equilibrium play is invariant for a wide class of preferences in supergames consisting of the sequential play of win-loss games.

Our first result, Theorem 1, is closely related to an important result due to Benoit and Krishna (1987) for repeated games in which the same stage game is played successively T times and in which utility is the average, or sum, of the stage game utilities. They show that if in every Nash equilibrium of the stage game each player obtains his minimax payoff, then in every Nash equilibrium of the supergame, play at each stage will be a Nash equilibrium of the stage game.

In contrast to our result, Benoit and Krishna (1987) employ strong restrictions on the players' preferences. The additivity of utility across stages implies that, when the stage game is a win-loss game, the players are risk neutral in the supergame (i.e., each player is indifferent between lotteries over supergame outcomes in which the expected number of wins is the same). In the twice repeated O'Neill game, for example, additivity of utility implies that every player is indifferent between the outcome WL and the lottery which gives WW or LL with equal probability, and that every player is also indifferent to the *timing* of wins and losses, i.e., $WL \sim LW$. Since in every Nash equilibrium of a two-player strictly competitive win-loss game each player obtains his minimax payoff, when the stage games are win-loss games our result generalizes Benoit and Krishna's by dispensing with restrictions regarding players' risk attitudes or time preferences.⁶

Our results on the subgame perfect equilibria of supergames consisting of the sequential play of win-loss stage games are related to well-known results for finitely repeated games in which utility is additive across stages: (i) it is a Nash equilibrium of the repeated game to play at each stage the stage game's Nash equilibrium, and (ii) if the stage game has a unique Nash equilibrium then the repeated game has a unique subgame perfect equilibrium in which play at each stage, on and off the equilibrium path, is the Nash equilibrium of the stage game (see Benoit and Krishna, 1985). We obtain analogous results when each stage game in the sequence of stage games is a win-loss game, again without assumptions regarding the players' attitude toward risk and their time preferences.

⁶The proof of Benoit and Krishna's (1987) result can be seen to go through provided that utility in the supergame is a weighted sum of the stage game utilities, i.e., utility is additively separable. Monotonicity is weaker than additive separability and allows, for example, the extra utility obtained by winning rather than losing at a stage to depend on the outcome at other stages.

Our last result concerns maximin play in two-player supergames consisting of the sequential play of strictly competitive win–loss stage games. Theorem 4 establishes that if a player’s utility function over supergame outcomes is monotone, then the behavioral strategy which calls for maximin play at each stage is a maximin behavioral strategy, i.e., this behavioral strategy maximizes the expected payoff that a player can guarantee himself in the supergame. We also show that it is a Nash (and subgame perfect) equilibrium of the supergame when both players follow a maximin behavioral strategy of this kind. Furthermore, a player’s expected payoff is the same in every Nash equilibrium of the supergame, and is equal to his maximin payoff. Hence, even though the supergame need not be equivalent to a zero-sum game, it inherits some of the properties of its constituent stage games which are.

2. WIN–LOSS GAMES

A **win–loss game** G is defined as

$$G = (N, (A_i)_{i \in N}, (o_i)_{i \in N}),$$

where N is the set of players, and for each $i \in N$, A_i is player i ’s set of actions (or pure strategies), o_i is a function mapping $A = \times_{i \in N} A_i$ to $\{\mathcal{W}_i, \mathcal{L}_i\}$ the set of outcomes for player i . We assume each player i prefers the outcome \mathcal{W}_i to the outcome \mathcal{L}_i and, when comparing two lotteries on $\{\mathcal{W}_i, \mathcal{L}_i\}$, prefers the lottery in which \mathcal{W}_i has higher probability. We refer to the two outcomes \mathcal{W}_i and \mathcal{L}_i , respectively, as a “win” and a “loss” for player i . Assume that N and A_i for each $i \in N$ are finite. For any finite set Z , denote by ΔZ the set of probability distributions over Z . We denote a mixed strategy for player i as $\sigma_i \in \Delta A_i$, and we write $\sigma_i(a_i)$ for the probability that player i chooses $a_i \in A_i$. To reduce notation, for $a_{-i} \in \times_{j \in N \setminus \{i\}} A_j$ we sometimes write $\sigma_{-i}(a_{-i})$ for $\prod_{j \in N \setminus \{i\}} \sigma_j(a_j)$. A mixed-strategy profile is denoted by $\sigma = (\sigma_i)_{i \in N}$. A win–loss game is **strictly competitive** if for each $a \in A$ there is some $i \in N$ such that $o_i(a) = \mathcal{W}_i$ and $o_j(a) = \mathcal{L}_j$ for all $j \neq i$.

Although utility numbers are not specified in win–loss games, it is nonetheless unambiguous to refer to their Nash equilibria. We say that $\bar{\sigma} = (\bar{\sigma}_i)_{i \in N} \in \times_{i \in N} \Delta A_i$ is a **Nash equilibrium** if for each $i \in N$ and every $\sigma_i \in \Delta A_i$,

$$\sum_{\{a \in A | o_i(a) = \mathcal{W}_i\}} \bar{\sigma}_i(a_i) \bar{\sigma}_{-i}(a_{-i}) \geq \sum_{\{a \in A | o_i(a) = \mathcal{W}_i\}} \sigma_i(a_i) \bar{\sigma}_{-i}(a_{-i}).$$

In other words, in a Nash equilibrium each player’s strategy maximizes his probability of winning given the strategies of the other players.

Two-player strictly competitive win–loss games have the property that for any assignment of utility numbers to outcomes such that each player’s utility of winning is greater than his utility of losing, the resulting game is equivalent to a zero-sum game.⁷ The Minimax Theorem, as it applies to two-person strictly competitive win–loss games, can be stated as follows.

MINIMAX THEOREM. *Let $G = (\{1, 2\}, (A_1, A_2), (o_1, o_2))$ be a two-person strictly competitive win–loss game. Then $(\bar{\sigma}_1, \bar{\sigma}_2)$ is a Nash equilibrium of G if and only if, for each $i, j \in \{1, 2\}$, $i \neq j$,*

$$\bar{\sigma}_i \in \arg \max_{\sigma_i \in \Delta A_i} \min_{\sigma_j \in \Delta A_j} \sum_{\{a \in A | o_i(a) = \mathcal{W}_i\}} \sigma_i(a_i) \sigma_j(a_j), \quad (1)$$

and

$$\bar{\sigma}_j \in \arg \min_{\sigma_j \in \Delta A_j} \max_{\sigma_i \in \Delta A_i} \sum_{\{a \in A | o_i(a) = \mathcal{W}_i\}} \sigma_i(a_i) \sigma_j(a_j). \quad (2)$$

Furthermore, if $(\bar{\sigma}_1, \bar{\sigma}_2)$ is a Nash equilibrium of G , then for each $i \in \{1, 2\}$ that

$$\begin{aligned} \sum_{\{a \in A | o_i(a) = \mathcal{W}_i\}} \bar{\sigma}_1(a_1) \bar{\sigma}_2(a_2) &= \max_{\sigma_i \in \Delta A_i} \min_{\sigma_j \in \Delta A_j} \sum_{\{a \in A | o_i(a) = \mathcal{W}_i\}} \sigma_i(a_i) \sigma_j(a_j) \\ &= \min_{\sigma_j \in \Delta A_j} \max_{\sigma_i \in \Delta A_i} \sum_{\{a \in A | o_i(a) = \mathcal{W}_i\}} \sigma_i(a_i) \sigma_j(a_j). \end{aligned}$$

Proof. The statement and proof of this theorem are obtained by a straightforward adaptation of Theorem 3.2 in Myerson (1991). ■

We refer to a mixed-strategy $\bar{\sigma}_i$ which satisfies (1) as a **maximin strategy** for player i in G , and we refer to a mixed-strategy $\bar{\sigma}_j$ which satisfies (2) as a **minimax strategy** for player j in G . Since player i wins if and only if player j loses, $\bar{\sigma}_i$ is a maximin strategy for player i if and only if $\bar{\sigma}_i$ is a minimax strategy for player i . Hence, by the Minimax Theorem, $(\bar{\sigma}_1, \bar{\sigma}_2)$ is a Nash equilibrium if and only if $(\bar{\sigma}_1, \bar{\sigma}_2)$ is maximin-strategy profile.

In the next section, we consider a sequence G^1, \dots, G^T of win–loss games. For the mixed-strategy profile σ^t in G^t , we write $v_i^t(\omega_i^t | \sigma^t)$ for the probability that the outcome for player i is $\omega_i^t \in \{\mathcal{W}_i^t, \mathcal{L}_i^t\}$, i.e.,

$$v_i^t(\omega_i^t | \sigma^t) = \sum_{\{a^t \in A^t | o_i^t(a^t) = \omega_i^t\}} \sigma_i^t(a_i^t) \sigma_{-i}^t(a_{-i}^t).$$

By the Minimax Theorem if $\bar{\sigma}^t$ and $\hat{\sigma}^t$ are both Nash equilibria (maximin-strategy profiles) of a two-player strictly competitive game G^t , then $v_i^t(\omega_i^t | \bar{\sigma}^t) = v_i^t(\omega_i^t | \hat{\sigma}^t)$, i.e., all Nash equilibria of G^t have the same value. In this case, we write $\bar{v}_i^t(\omega_i^t)$ for the Nash equilibrium probability of ω_i^t in G^t .

⁷Following Myerson (1991) we say the games $(N, (A_i)_{i \in N}, (u_i)_{i \in N})$ and $(N, (A_i)_{i \in N}, (\hat{u}_i)_{i \in N})$ are equivalent if and only if, for every player i , there exist numbers $\alpha_i > 0$ and β_i such that $\hat{u}_i(a) = \alpha_i u_i(a) + \beta_i \forall a \in A$.

3. REPEATED WIN-LOSS GAMES

We analyze supergames consisting of the sequential play of T win-loss games where, following each stage, every player observes the profile of actions taken. Let G^1, \dots, G^T be a sequence of win-loss games, where in the t th game $G^t = (N, (A_i^t)_{i \in N}, (o_i^t)_{i \in N})$ a “win” and “loss” for player i are denoted by \mathcal{W}_i^t and \mathcal{L}_i^t , respectively. An outcome for player i in the supergame consists of a sequence $(\omega_i^1, \dots, \omega_i^T)$, where $\omega_i^t \in \{\mathcal{W}_i^t, \mathcal{L}_i^t\}$ is the outcome for player i at the t th stage. Denote by $\Omega_i = \times_{t=1}^T \{\mathcal{W}_i^t, \mathcal{L}_i^t\}$ the set of supergame outcomes for player i . Let $(u_i)_{i \in N}$ be a profile of von Neumann–Morgenstern utility functions where, for each $i \in N$, u_i is a real-valued function on Ω_i . Then the **supergame derived from G^1, \dots, G^T and $(u_i)_{i \in N}$** is given by

$$\Gamma = (N, T, ((A_i^t, H^t)_{t=1}^T, u_i)_{i \in N}),$$

where N is the set of players, T is the number of stages and, for each $t \in \{1, \dots, T\}$ and each $i \in N$, A_i^t is the set of actions for player i at the t th stage, $H^t = \times_{j=1}^{t-1} (\times_{i \in N} A_i^j)$ is the set of possible histories at the t th stage, and player i ’s utility for terminal history (a^1, \dots, a^T) is $u_i(o_i^1(a^1), \dots, o_i^T(a^T))$.

We say that $u_i : \Omega_i \rightarrow \mathbb{R}$ is **monotone** if for every $\hat{\omega}_i, \bar{\omega}_i \in \Omega_i$ such that $\hat{\omega}_i^s = \mathcal{W}_i^s$ and $\bar{\omega}_i = \mathcal{L}_i^s$ for some s and $\hat{\omega}_i^t = \bar{\omega}_i^t$ for all $t \neq s$ we have $u_i(\hat{\omega}_i) > u_i(\bar{\omega}_i)$. In other words, u_i is monotone if, whenever two supergame outcomes differ only at a single stage s , then u_i assigns a higher utility to the outcome in which player i wins at stage s than to the one where he loses.

A behavioral strategy for player i in the supergame can be written as $b_i = (b_i^1, \dots, b_i^T)$, where $b_i^t : H^t \rightarrow \Delta A_i^t$ for each t . For $h^t \in H^t$ we write $b_i^t(a_i^t|h^t)$ for the probability that player i , having observed the history h^t , chooses action $a_i^t \in A_i^t$; for $a_{-i}^t \in \times_{j \in N \setminus \{i\}} A_j^t$ we write $b_{-i}^t(a_{-i}^t|h^t)$ for $\prod_{j \in N \setminus \{i\}} b_j^t(a_j^t|h^t)$; and for $a^t \in \times_{i \in N} A_i^t$ we write $b^t(a^t|h^t)$ for $\prod_{i \in N} b_i^t(a_i^t|h^t)$. A behavioral strategy profile in the supergame is denoted by $b = (b_i)_{i \in N}$. Let B_i denote the set of all behavioral strategies for player i , and let $B_{-i} = \times_{j \in N \setminus \{i\}} B_j$. Given a behavioral strategy profile b , the expected utility to player i in subgame h^t is defined recursively as

$$\begin{aligned} U_i^t(b^t, \dots, b^T | h^t) &= \sum_{a^t \in A^t} b_i^t(a_i^t | h^t) b_{-i}^t(a_{-i}^t | h^t) \\ &\quad \times U_i^{t+1}(b^{t+1}, \dots, b^T | h^t, (a_i^t, a_{-i}^t)), \end{aligned}$$

where

$$U_i^{T+1}(h^T, a^T) = u_i(o_i^1(a^1), \dots, o_i^T(a^T)).$$

The expected utility to player i in the supergame overall is $U_i^1(b|h^1)$, where h^1 is the null history, which we simply write as $U_i(b)$.

The behavioral strategy profile \bar{b} is a **Nash equilibrium of the supergame** if for each $i \in N$ and every $b_i \in B_i$,

$$U_i(\bar{b}_i, \bar{b}_{-i}) \geq U_i(b_i, \bar{b}_{-i}).$$

If b is a Nash equilibrium of the supergame, we say that $h^t = (a^1, \dots, a^{t-1})$ is an equilibrium-path history (or on the equilibrium path) if

$$b^1(a^1)b^2(a^2|a^1) \cdots b^{t-1}(a^{t-1}|a^1, \dots, a^{t-2}) > 0.$$

Finally, we say that \bar{b}_i is a **maximin behavioral strategy** for player i if

$$\bar{b}_i \in \arg \max_{b_i \in B_i} \min_{b_{-i} \in B_{-i}} U_i(b_i, b_{-i}),$$

i.e., \bar{b}_i maximizes player i 's payoff under the assumption that, whatever behavioral strategy player i chooses, the behavioral strategies of the other players are chosen to minimize i 's payoff. The payoff player i achieves from following a maximin behavioral strategy is the largest payoff that he can guarantee himself.

Nash Equilibrium

Our first result is that given a sequence of two-player strictly competitive win–loss games and given a profile of monotone von Neumann–Morgenstern utility functions over supergame outcomes, then any Nash equilibrium of the supergame consisting of the sequential play of the stage games has the players (on the equilibrium path) play, at each stage, one of the stage game's Nash equilibria. All proofs are postponed to the appendix.

THEOREM 1. *Let $\{G^t\}_{t=1}^T = \{\{1, 2\}, (A_1^t, A_2^t), (o_1^t, o_2^t)\}_{t=1}^T$ be a sequence of two-player strictly competitive win–loss games and, for each $i \in \{1, 2\}$, let $u_i : \Omega_i \rightarrow \mathbb{R}$ be monotone. If b is a Nash equilibrium of the supergame derived from $\{G^t\}_{t=1}^T$ and (u_1, u_2) , then for each t and each equilibrium-path history $h^t \in H^t$, we have that $(b_1^t(h^t), b_2^t(h^t)) \in \Delta A_1^t \times \Delta A_2^t$ is a Nash equilibrium of G^t .*

By Theorem 1, the probability the outcome for player i is ω_i^t at stage t (on the equilibrium path) is the same in every Nash equilibrium of the supergame, and is equal to $\bar{v}_i^t(\omega_i^t)$. (Recall by the Minimax Theorem that every Nash equilibrium of G^t has the same value.) Define \bar{V}_i to be the expected utility to player i in the supergame when, at each stage t , the probability of ω_i^t is $\bar{v}_i^t(\omega_i^t)$, i.e.,

$$\bar{V}_i \equiv \sum_{(\omega_i^1, \dots, \omega_i^T) \in \Omega_i} \bar{v}_i^1(\omega_i^1) \cdots \bar{v}_i^T(\omega_i^T) u_i(\omega_i^1, \dots, \omega_i^T).$$

We have the following corollary.

COROLLARY 1. *Under the assumptions of Theorem 1, for each player $i \in \{1, 2\}$, player i 's expected utility is the same, and equal to \bar{V}_i , in every Nash equilibrium of the supergame.*

The following corollary is an immediate consequence of Theorem 1.

COROLLARY 2. *Suppose in addition to the assumptions of Theorem 1 that each stage game G^t has a unique Nash equilibrium σ^t . If b is a Nash equilibrium of the supergame derived from $\{G^t\}_{t=1}^T$ and $(u_i)_{i \in N}$, then for each t and each equilibrium-path history $h^t \in H^t$ we have $b^t(h^t) = \sigma^t$.*

By Corollary 2 if, in addition to the assumptions of Theorem 1, every stage game has a unique Nash equilibrium, then all Nash equilibria of the supergame are observationally equivalent. If every stage game's unique Nash equilibrium is also completely mixed, then every supergame history is on the equilibrium path. Therefore, we have the following corollary.

COROLLARY 3. *Suppose in addition to the assumptions of Theorem 1 that each stage game G^t has a unique and completely mixed Nash equilibrium σ^t . Then the supergame derived from $\{G^t\}_{t=1}^T$ and $(u_i)_{i \in N}$ has a unique Nash equilibrium b and, furthermore, for each t and $h^t \in H^t$ we have $b^t(h^t) = \sigma^t$.*

To illustrate these results, we consider the twice repeated Matching Pennies game.

EXAMPLE 1 (Twice Repeated Matching Pennies game (TRMPG)). For $t \in \{1, 2\}$, let G^t be the matching pennies game, i.e., $G^t = (\{1, 2\}, (A_1^t, A_2^t), (o_1^t, o_2^t))$, where $A_i^t = \{H^t, T^t\}$, $o_1^t(a^t) = \mathcal{W}_1^t$ and $o_2^t(a^t) = \mathcal{L}_2^t$ if $a_1^t = a_2^t$, and $o_1^t(a^t) = \mathcal{L}_1^t$ and $o_2^t(a^t) = \mathcal{W}_2^t$ otherwise. In the TRMPG the set of supergame outcomes for each player i is $\{(\mathcal{W}_i^1, \mathcal{W}_i^2), (\mathcal{W}_i^1, \mathcal{L}_i^2), (\mathcal{L}_i^1, \mathcal{W}_i^2), (\mathcal{L}_i^1, \mathcal{L}_i^2)\}$. We assume that player i 's preferences are monotone, i.e., $u_i(\mathcal{W}_i^1, \mathcal{W}_i^2) = 1$, $u_i(\mathcal{W}_i^1, \mathcal{L}_i^2) = \delta_i$, $u_i(\mathcal{L}_i^1, \mathcal{W}_i^2) = \varepsilon_i$, and $u_i(\mathcal{L}_i^1, \mathcal{L}_i^2) = 0$, where $\delta_i, \varepsilon_i \in (0, 1)$. Monotonicity is very weak, imposing almost no restriction on preferences over supergame outcomes except that winning would be better than losing at each stage.

The matching pennies game is a two-player strictly competitive win–loss game and has a unique Nash equilibrium in which each player chooses each of his actions with probability 1/2. By Corollary 3, the TRMPG also has a unique Nash equilibrium in which at each stage each player chooses each of his actions with probability 1/2. This conclusion holds regardless of whether player i prefers winning and then losing to losing and then winning (or vice versa), i.e., regardless of whether $\delta_i > \varepsilon_i$ or $\delta_i < \varepsilon_i$. It also holds whether player i is risk averse in the number of wins (i.e., $\delta_i = \varepsilon_i > 1/2$) or risk loving (i.e., $\delta_i = \varepsilon_i < 1/2$).

As is clear from Example 1, even if each stage game is equivalent to a zero-sum game, the resulting supergame need not be. (The *TRMPG* is equivalent to a zero-sum game only if $\delta_1 + \varepsilon_2 = 1$ and $\delta_2 + \varepsilon_1 = 1$.) Indeed, the supergame need not even be strictly competitive. If both players are risk averse, for example, each would obtain a payoff greater than his Nash payoff if they agreed to coordinate their actions so that player 1 wins at the first stage and loses at the second stage.⁸

Subgame Perfect Equilibrium

Our next results apply to supergames consisting of the sequential play of win–loss (but not necessarily strictly competitive) stage games with any finite number of players. Theorem 2 establishes that any behavioral strategy profile in which the players, at each stage t , play a Nash equilibrium of the stage game G^t is a subgame perfect (and Nash) equilibrium in the supergame for any profile of monotone utility functions over supergame outcomes.

THEOREM 2. *Let $\{G^t\}_{t=1}^T = \{(N, (A_i^t)_{i \in N}, (o_i^t)_{i \in N})\}_{t=1}^T$ be a sequence of win–loss games and, for each $i \in N$, let $u_i : \Omega_i \rightarrow \mathbb{R}$ be monotone. For each t let $\bar{\sigma}^t$ be a Nash equilibrium of G^t . Then the behavioral strategy profile b , in which $b^t(h^t) = \bar{\sigma}^t$ for each t and each $h^t \in H^t$, is a subgame perfect equilibrium of the supergame derived from $\{G^t\}_{t=1}^T$ and $(u_i)_{i \in N}$.*

Theorem 2 leaves open the possibility that there are subgame perfect equilibria of the supergame in which for some history the players do not play a Nash equilibrium of the current stage. Theorem 3 shows that this is not possible when each stage game has a unique Nash equilibrium. In particular, given a sequence of win–loss games, each game having a unique Nash equilibrium, and given any profile of monotone utility functions over supergame outcomes, then the supergame consisting of the sequential play of the win–loss games has a unique subgame perfect equilibrium: at each stage, and both on and off the equilibrium path, the players play the Nash equilibrium of the stage game.

THEOREM 3. *Let $\{G^t\}_{t=1}^T = \{(N, (A_i^t)_{i \in N}, (o_i^t)_{i \in N})\}_{t=1}^T$ be a sequence of win–loss games and, for each $i \in N$, let $u_i : \Omega_i \rightarrow \mathbb{R}$ be monotone. If each stage game G^t has a unique Nash equilibrium $\bar{\sigma}^t$, then the supergame derived from $\{G^t\}_{t=1}^T$ and $(u_i)_{i \in N}$ has a unique subgame perfect equilibrium b . Furthermore $b^t(h^t) = \bar{\sigma}^t$ for every $h^t \in H^t$.*

⁸Of course, such an agreement would not be self enforcing since (by Theorem 1) it is not Nash.

Together Theorems 2 and 3 establish the irrelevance of risk attitudes and time preferences in a broader class of games (i.e., game with more than two players) than did Theorem 1 when the solution concept is subgame perfection.

Maximin Strategies

Our next result applies to two-player games and establishes that playing at each stage t a maximin strategy of G^t is a maximin behavioral strategy in the supergame.

THEOREM 4. *Let $\{G^t\}_{t=1}^T = \{\{1, 2\}, (A_1^t, A_2^t), (o_1^t, o_2^t)\}_{t=1}^T$ be a sequence of two-player strictly competitive win-loss games and, for each $i \in \{1, 2\}$, let $u_i : \Omega_i \rightarrow \mathbb{R}$ be monotone. For each t let $\bar{\sigma}_i^t$ be a maximin strategy for player i in G^t . Then the behavioral strategy \bar{b}_i , in which $\bar{b}_i^t(h^t) = \bar{\sigma}_i^t$ for each t and each $h^t \in H^t$, is a maximin behavioral strategy for player i in the supergame derived from $\{G^t\}_{t=1}^T$ and $(u_i)_{i \in N}$. Player i 's maximin payoff is \bar{V}_i , i.e., $\bar{V}_i = \max_{b_i \in B_i} \min_{b_{-i} \in B_{-i}} U_i(b_1, b_2)$.*

Our results for supergames consisting of the sequential play of two-player strictly competitive win-loss games establish that even though such games need not be zero sum, they do inherit some of the properties of the constituent stage games which are equivalent to zero-sum games: (i) it is a Nash equilibrium of the supergame when both players follow the maximin behavioral strategy of playing maximin at each stage (Theorem 2), and (ii) a player's expected utility is the same in every Nash equilibrium of the supergame, and equal to his maximin payoff (Theorem 4 and Corollary 1).

4. DISCUSSION

In this section we discuss examples establishing the necessity of the assumptions of Theorem 1. We first show that it is essential that the stage games be strictly competitive.

EXAMPLE 2. For $t \in \{1, 2\}$ let G^t be the two-player win-loss game given below.

	H_2^t	T_2^t	P_2^t
H_1^t	$\mathcal{W}_1^t, \mathcal{L}_2^t$	$\mathcal{L}_1^t, \mathcal{W}_2^t$	$\mathcal{L}_1^t, \mathcal{L}_2^t$
T_1^t	$\mathcal{L}_1^t, \mathcal{W}_2^t$	$\mathcal{W}_1^t, \mathcal{L}_2^t$	$\mathcal{L}_1^t, \mathcal{L}_2^t$

Each game G^t has a unique Nash equilibrium $(1/2H_1^t + 1/2T_1^t, 1/2H_2^t + 1/2T_2^t)$, but is not strictly competitive since both players lose if player 2 chooses P_2^t .

Assume that $u_i : \times_{t=1}^2 \{\mathcal{W}_i^t, \mathcal{L}_i^t\} \rightarrow \mathbb{R}$ is monotone for each player i and that $u_1(\mathcal{L}_1^1, \mathcal{L}_1^2)/2 + u_1(\mathcal{L}_1^1, \mathcal{W}_1^2)/2 > u_1(\mathcal{W}_1^1, \mathcal{L}_1^2)$. We show that the supergame derived from $\{G^1, G^2\}$ and (u_1, u_2) has a Nash equilibrium in which, at the first stage, play is not a Nash equilibrium of the stage game. In particular, the following behavioral strategy profile is a Nash equilibrium: at the first stage players 1 and 2 choose H_1^1 and T_2^1 , respectively; at the second stage both players play the Nash equilibrium of the stage game unless player 1 chose T_1^1 at the first stage, in which case player 2 chooses P_2^2 . In this Nash equilibrium player 1's payoff is $u_1(\mathcal{L}_1^1, \mathcal{L}_1^2)/2 + u_1(\mathcal{L}_1^1, \mathcal{W}_1^2)/2$. Although H_1^1 is not a best response to T_2^1 in G^1 , by deviating to T_1^1 at the first stage player 1's payoff is only $u_1(\mathcal{W}_1^1, \mathcal{L}_1^2)$ since he is punished by player 2 at the second stage. (By Theorem 4 this Nash equilibrium is not subgame perfect.)

Our next example establishes that it is also essential for Theorem 1 that the stage games be two-player games.

EXAMPLE 3. For $t \in \{1, 2, 3\}$ let G^t be the three-player strictly competitive win-loss game given by

	H_3^t		T_3^t	
H_2^t	$\mathcal{W}_1^t, \mathcal{L}_2^t, \mathcal{L}_3^t$	$\mathcal{L}_1^t, \mathcal{L}_2^t, \mathcal{W}_3^t$	H_2^t	T_2^t
H_1^t	$\mathcal{W}_1^t, \mathcal{L}_2^t, \mathcal{L}_3^t$	$\mathcal{L}_1^t, \mathcal{L}_2^t, \mathcal{W}_3^t$	H_1^t	$\mathcal{W}_1^t, \mathcal{L}_2^t, \mathcal{L}_3^t$
T_1^t	$\mathcal{L}_1^t, \mathcal{W}_2^t, \mathcal{L}_3^t$	$\mathcal{W}_1^t, \mathcal{L}_2^t, \mathcal{L}_3^t$	T_1^t	$\mathcal{L}_1^t, \mathcal{L}_2^t, \mathcal{W}_3^t$

In G^t player 1 chooses the row, player 2 chooses the column, and player 3 chooses the matrix. Each game G^t has a pure-strategy Nash equilibrium (H_1^t, H_2^t, H_3^t) in which player 1 wins, a pure-strategy Nash equilibrium (T_1^t, T_2^t, T_3^t) in which player 3 wins, but has no pure-strategy Nash equilibrium in which player 2 wins.

Assume that $u_i : \times_{t=1}^3 \{\mathcal{W}_i^t, \mathcal{L}_i^t\} \rightarrow \mathbb{R}$ is monotone for each player i and, furthermore, that $u_1(\mathcal{L}_1^1, \mathcal{W}_1^2, \mathcal{L}_1^3) > u_1(\mathcal{W}_1^1, \mathcal{L}_1^2, \mathcal{L}_1^3)$ and $u_3(\mathcal{L}_3^1, \mathcal{L}_3^2, \mathcal{W}_3^3) > u_3(\mathcal{W}_3^1, \mathcal{L}_3^2, \mathcal{L}_3^3)$. We show that the supergame derived from $\{G^t\}_{t=1}^3$ and $(u_i)_{i \in N}$ has a Nash equilibrium in which player 2 wins at the first stage. Specifically, consider the behavioral strategy profile in which at the first stage the players choose (T_1^1, H_2^1, H_3^1) , at the second stage they choose (H_1^2, H_2^2, H_3^2) if player 1 chose T_1^1 at the first stage and they choose (T_1^2, T_2^2, T_3^2) otherwise, and at the third stage they choose (T_1^3, T_2^3, T_3^3) if player 3 chose H_3^1 at the first stage and they choose (H_1^3, H_2^3, H_3^3) otherwise. In this Nash equilibrium, player 2 wins at the first stage and players 1 and 3 win at the second and third stage, respectively. Play at the first stage is not Nash in G^1 but is supported by punishing a deviation by player 1 (player 3) at the first stage with a certain loss in the second (third) stage.

The key feature of these examples is that each stage game has a Nash equilibrium in which some player's probability of winning is more than his minimax probability of winning.⁹ Equilibrium play in the supergame, which is not Nash for some stage, is sustained by threats of future punishment.

5. CONCLUSION

We conclude by relating our results to the experimental literature on games. First, as in O'Neill (1987), experiments by Rapoport and Boebel (1992), Mookherjee and Sopher (1994), and Shachat (1996) have fixed pairs of subjects repeatedly play a win–loss game (with a unique Nash equilibrium in completely mixed strategies). Our results show that these papers have adequately controlled for the subjects' risk attitudes and time preferences in the supergame. Second, by identifying classes of stage games for which Nash (or subgame perfect) play in the supergame is at each stage a Nash equilibrium of the stage game, our results provide guidance on the design of experiments which control for risk attitudes and time preferences. Finally, our results allow the supergame to consist of a sequence of different stage games. This is useful to the experimenter who may wish to confront subjects with a sequence of different stage games, as when subjects alternate between choosing the row and the column in a nonsymmetric game.

APPENDIX

We begin by defining terms, stating two preliminary lemmas, and proving a lemma on continuation payoffs. Let $\{G'\}_{t=1}^T = \{(N, (A_i^t)_{i \in N}, (o_i^t)_{i \in N})\}_{t=1}^T$ be a sequence of win–loss games and, for each $i \in N$, let $u_i : \Omega_i \rightarrow \mathbb{R}$ be monotone. In the supergame derived from $\{G'\}_{t=1}^T$ and $(u_i)_{i \in N}$ the expected utility of player i at stage t if player i 's record of wins and losses is $\omega_i^1, \dots, \omega_i^{t-1}$ and if at each stage $k \geq t$ actions are chosen according to the mixed-strategy profile $\sigma^k = (\sigma_i^k)_{i \in N}$ is denoted by $V_i^t(\sigma^t, \dots, \sigma^T | \omega_i^1, \dots, \omega_i^{t-1})$ and is defined recursively for $t \leq T$ as

$$\begin{aligned} V_i^t(\sigma^t, \dots, \sigma^T | \omega_i^1, \dots, \omega_i^{t-1}) \\ = \sum_{\omega_i^t \in \{\mathcal{W}_i^t, \mathcal{L}_i^t\}} v_i^t(\omega_i^t | \sigma^t) V_i^{t+1}(\sigma^{t+1}, \dots, \sigma^T | \omega_i^1, \dots, \omega_i^{t-1}, \omega_i^t), \end{aligned}$$

⁹For example, in the stage game of Example 2, player 1's Nash equilibrium winning probability is 1/2 while his minimax winning probability is zero.

where

$$V_i^{T+1}(\omega_i^1, \dots, \omega_i^T) = u_i(\omega_i^1, \dots, \omega_i^T).$$

LEMMA 1. Let $\{G^t\}_{t=1}^T$ be a sequence of win-loss games and, for each $i \in N$, let $u_i : \Omega_i \rightarrow \mathbb{R}$ be monotone. Then for each $t > 1$, every $(\omega_i^1, \dots, \omega_i^{t-2}) \in \times_{j=1}^{t-2} \{\mathcal{W}_i^j, \mathcal{L}_i^j\}$, and every $(\sigma^t, \dots, \sigma^T)$ we have

$$V_i^t(\sigma^t, \dots, \sigma^T | \omega_i^1, \dots, \omega_i^{t-2}, \mathcal{W}_i^{t-1}) > V_i^t(\sigma^t, \dots, \sigma^T | \omega_i^1, \dots, \omega_i^{t-2}, \mathcal{L}_i^{t-1}).$$

Proof. The proof is straightforward and is omitted. ■

LEMMA 2. Let $\{G^t\}_{t=1}^T$ be a sequence of two-player strictly competitive win-loss games and, for each $i \in N$, let $u_i : \Omega_i \rightarrow \mathbb{R}$ be monotone. For each k let $\hat{\sigma}^k = (\hat{\sigma}_1^k, \hat{\sigma}_2^k)$ and $\bar{\sigma}^k = (\bar{\sigma}_1^k, \bar{\sigma}_2^k)$ be profiles of maximin strategies in G^k . Then for each $t \geq 1$ and every record of wins and losses $\omega_i^1, \dots, \omega_i^{t-1}$ we have

$$V_i^t(\bar{\sigma}^t, \dots, \bar{\sigma}^T | \omega_i^1, \dots, \omega_i^{t-1}) = V_i^t(\hat{\sigma}^t, \dots, \hat{\sigma}^T | \omega_i^1, \dots, \omega_i^{t-1}).$$

Proof. The proof is straightforward and is omitted. ■

The next lemma on continuation payoffs applies to supergames consisting of a sequence of two-player strictly competitive win-loss games. We show that at every stage, and regardless of the other player's behavioral strategy, by playing one of his maximin strategies at the current and every subsequent stage, a player obtains an expected utility at least as great as what he would obtain were both players to both employ maximin strategies at the current and subsequent stages.

CONTINUATION-PAYOUT LEMMA. Let $\{G^t\}_{t=1}^T$ be a sequence of two-player strictly competitive win-loss games and, for each t , let $\bar{\sigma}^t = (\bar{\sigma}_1^t, \bar{\sigma}_2^t)$ be a profile of maximin strategies for G^t . Let (b_1, b_2) be a behavioral strategy profile in the supergame derived from $\{G^t\}_{t=1}^T$ and (u_1, u_2) , where u_i is monotone for each $i \in \{1, 2\}$. If $b_i^k(h^k) = \bar{\sigma}_i^k$ for each $h^k \in H^k$ and $k \geq t$, then for each history $h^t = (a^1, \dots, a^{t-1}) \in H^t$ we have

$$U_i^t(b^t, \dots, b^T | h^t) \geq V_i^t(\bar{\sigma}^t, \dots, \bar{\sigma}^T | o_i^1(a^1), \dots, o_i^{t-1}(a^{t-1})).$$

Proof. For each t and history $h^t \in H^t$, if $\bar{\sigma}^t$ is a profile of maximin strategies then we have

$$\begin{aligned} v_i^t(\mathcal{W}_i^t | \bar{\sigma}^t) &= \sum_{\{a^t \in A^t | o_i^t(a^t) = \mathcal{W}_i^t\}} \bar{\sigma}_i^t(a_i^t) \bar{\sigma}_j^t(a_j^t) \\ &= \min_{\sigma_j^t \in \Delta A_j^t} \sum_{\{a^t \in A^t | o_i^t(a^t) = \mathcal{W}_i^t\}} \bar{\sigma}_i^t(a_i^t) \sigma_j^t(a_j^t) \\ &\leq \sum_{\{a^t \in A^t | o_i^t(a^t) = \mathcal{W}_i^t\}} \bar{\sigma}_i^t(a_i^t) b_j^t(a_j^t | h^t). \end{aligned} \tag{3}$$

The first equality follows from the definition of $v_i^t(\omega_i^t | \sigma^t)$. By the Minimax Theorem $\bar{\sigma}^t$ is a Nash equilibrium, which implies the second equality.

The proof is by induction. Mildly abusing notation, for $h^t = (a^1, \dots, a^{t-1})$ we write $o_i(h^t)$ for the vector $(o_i^1(a^1), \dots, o_i^{t-1}(a^{t-1}))$ of stage game outcomes given history h^t . For each k , let $(\bar{\sigma}_1^k, \bar{\sigma}_2^k)$ be a profile of maximin strategies for G^k . Let $P(t)$ be the proposition that “If $b_i^k(h^k) = \bar{\sigma}_i^k$ for each $h^k \in H^k$ and $k \geq t$, then for each history $h^t = (a^1, \dots, a^{t-1}) \in H^t$ we have

$$U_i^t(b^t, \dots, b^T | h^t) \geq V_i^t(\bar{\sigma}^t, \dots, \bar{\sigma}^T | o_i(h^t)).$$

We first show that $P(T)$ is true. Let $b_i^T(h^T) = \bar{\sigma}_i^T$ for each $h^T \in H^T$, and let $\hat{h}^T = (\hat{a}^1, \dots, \hat{a}^{T-1}) \in H^T$ be arbitrary. We have that

$$U_i^T(b^T | \hat{h}^T) = \sum_{a^T \in A^T} \bar{\sigma}_i^T(a_i^T) b_j^T(a_j^T | \hat{h}^T) u_i(o_i(\hat{h}^T), o_i^T(a^T)) \quad (4)$$

$$\begin{aligned} &= \sum_{\omega_i^T \in \{\mathcal{W}_i^T, \mathcal{L}_i^T\}} u_i(o_i(\hat{h}^T), \omega_i^T) \sum_{\{a^T \in A^T | o_i^T(a^T) = \omega_i^T\}} \bar{\sigma}_i^T(a_i^T) b_j^T(a_j^T | \hat{h}^T) \\ &\geq \sum_{\omega_i^T \in \{\mathcal{W}_i^T, \mathcal{L}_i^T\}} v_i^T(\omega_i^T | \bar{\sigma}^T) u_i(o_i(\hat{h}^T), \omega_i^T) \end{aligned} \quad (5)$$

$$= V_i^T(\bar{\sigma}^T | o_i(\hat{h}^T)) \quad (6)$$

Equality (4) follows from the definition of $U_i^T(b^T | \hat{h}^T)$ and that $b_i^T(h^T) = \bar{\sigma}_i^T$ for each $h^T \in H^T$. We have

$$\sum_{\{a^T \in A^T | o_i^T(a^T) = \mathcal{W}_i^T\}} \bar{\sigma}_i^T(a_i^T) b_j^T(a_j^T | \hat{h}^T) \geq v_i^T(\mathcal{W}_i^T | \bar{\sigma}^T)$$

by (3), and

$$u_i(o_i(\hat{h}^T), \mathcal{W}_i^T) > u_i(o_i(\hat{h}^T), \mathcal{L}_i^T),$$

since u_i is monotone, which together imply (5). Finally, equality (6) follows by the definition of $V_i^T(\bar{\sigma}^T | o_i(\hat{h}^T))$.

Assume that $P(t+1)$ is true for $t+1 \leq T$. We show that $P(t)$ is true. Let $b_i^k(h^k) = \bar{\sigma}_i^k$ for each $h^k \in H^k$ and $k \geq t$, and let $\hat{h}^t = (\hat{a}^1, \dots, \hat{a}^{t-1}) \in H^t$ be arbitrary. We have that

$$\begin{aligned} U_i^t(b^t, \dots, b^T | \hat{h}^t) &= \sum_{a^t \in A^t} \bar{\sigma}_i^t(a_i^t) b_j^t(a_j^t | \hat{h}^t) U_i^{t+1}(b^{t+1}, \dots, b^T | \hat{h}^t, a^t) \\ &\geq \sum_{a^t \in A^t} \bar{\sigma}_i^t(a_i^t) b_j^t(a_j^t | \hat{h}^t) \\ &\quad \times V_i^{t+1}(\bar{\sigma}^{t+1}, \dots, \bar{\sigma}^T | o_i(\hat{h}^t), o_i^t(a^t)) \\ &= \sum_{\omega_i^t \in \{\mathcal{W}_i^t, \mathcal{L}_i^t\}} V_i^{t+1}(\bar{\sigma}^{t+1}, \dots, \bar{\sigma}^T | o_i(\hat{h}^t), \omega_i^t) \end{aligned} \quad (7)$$

$$\begin{aligned}
& \times \sum_{\{a^t \in A^t | o_i^t(a^t) = \omega_i^t\}} \bar{\sigma}_i^t(a_i^t) b_j^t(a_j^t | \hat{h}^t) \\
& \geq \sum_{\omega_i^t \in \{\mathcal{W}_i^t, \mathcal{L}_i^t\}} v_i^t(\omega_i^t | \bar{\sigma}^t) \\
& \quad \times V_i^{t+1}(\bar{\sigma}^{t+1}, \dots, \bar{\sigma}^T | o_i(\hat{h}^t), \omega_i^t) \\
& = V_i^t(\bar{\sigma}^t, \dots, \bar{\sigma}^T | o_i(\hat{h}^t)). \tag{8}
\end{aligned}$$

Inequality (7) holds since $P(t+1)$ is true. We have

$$\sum_{\{a^t \in A^t | o_i(a^t) = \mathcal{W}_i^t\}} \bar{\sigma}_i^t(a_i^t) b_j^t(a_j^t | \hat{h}^t) \geq v_i^t(\mathcal{W}_i^t | \bar{\sigma}^t)$$

by (3), and

$$V_i^{t+1}(\bar{\sigma}^{t+1}, \dots, \bar{\sigma}^T | o_i(\hat{h}^t), \mathcal{W}_i^t) > V_i^{t+1}(\bar{\sigma}^{t+1}, \dots, \bar{\sigma}^T | o_i(\hat{h}^t), \mathcal{L}_i^t),$$

by Lemma 1, which together imply (8). Hence $U_i^t(b^t, \dots, b^T | \hat{h}^t) \geq V_i^t(\bar{\sigma}^t, \dots, \bar{\sigma}^T | o_i(\hat{h}^t))$ and the Lemma is established. ■

Proof of Theorem 1. Let b be a Nash equilibrium of the supergame derived from $\{G^t\}_{t=1}^T$ and (u_1, u_2) . Suppose contrary to the Theorem that there is some t and equilibrium-path history h^t such that $b^t(h^t) \in \Delta A_1^t \times \Delta A_2^t$ is not a Nash equilibrium of G^t . Let t be the last such period and let $\tilde{h}^t \in H^t$ be an equilibrium-path history such that $b^t(\tilde{h}^t)$ is not a Nash equilibrium of G^t .

Then player 1, say, has a strategy $\hat{\sigma}_1^t$ such that

$$\sum_{\{a^t \in A^t | o_1^t(a^t) = \mathcal{W}_1^t\}} \hat{\sigma}_1^t(a_1^t) b_2^t(a_2^t | \tilde{h}^t) > \sum_{\{a^t \in A^t | o_1^t(a^t) = \mathcal{W}_1^t\}} b_1^t(a_1^t | \tilde{h}^t) b_2^t(a_2^t | \tilde{h}^t). \tag{9}$$

For each $k > t$ let $(\bar{\sigma}_1^k, \bar{\sigma}_2^k)$ be a profile of maximin strategies for G^k . Consider the strategy \dot{b}_1 in which player 1 follows b_1 unless history \tilde{h}^t is reached, in which case he follows $\hat{\sigma}_1^t$ and, for every continuation history $(\tilde{h}^t, a^t, \dots, a^{k-1})$, he plays the maximin strategy $\bar{\sigma}_1^k$.

Let $\dot{b} = (\dot{b}_1, b_2)$. Abusing notation, for $h^t = (a^1, \dots, a^{t-1})$ we write $o_1(h^t)$ for the vector $(o_1^1(a^1), \dots, o_1^{t-1}(a^{t-1}))$. We show that $U_1^t(\dot{b}^t, \dots, \dot{b}^T | \tilde{h}^t) > U_1^t(b^t, \dots, b^T | \tilde{h}^t)$. We have that

$$\begin{aligned}
U_1^t(\dot{b}^t, \dots, \dot{b}^T | \tilde{h}^t) &= \sum_{a^t \in A^t} \hat{\sigma}_1^t(a_1^t) b_2^t(a_2^t | \tilde{h}^t) U_1^{t+1}(\dot{b}^{t+1}, \dots, \dot{b}^T | \tilde{h}^t, a^t) \\
&\geq \sum_{a^t \in A^t} \hat{\sigma}_1^t(a_1^t) b_2^t(a_2^t | \tilde{h}^t) V_1^{t+1}(\bar{\sigma}^{t+1}, \dots, \bar{\sigma}^T | o_1(\tilde{h}^t), o_1^t(a^t)) \\
&= \sum_{\omega_1^t \in \{\mathcal{W}_1^t, \mathcal{L}_1^t\}} V_1^{t+1}(\bar{\sigma}^{t+1}, \dots, \bar{\sigma}^T | o_1(\tilde{h}^t), \omega_1^t)
\end{aligned}$$

$$\begin{aligned}
& \times \sum_{a^t \in A^t | o_1^t(a^t) = \omega_1^t} \hat{\sigma}_1^t(a_1^t) b_2^t(a_2^t | \tilde{h}^t) \\
& > \sum_{\omega_1^t \in \{\mathcal{W}_1^t, \mathcal{L}_1^t\}} V_1^{t+1}(\bar{\sigma}^{t+1}, \dots, \bar{\sigma}^T | o_1(\tilde{h}^t), \omega_1^t) \\
& \quad \times \sum_{a^t \in A^t | o_1^t(a^t) = \omega_1^t} b_1^t(a_1^t | \tilde{h}^t) b_2^t(a_2^t | \tilde{h}^t) \\
& = \sum_{a^t \in A^t} b_1^t(a_1^t | \tilde{h}^t) b_2^t(a_2^t | \tilde{h}^t) U_1^{t+1}(b^{t+1}, \dots, b^T | \tilde{h}^t, a^t) \\
& = U_1^t(b^t, \dots, b^T | \tilde{h}^t).
\end{aligned}$$

The first and last equalities follow from definition of U_1^t . The weak inequality follows by the Continuation-Payoff Lemma since $\dot{b}_i^k(h^k) = \bar{\sigma}_i^k$ for each $h^k \in H^k$ and $k > t$. The strict inequality follows from (9) and that

$$V_1^{t+1}(\bar{\sigma}^{t+1}, \dots, \bar{\sigma}^T | o_1(\tilde{h}^t), \mathcal{W}_1^t) > V_1^{t+1}(\bar{\sigma}^{t+1}, \dots, \bar{\sigma}^T | o_1(\tilde{h}^t), \mathcal{L}_1^t).$$

The second equality is obvious. The third equality holds since for every a^t such that $b_1^t(a_1^t | \tilde{h}^t) b_2^t(a_2^t | \tilde{h}^t) > 0$ (i.e., for every equilibrium path history (\tilde{h}^t, a^t) we have $V_1^{t+1}(\bar{\sigma}^{t+1}, \dots, \bar{\sigma}^T | o_1(\tilde{h}^t), o_1(a^t)) = U_1^{t+1}(b^{t+1}, \dots, b^T | \tilde{h}^t, a^t)$). This holds since for every $k > t$ and equilibrium-path history $h^k \in H^k$ we have $v_1^k(\omega_1^k | b^k(h^k)) = v_1^k(\omega_1^k | \bar{\sigma}^k)$ for $\omega_1^k \in \{\mathcal{W}_1^k, \mathcal{L}_1^k\}$.

In sum, we have $U_1^t(\dot{b}^t, \dots, \dot{b}^T | \tilde{h}^t) > U_1^t(b^t, \dots, b^T | \tilde{h}^t)$ for equilibrium-path history \tilde{h}^t , which contradicts that (b_1, b_2) is a Nash equilibrium of the supergame. ■

Proof of Theorem 2. For each t let $\bar{\sigma}^t$ be a Nash equilibrium of G^t . Let b be the behavioral strategy in which $b^t(h^t) = \bar{\sigma}^t$ for each t and each $h^t \in H^t$, and let \dot{b}_i be an arbitrary behavioral strategy for Player i . Let $\dot{b} = (\dot{b}_i, b_{-i})$.

Let $P(t)$ be the proposition that “For each history $h^t \in H^t$ we have

$$U_i^t(\dot{b}^t, \dots, \dot{b}^T | h^t) \leq V_i^t(\bar{\sigma}^t, \dots, \bar{\sigma}^T | o_i(h^t)).$$

If $P(t)$ is true for each $i \in N$, then the restriction of b to subgame h^t is a Nash equilibrium of the subgame since

$$U_i^t(\dot{b}^t, \dots, \dot{b}^T | h^t) \leq V_i^t(\bar{\sigma}^t, \dots, \bar{\sigma}^T | o_i(h^t)) = U_i^t(b^t, \dots, b^T | h^t)$$

for each behavioral strategy \dot{b}_i of player i . We show that b is a subgame perfect equilibrium of the supergame by establishing that $P(t)$ is true for each t and each $h^t \in H^t$.

$P(T + 1)$ is trivially true since

$$U_i^{T+1}(h^{T+1}) = u_i(o_i(h^{T+1})) = V_i^{T+1}(o_i(h^{T+1})).$$

Assume that $P(t+1)$ is true for $t \leq T$, i.e., for each history $h^{t+1} \in H^{t+1}$ we have

$$U_i^{t+1}(b^{t+1}, \dots, b^T | h^{t+1}) \leq V_i^{t+1}(\bar{\sigma}^{t+1}, \dots, \bar{\sigma}^T | o_i(h^{t+1})).$$

We show that $P(t)$ is true. For any $h^t \in H^t$ we have that

$$\begin{aligned} U_i^t(b^t, \dots, b^T | h^t) &= \sum_{a^t \in A^t} \dot{b}_i^t(a_i^t | h^t) \bar{\sigma}_{-i}^t(a_{-i}^t) U_i^{t+1}(b^{t+1}, \dots, b^T | h^t, a^t) \\ &\leq \sum_{a^t \in A^t} \dot{b}_i^t(a_i^t | h^t) \bar{\sigma}_{-i}^t(a_{-i}^t) V_i^{t+1}(\bar{\sigma}^{t+1}, \dots, \bar{\sigma}^T | o_i(h^t), o_i^t(a^t)) \\ &= \sum_{\omega_i^t \in \{\mathcal{W}_i^t, \mathcal{L}_i^t\}} V_i^{t+1}(\bar{\sigma}^{t+1}, \dots, \bar{\sigma}^T | o_i(h^t), \omega_i^t) \sum_{\{a^t \in A^t | o_i^t(a^t) = \omega_i^t\}} \dot{b}_i^t(a_i^t | h^t) \bar{\sigma}_{-i}^t(a_{-i}^t) \\ &\leq \sum_{\omega_i^t \in \{\mathcal{W}_i^t, \mathcal{L}_i^t\}} V_i^{t+1}(\bar{\sigma}^{t+1}, \dots, \bar{\sigma}^T | o_i(h^t), \omega_i^t) \sum_{\{a^t \in A^t | o_i^t(a^t) = \omega_i^t\}} \bar{\sigma}_i^t(a_i^t) \bar{\sigma}_{-i}^t(a_{-i}^t) \\ &= \sum_{\omega_i^t \in \{\mathcal{W}_i^t, \mathcal{L}_i^t\}} V_i^t(\bar{\sigma}^t, \dots, \bar{\sigma}^T | o_i(h^t)). \end{aligned}$$

The first equality follows from the definition of $U_i^t(b^t, \dots, b^T | h^t)$ and since $b_{-i}(a_{-i} | h^t) = \bar{\sigma}_{-i}^t(a_{-i})$. The first inequality follows from the induction hypothesis. The second follows since $\bar{\sigma}^t$ is a Nash equilibrium of G^t implies

$$\sum_{\{a^t \in A^t | o_i^t(a^t) = \mathcal{W}_i^t\}} \dot{b}_i^t(a_i^t | h^t) \bar{\sigma}_{-i}^t(a_{-i}^t) \leq \sum_{\{a^t \in A^t | o_i^t(a^t) = \mathcal{W}_i^t\}} \bar{\sigma}_i^t(a_i^t) \bar{\sigma}_{-i}^t(a_{-i}^t),$$

and since $V_i^{t+1}(\bar{\sigma}^{t+1}, \dots, \bar{\sigma}^T | o_i(h^t), \mathcal{W}_i^t) > V_i^{t+1}(\bar{\sigma}^{t+1}, \dots, \bar{\sigma}^T | o_i(h^t), \mathcal{L}_i^t)$. ■

Proof of Theorem 3. Let b be a subgame perfect equilibrium of the supergame derived from $\{G^t\}_{t=1}^T$ and $(u_i)_{i \in N}$. For each t let $\bar{\sigma}^t$ be the (unique) Nash equilibrium of G^t . Suppose contrary to the Theorem that there is some t and some history h^t such that $b^t(h^t)$ is not a Nash equilibrium of G^t . Let t be the last such period and let \tilde{h}^t be such that $b^t(\tilde{h}^t)$ is not a Nash equilibrium of G^t .

Then for each $h^{t+1} \in H^{t+1}$ we have

$$U_i^{t+1}(b^{t+1}, \dots, b^T | h^{t+1}) = V_i^{t+1}(\bar{\sigma}^{t+1}, \dots, \bar{\sigma}^T | o_i^1(a^1), \dots, o_i^t(a^t)). \quad (10)$$

Since $b^t(\tilde{h}^t)$ is not a Nash equilibrium of G^t , there is a player i who has a strategy \hat{o}_i^t such that

$$\sum_{\{a^t \in A^t | o_i^t(a^t) = \mathcal{W}_i^t\}} \hat{o}_i^t(a_i^t) b_{-i}^t(a_{-i}^t | \tilde{h}^t) > \sum_{\{a^t \in A^t | o_i^t(a^t) = \mathcal{W}_i^t\}} b_i^t(a_i^t | \tilde{h}^t) b_{-i}^t(a_{-i}^t | \tilde{h}^t). \quad (11)$$

We show that (b^t, \dots, b^T) is not a Nash equilibrium in subgame \tilde{h}^t . Consider the strategy \bar{b}_i in which player 1 follows b_i except for history \tilde{h}^t , in which case he follows $\hat{\sigma}_i^t$. Let $\bar{b} = (\bar{b}_i, b_{-i})$. Abusing notation, for $h^t = (a^1, \dots, a^{t-1})$ we write $o_i(h^t)$ for the vector $(o_i^1(a^1), \dots, o_i^{t-1}(a^{t-1}))$. We show that $U_i^t(\bar{b}^t, \dots, \bar{b}^T | \tilde{h}^t) > U_i^t(b^t, \dots, b^T | \tilde{h}^t)$, and therefore b is not a subgame perfect equilibrium. We have that

$$\begin{aligned} U_i^t(\bar{b}^t, \dots, \bar{b}^T | \tilde{h}^t) &= \sum_{a^t \in A^t} \hat{\sigma}_i^t(a_i^t) b_{-i}^t(a_{-i}^t | \tilde{h}^t) U_i^{t+1}(\bar{b}_i^{t+1}, b_{-i}^{t+1}, \dots, \bar{b}_i^T, b_{-i}^T | \tilde{h}^t, a^t) \\ &= \sum_{a^t \in A^t} \hat{\sigma}_i^t(a_i^t) b_{-i}^t(a_{-i}^t | \tilde{h}^t) V_i^{t+1}(\bar{\sigma}^{t+1}, \dots, \bar{\sigma}^T | o_i(\tilde{h}^t), o_i^t(a^t)) \\ &= \sum_{\omega_i^t \in \{\mathcal{W}_i^t, \mathcal{L}_i^t\}} V_i^{t+1}(\bar{\sigma}^{t+1}, \dots, \bar{\sigma}^T | o_i(\tilde{h}^t), \omega_i^t) \sum_{a^t \in A^t | o_i^t(a^t) = \omega_i^t} \hat{\sigma}_i^t(a_i^t) b_{-i}^t(a_{-i}^t | \tilde{h}^t) \\ &> \sum_{\omega_i^t \in \{\mathcal{W}_i^t, \mathcal{L}_i^t\}} V_i^{t+1}(\bar{\sigma}^{t+1}, \dots, \bar{\sigma}^T | o_i(\tilde{h}^t), \omega_i^t) \sum_{a^t \in A^t | o_1^t(a^t) = \omega_1^t} b_i^t(a_i^t | \tilde{h}^t) b_{-i}^t(a_{-i}^t | \tilde{h}^t) \\ &= U_i^t(b^t, \dots, b^T | \tilde{h}^t). \end{aligned}$$

The first equality follows from the definition of $U_i^t(\bar{b}^t, \dots, \bar{b}^T | \tilde{h}^t)$. The second equality follows from (10). The third equality is obvious. The strict inequality follows from (11) and, by Lemma 1, that

$$V_i^{t+1}(\bar{\sigma}^{t+1}, \dots, \bar{\sigma}^T | o_i(\tilde{h}^t), \mathcal{W}_i^t) > V_i^{t+1}(\bar{\sigma}^{t+1}, \dots, \bar{\sigma}^T | o_i(\tilde{h}^t), \mathcal{L}_i^t).$$

The last equality also follows from (10). Hence

$$U_i^t(\bar{b}^t, \dots, \bar{b}^T | \tilde{h}^t) > U_i^t(b^t, \dots, b^T | \tilde{h}^t),$$

which contradicts that (b_1, b_2) induces a Nash equilibrium in subgame \tilde{h}^t . This establishes the theorem. ■

Proof of Theorem 4. For each t let $\bar{\sigma}^t = (\bar{\sigma}_1^t, \bar{\sigma}_2^t)$ be a profile of maximin strategies for G^t . For each player i , each t , and every history $h^t \in H^t$, let \bar{b}_i be the behavioral strategy which satisfies $\bar{b}_i^t(h^t) = \bar{\sigma}_i^t$. We show that \bar{b}_1 is a maximin behavioral strategy for player 1. (The symmetric argument establishes that \bar{b}_2 is a maximin behavioral strategy for Player 2.)

For each stage game G^t , define player 2's best response correspondence in the usual way, i.e., for each $\sigma_1^t \in \Delta A_1^t$ define

$$BR_2^t(\sigma_1^t) = \arg \max_{\sigma_2^t \in \Delta A_2^t} \sum_{\{a^t \in A^t | o_2(a^t) = \mathcal{W}_2^t\}} \sigma_1^t(a_1^t) \sigma_2^t(a_2^t).$$

Note that for $\sigma_1^t \in \Delta A_1^t$ and $\sigma_2^t \in BR_2^t(\sigma_1^t)$, since G^t is strictly competitive we have

$$\sum_{\{a^t \in A^t | o_1(a^t) = \mathcal{W}_1^t\}} \sigma_1^t(a_1^t) \sigma_2^t(a_2^t) \leq \sum_{\{a^t \in A^t | o_1(a^t) = \mathcal{W}_1^t\}} \sigma_1^t(a_1^t) \bar{\sigma}_2^t(a_2^t).$$

Furthermore, since $(\bar{\sigma}_1^t, \bar{\sigma}_2^t)$ is a Nash equilibrium (by the Minimax Theorem), we have

$$\sum_{\{a^t \in A^t | o_1(a^t) = \mathcal{W}_1^t\}} \sigma_1^t(a_1^t) \bar{\sigma}_2^t(a_2^t) \leq \sum_{\{a^t \in A^t | o_1(a^t) = \mathcal{W}_1^t\}} \bar{\sigma}_1^t(a_1^t) \bar{\sigma}_2^t(a_2^t).$$

These two inequalities imply, for $\sigma_1^t \in \Delta A_1^t$ and $\sigma_2^t \in BR_2^t(\sigma_1^t)$, that

$$\sum_{\{a^t \in A^t | o_1(a^t) = \mathcal{W}_1^t\}} \sigma_1^t(a_1^t) \sigma_2^t(a_2^t) \leq \sum_{\{a^t \in A^t | o_1(a^t) = \mathcal{W}_1^t\}} \bar{\sigma}_1^t(a_1^t) \bar{\sigma}_2^t(a_2^t). \quad (12)$$

For each behavioral strategy b_1 , define the set $MBR_2(b_1)$ of “myopic” best responses for player 2 as

$$MBR_2(b_1) = \{b_2 \in B_2 | \text{for each } t \text{ and } h^t \in H^t : b_2^t(h^t) \in BR_2^t(b_1^t(h^t))\}.$$

If $b_2 \in MBR_2(b_1)$, then at each history player 2’s strategy calls for him to maximize his probability of winning at the current stage given player 1’s mixture over actions. Clearly MBR_2 is a nonempty correspondence.

It is straightforward to see that $\bar{V}_1 = V_1^1(\bar{\sigma}^1, \dots, \bar{\sigma}^T)$. Let $b_1 \in B_1$ and $b_2 \in MBR_2(b_1)$. We first establish that

$$U_1^1(b_1, b_2 | h^1) \leq \bar{V}_1,$$

i.e., for any behavioral strategy b_1 that player 1 follows, player 2 can hold player 1 to a payoff of \bar{V}_1 or less by following a myopic best response.

For $t \leq T+1$, let $P(t)$ be the proposition that “If $b_1 \in B_1$, $b_2 \in MBR_2(b_1)$, and $h^t = (a^1, \dots, a^{t-1}) \in H^t$, then

$$U_1^t(b_1^t, b_2^t, \dots, b_1^T, b_2^T | h^t) \leq V_1^t(\bar{\sigma}^t, \dots, \bar{\sigma}^T | o_1(h^t)).$$

We first show that $P(T+1)$ is true. Let $b_1 \in B_1$, let $b_2 \in MBR_2(b_1)$, let $h^{T+1} = (a^1, \dots, a^T) \in H^{T+1}$. We have that

$$U_1^{T+1}(h^{T+1}) = u_1(o_1^1(a^1), \dots, o_1^T(a^T)) = V_1^{T+1}(o_1(h^{T+1})),$$

which establishes $P(T+1)$. Assume that $P(t+1)$ is true for $t \leq T$. We show that $P(t)$ is true. Let $b_1 \in B_1$, let $b_2 \in MBR_2(b_1)$, let $h^t = (a^1, \dots, a^{t-1}) \in H^t$. Denote (b_1^t, b_2^t) by b^t . We have

$$U_1^t(b^t, \dots, b^T | h^t)$$

$$= \sum_{a^t \in A^t} b_1^t(a_1^t | h^t) b_2^t(a_2^t | h^t) U_1^{t+1}(b^{t+1}, \dots, b^T | h^t, a^t)$$

$$\leq \sum_{a^t \in A^t} b_1^t(a_1^t | h^t) b_2^t(a_2^t | h^t) V_1^{t+1}(\bar{\sigma}^{t+1}, \dots, \bar{\sigma}^T | o_1(h^t), o_1^t(a^t))$$

$$= \sum_{\omega_1^t \in \{\mathcal{W}_1^t, \mathcal{L}_1^t\}} V_1^{t+1}(\bar{\sigma}^{t+1}, \dots, \bar{\sigma}^T | o_1(h^t), \omega_1^t)$$

$$\begin{aligned}
& \times \sum_{\{a^t \in A^t | o_1^t(a^t) = \omega_1^t\}} b_1^t(a_1^t|h^t) b_2^t(a_2^t|h^t) \\
& \leq \sum_{\omega_1^t \in \{\mathcal{W}_1^t, \mathcal{L}_1^t\}} V_1^{t+1}(\bar{\sigma}^{t+1}, \dots, \bar{\sigma}^T | o_1(h^t), \omega_1^t) \sum_{\{a^t \in A^t | o_1^t(a^t) = \omega_1^t\}} \bar{\sigma}_1^t(a_1^t) \bar{\sigma}_2^t(a_2^t) \\
& = V_1^t(\bar{\sigma}^t, \dots, \bar{\sigma}^T | o_1(h^t)).
\end{aligned}$$

The first inequality follows from the induction hypothesis. Since b_2 is a myopic best response to b_1 , we have that $b_2^t(h^t) \in BR_2^t(b_1^t(h^t))$ and hence

$$\sum_{\{a^t \in A^t | o_1(a^t) = \mathcal{W}_1^t\}} b_1^t(a_1^t|h^t) b_2^t(a_2^t|h^t) \leq \sum_{\{a^t \in A^t | o_1(a^t) = \mathcal{W}_1^t\}} \bar{\sigma}_1^t(a_1^t) \bar{\sigma}_2^t(a_2^t),$$

i.e., (12) holds replacing σ_1^t with $b_1^t(a_1^t|h^t)$ and σ_2^t with $b_2^t(a_2^t|h^t)$. Furthermore, by Lemma 1

$$V_1^{t+1}(\bar{\sigma}^{t+1}, \dots, \bar{\sigma}^T | o_1(h^t), \mathcal{W}_1^t) > V_1^{t+1}(\bar{\sigma}^{t+1}, \dots, \bar{\sigma}^T | o_1(h^t), \mathcal{L}_1^t).$$

Hence the second inequality holds. This establishes that $P(t)$ is true.

We have, in particular, for $t = 1$ that

$$U_1^1(b^1, \dots, b^T | h^1) \leq V_1^1(\bar{\sigma}^1, \dots, \bar{\sigma}^T) = \bar{V}_1.$$

Hence, for any behavioral strategy b_1 for player 1, a myopic best response to b_1 by player 2 holds player 1 to a payoff of \bar{V}_1 or less. By the Continuation-Payoff Lemma we have for every b_2 that

$$U_1^1(\bar{b}_1, b_2 | h^1) \geq V_1^1(\bar{\sigma}^1, \dots, \bar{\sigma}^T) = \bar{V}_1,$$

i.e., with the behavioral strategy \bar{b}_1 player 1 can guarantee himself a payoff of \bar{V}_1 . Hence \bar{b}_1 is a maximin behavioral strategy for player 1. ■

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