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Matching and bargaining models of markets: approximating small markets by large markets★

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Summary. We show that the equilibrium of a matching and bargaining model of a market in which there is a finite number of agents at each date need not be near the equilibrium of a market with a continuum of agents, although matching probabilities are the same in both markets. Holding the matching process fixed, as the finite market becomes large its equilibrium approaches the equilibrium of its continuum limit.

JEL Classification Numbers: C73, C78.

1 Introduction

In some recently developed market models trade is conducted through sequential pairwise random matching between agents who bargain strategically over prices. These models provide a means of determining whether decentralized trading processes lead to nearly competitive outcomes when trading frictions are small (see, e.g., Gale (1987) and Rubinstein and Wolinsky (1985)), and explain how bargaining procedures, rates of impatience, and outside options affect market outcomes (see Wolinsky (1987)).

Our focus is on markets where trade is carried out through matching and bargaining and where, at each date $t = 0, 1, \dots,$ an exogenous finite number of agents enters. We refer to such markets as "small." In small markets, an agent's decision to force a match to end without trade causes the number of agents in the market at subsequent dates to be larger than it would have been. Since matching probabilities are determined by the number of agents, an individual's decision whether or not to trade also influences matching probabilities at subsequent dates.

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In the present paper, we consider two questions. First, is the equilibrium of a small market near the equilibrium of a limit market with a continuum of agents? The existing literature on markets in which an infinite measure of agents eventually enters assumes that there is a continuum of agents (or that the market behaves as if there were). In that case, whether or not a given match ends with trade has no effect on the measure of each type of agent in the market at the next date.[1] We refer to markets with this property as "large." The second question of interest is whether the equilibrium of a small market converges to the equilibrium of a limit market with a continuum of agents as the small market becomes large.

Similar types of questions have been intensively investigated for exchange economies. Anderson (1978) gives a bound on the non-competitiveness of core allocations in a fixed finite economy. Hildenbrand (1974) gives conditions under which the core converges to the competitive equilibrium as a finite economy converges to its continuum limit. In the present paper, in the context of a matching and bargaining model, we show that there is no analog to the Anderson-type bound on the difference between the equilibrium of a small market and the equilibrium of the associated large market. We also obtain a Hildenbrand-like result on the convergence of small market equilibria to large market equilibria.

To address the questions we pose on nearness and convergence of equilibria, we study a small market for a one-parameter family of matching processes. At each date, n buyers and n sellers enter and there is an initial excess of $\Delta > 0$ sellers. We find that the equilibrium of the small market need not be near the equilibrium of the associated large market. In particular, for any share of the gains to trade arbitrarily close to but less than $\frac{1}{2}$, there is *some* matching process in the family such that in the equilibrium of the small market sellers obtain at least this share of the gains to trade as the discount factor approaches 1. (This is the case even though n and Δ may be arbitrarily large.) In contrast, in the associated large market sellers obtain a share of the surplus $\frac{n}{2n+\Delta}$, which is less than $\frac{1}{2}$, as the discount factor approaches 1.

When comparing the equilibria of small and large markets, we compare markets in which the matching probability of each type of agent is the same across markets. Therefore, given a small market and a matching process the associated large market is one with matching probabilities equal to the equilibrium-path matching probabilities of the small market. Thus, differences in equilibria arise solely from the ability of agents in small markets to influence matching probabilities.

In answer to the convergence question, we find that as the small market becomes large, holding the matching process fixed, the small market equilibrium converges to the equilibrium of a large market.

[1] In Rubinstein and Wolinsky (1985), for example, matching probabilities are a primitive of the model. Osborne and Rubinstein (1990), p. 141, in their discussion of this model, assert that taking matching probabilities as a primitive is appropriate for large markets.

2 Small and large markets

Our model of a small market specifies the following elements: (i) the entry process of agents, (ii) the agents' preferences and endowments, (iii) the matching technology, (iv) the bargaining game played by matched agents, (v) the agents' information about their own market experience, and (vi) the agents' information about the number of buyers and sellers in the market.

2.1 Small markets

At date zero a small market has n buyers and $n + \Delta$ sellers. At each subsequent date a stream of n buyers and n sellers enters the market. An agent who enters at date T is said to be a generation T agent. We consider the case where $\Delta > 0$. The case where there is initially an excess of buyers is symmetric.

In a small market each seller is endowed with a single indivisible unit of the good and has a reservation price of zero. Each buyer is endowed with a unit of money and demands a single unit of the indivisible good with a reservation price of one. Thus, there is a one-unit gain to trade in any match. Agents are von Neumann-Morgenstern expected utility maximizers and discount the future using the discount factor $\delta < 1$. A generation T agent who receives a share z of the unit gain to trade at date $t \geq T$ has utility $\delta^{t-T} z$.

Of the agents in the market following entry, n buyers (and n sellers) are new entrants and k buyers (and $k + \Delta$ sellers) remain from previous periods. The matching process, parameterized by an integer $k^* \geq 0$, operates as follows: when $n + k$ buyers (and, therefore, $n + k + \Delta$ sellers) are in the market, the process produces n matches if $k \leq k^*$ and $n + k - k^*$ matches if $k > k^*$. The matching probabilities of sellers and buyers when $n + k$ buyers are in the market are denoted by α_k and β_k, respectively, where

$$\alpha_k = \begin{cases} \frac{n}{n+\Delta+k} & \text{if } k \leq k^* \\ \frac{n+k-k^*}{n+\Delta+k} & \text{if } k > k^*, \end{cases}$$

and

$$\beta_k = \begin{cases} \frac{n}{n+k} & \text{if } k \leq k^* \\ \frac{n+k-k^*}{n+k} & \text{if } k > k^*. \end{cases}$$

When $k^* = 0$ then all the agents on the short side of the market are matched. At the other extreme, were $k^* = \infty$, then n matches would be produced regardless of the number of agents in the market.[2] If k^* matches have ended without trade, then the initial matching probability of sellers and buyers α_0 and β_0 are reduced to α_{k^*} and β_{k^*}, respectively. As will become evident, k^* indexes the manipulability of the matching probabilities, matching probabilities being more manipulable as k^* grows.

[2] The case $k^* = \infty$ is illustrative only, since we assume throughout that k^* is finite.

Once a buyer and seller are (anonymously) matched, each has probability one half of being selected as proposer. A proposer offers to his partner a share of the unit gain to trade. If his partner agrees, the surplus is divided as proposed and both exit. If his partner disagrees, then no trade takes place, the match is broken, and both agents remain in the market at the next date. Each agent has perfect recall of his own market experience, i.e., whether or not he was matched at each date and in each match his own and his partner's play of the bargaining game. In addition, at each date each agent in the market is informed of the number of buyers and sellers in the market, and thus at each date each agent knows his probability of matching.

Small markets differ only in their matching technology. We refer to the small market with matching technology k^* as *small market* k^*.

2.2 Large markets

In a large market the agents' preferences and endowments, the bargaining game played by matched agents, and the agents' information about their own market experience are the same as described above for small markets. Unlike in a small market, in a large market the matching probabilities of sellers and buyers, denoted by α and β, respectively, are exogenously specified. This completes the description of a large market.

A large market can be viewed as one in which there is a stock of agents of each type in the market and a matching technology which maps stocks into matching probabilities (neither the stocks nor the matching technology are explicitly modeled). The restriction to stationary matching probabilities is justified when there is a continuum of agents in the market (and so an agent cannot influence matching probabilities) and the market is in a steady state.[3]

As we focus on the effects of the ability of agents in small markets to influence matching probabilities, when comparing equilibrium outcomes in a small and large market, we compare markets in which the equilibrium-path matching probabilities of the small market are the same as the (exogenously specified) matching probabilities in the large market. Therefore, for a small market it is useful to introduce the notion of its associated large market, the comparison of equilibria to be made between these two markets. Given a small market k^*, its **associated large market** is the large market which has as its matching probability for sellers (buyers) the equilibrium-path matching probability of sellers (buyers) in small market k^*.

[3] Rubinstein and Wolinsky (1985) models a large market in which the players have rich bargaining opportunities.

3 Nearness of equilibria of small and large markets

An agent's strategy is semi-stationary if the agent's offer, when selected to propose, depends only on the number of buyers in the market and the agent's response to an offer depends only on the number of buyers and the offer received. For each k^*, small market k^* has a unique market equilibrium in semi-stationary strategies, and in this equilibrium each match ends with trade.[4] We omit the game theoretic details of this argument as they contribute no insight into the questions of interest here. (The argument can be obtained by adapting the proof in Wooders (1994) of the same result for a slightly different model.) Instead we focus on the value equations characterizing the market equilibrium.

For small market k^* let $V_i^k(k^*)$ denote the value to an agent of type i when the agent is unmatched, prior to random matching, and $n+k$ buyers are in the market. Suppressing the dependence of V_i^k on k^*, the value equations for small market k^* are

$$\text{for } 0 \leq k \leq k^*: \; V_S^k = \alpha_k \left[\frac{1}{2}\left(1-\delta V_B^{k+1}\right) + \frac{1}{2}\delta V_S^{k+1}\right] + \left(1-\alpha_k\right)\delta V_S^k$$
$$V_B^k = \beta_k \left[\frac{1}{2}\left(1-\delta V_S^{k+1}\right) + \frac{1}{2}\delta V_B^{k+1}\right] + \left(1-\beta_k\right)\delta V_B^k,$$

and

$$\text{for } k > k^*: \quad V_S^k = \alpha_k \left[\frac{1}{2}\left(1-\delta V_B^{k^*+1}\right) + \frac{1}{2}\delta V_S^{k^*+1}\right] + \left(1-\alpha_k\right)\delta V_S^{k^*}$$
$$V_B^k = \beta_k \left[\frac{1}{2}\left(1-\delta V_S^{k^*+1}\right) + \frac{1}{2}\delta V_B^{k^*+1}\right] + \left(1-\beta_k\right)\delta V_B^{k^*}. \quad (1)$$

These equations indicate that if there are $n+k$ buyers in the market and $k \leq k^*$ then there are n matches, and $n+k+1$ buyers in the market at the next period if exactly one of these matches ends in disagreement. If there are $n+k$ buyers and $k > k^*$ then there are $n+k-k^*$ matches, and $n+k^*+1$ buyers in the market at the next period if exactly one of these matches ends in disagreement. Thus if there are $n+k$ buyers in the market, a matched seller's disagreement payoff is δV_S^{k+1} if $k \leq k^*$ and $\delta V_S^{k^*+1}$ if $k > k^*$.

Although values are defined by an *infinite* system of simultaneous linear equations, one can characterize the solution by focusing on the four equations for $k = k^*$ and $k = k^*+1$. Solving, one obtains

$$V_S^{k^*}(k^*) = \frac{\alpha_{k^*}(1+\delta B_{k^*})}{\delta[\alpha_{k^*}(1+\delta B_{k^*}) + \beta_{k^*}(1+\delta A_{k^*})] + (1-\delta)[2+\delta A_{k^*}+\delta B_{k^*}]}, \quad (2)$$

[4] Informally, a market equilibrium is a pair of semi-stationary strategies (f_*, g_*) such that for each seller the strategy f_* is optimal for all possible histories when all other sellers employ f_* and all buyers employ g_* (the symmetric statement must hold for buyers). For a formal definition see Rubinstein and Wolinsky (1985), or see Wooders (1994) for an adaptation of their definition to small markets.

and

$$V_B^{k^*}(k^*) = \frac{\beta_{k^*}(1+\delta A_{k^*})}{\delta[\alpha_{k^*}(1+\delta B_{k^*}) + \beta_{k^*}(1+\delta A_{k^*})] + (1-\delta)[2+\delta A_{k^*} + \delta B_{k^*}]}, \quad (3)$$

where $A_{k^*} = \alpha_{k^*} - \alpha_{k^*+1}$ and $B_{k^*} = \beta_{k^*} - \beta_{k^*+1}$.[5]

On the equilibrium path, at each date zero buyers remain in the market from previous periods since each buyer is matched and each match ends with trade. Therefore, the equilibrium payoffs to sellers and buyers are $V_S^0(k^*)$ and $V_B^0(k^*)$, respectively. On the equilibrium path, sellers when selected to propose offer $\delta V_B^1(k^*)$ and buyers when selected to propose offer $\delta V_S^1(k^*)$. If there are $n+k$ buyers in the market and $k > 0$, then the market has moved off the equilibrium path. In this case, sellers offer $\delta V_B^{k+1}(k^*)$ and buyers offer $\delta V_S^{k+1}(k^*)$.

Given (2), $V_S^0(k^*)$ can be computed recursively using

$$V_S^k(k^*) = \frac{\frac{1}{2}\alpha_k}{1-\delta(1-\alpha_k)}\left(1 - \delta V_B^{k+1}(k^*) + \delta V_S^{k+1}(k^*)\right), \quad (4)$$

which follows from (1) for $k \leq k^*$. The equilibrium payoff of buyers $V_B^0(k^*)$ is computed in the same fashion. It is immediate that when the discount factor is positive but less than one, $V_S^0(k^*)$ and $V_B^0(k^*)$ depend not only on equilibrium-path matching probabilities α_0 and β_0, but also on the off-the-equilibrium-path matching probabilities $\alpha_1, \ldots, \alpha_{k^*+1}$ and $\beta_1, \ldots, \beta_{k^*+1}$.

Proposition 1, which follows, shows that as the discount factor approaches 1, in the limit equilibrium payoffs depend only on off-the-equilibrium-path matching probabilities.

Proposition 1. *For each k^*, as the discount factor approaches* 1, *equilibrium payoffs in small market k^* depend, in the limit, only on the off-the-equilibrium-path-matching probabilities α_{k^*}, α_{k^*+1}, β_{k^*}, and β_{k^*+1}. In particular,* $\lim_{\delta\to 1} V_S^0(k^*) = \frac{\alpha_{k^*}(1+B_{k^*})}{\alpha_{k^*}(1+B_{k^*})+\beta_{k^*}(1+A_{k^*})}$ *and* $\lim_{\delta\to 1} V_B^0(k^*) = \frac{\beta_{k^*}(1+A_{k^*})}{\alpha_{k^*}(1+B_{k^*})+\beta_{k^*}(1+A_{k^*})}$.

Proof. Appendix.

To understand this result it is useful to recall that in a large market in which sellers and buyers are matched with probability α and β, respectively, the equilibrium payoff to sellers as frictions vanish is[6]

$$\frac{\alpha}{\alpha+\beta} = \frac{1}{1+\frac{\beta}{\alpha}}. \quad (5)$$

That is, as frictions vanish payoffs depend upon the ratio of matching probabilities, not their magnitudes. In small market k^*, by forcing k^* of his matches to end without trade, a seller permanently reduces the matching

[5] For completeness, we add that $V_S^{k^*+1}(k^*) = \frac{(\alpha_{k^*+1}+\delta A_{k^*})(1+\delta B_{k^*})}{\delta[\alpha_{k^*}(1+\delta B_{k^*})+\beta_{k^*}(1+\delta A_{k^*})]+(1-\delta)[2+\delta A_{k^*}+\delta B_{k^*}]}$ and $V_B^{k^*+1}(k^*) = \frac{(\beta_{k^*+1}+\delta B_{k^*})(1+\delta A_{k^*})}{\delta[\alpha_{k^*}(1+\delta B_{k^*})+\beta_{k^*}(1+\delta A_{k^*})]+(1-\delta)[2+\delta A_{k^*}+\delta B_{k^*}]}$.

[6] Equilibrium payoffs in the large market are characterized by the equations $V_S = \alpha\left[\frac{1}{2}(1-\delta V_B) + \frac{1}{2}\delta V_S\right] + (1-\alpha)\delta V_S$ and $V_B = \beta\left[\frac{1}{2}(1-\delta V_S) + \frac{1}{2}\delta V_B\right] + (1-\beta)\delta V_B$.

probability of sellers from α_0 to α_{k^*} and of buyers from β_0 to β_{k^*} (when all future matches end with trade). This also reduces the ratio of the matching probabilities from $\frac{\beta_0}{\alpha_0}$ to $\frac{\beta_{k^*}}{\alpha_{k^*}}$, which from (5) increases the seller's payoff. In a market equilibrium sellers receive offers sufficiently large so that such manipulation is not carried out. Thus in a small market, as frictions vanish, it is the matching probabilities that would prevail after manipulation that determine payoffs, not equilibrium-path matching probabilities.

We now show that the equilibrium of a small market with an arbitrary but finite number of agents need not be near the equilibrium of its associated large market. For each k^*, in small market k^* the equilibrium-path matching probability of sellers is $\alpha_0 = \frac{n}{n+\Delta} < 1$ and of buyers is $\beta_0 = 1$. Thus, every small market has the same associated large market. In this large market, the equilibrium payoff of sellers is $\frac{\alpha_0}{2-\delta(1-\alpha_0)}$ and of buyers is $\frac{1}{2-\delta(1-\alpha_0)}$. Corollary 1 shows that there are small markets whose equilibrium payoffs differ from equilibrium payoffs in the associated large market by an amount arbitrarily close to $\frac{1}{2} - \frac{\alpha_0}{1+\alpha_0}$.

Corollary 1. *Let ϵ be such that $0 < \epsilon < \frac{1}{2} - \frac{\alpha_0}{1+\alpha_0}$, but otherwise be arbitrary. There is a K such that $k^* \geq K$ implies the difference between equilibrium payoffs in small market k^* and equilibrium payoffs in its associated large market is more than ϵ as the discount factor approaches 1.*

Proof. Appendix.

We conclude that as frictions vanish, the equilibrium of a small market need not be near the equilibrium of the associated large market. The potential difference of the equilibria is increasing as α_0 decreases, or equivalently as the initial excess of sellers grows.

4 Convergence of equilibria

We now address the question of whether the equilibrium of a small market converges to the equilibrium of a large market as the small market becomes large. Our approach is to study the equilibria of replica markets as the number of replications becomes large. In the r^{th} replication of a small market, rn buyers and $rn + r\Delta$ sellers enter at date zero, and at each subsequent date rn buyers and rn sellers enter. For small market k^*, the matching process is held fixed across replicas in the sense that it produces a number of matches equal to the number of entrants when no more than k^* buyers remain from previous periods, and it produces an additional $k - k^*$ matches when $k > k^*$ buyers remain. In particular, in the r^{th} replica of small market k^*, when $rn + k$ buyers are in the market, the matching process produces rn matches if $k \leq k^*$ and $rn + k - k^*$ matches if $k > k^*$. For small market k^*, let $V_i^{k,r}(k^*)$ denote the value to an agent of type i in the r^{th} replica when $rn + k$ buyers are in the market.

In small market k^* and its replicas, on the equilibrium path each match ends with trade; thus in the r^{th} replica the equilibrium payoff of an agent of type i is $V_i^{0,r}(k^*)$. The equilibrium-path matching probability of sellers and buyers in the r^{th} replica is $\alpha_0 = \frac{n}{n+\Delta}$ and $\beta_0 = 1$, respectively. Since these probabilities are independent of r and k^*, each small market and its replicas has the same associated large market. Proposition 2 shows that as r becomes large, equilibrium-path payoffs in the r^{th} replica approach the payoffs in the associated large market.

Proposition 2. *For each k^*, as r becomes large, equilibrium payoffs in the r^{th} replica of small market k^* approach equilibrium payoffs in the associated large market. In particular,* $\lim_{r\to\infty} V_S^{0,r}(k^*) = \frac{\alpha_0}{2-\delta(1-\alpha_0)}$ *and* $\lim_{r\to\infty} V_B^{0,r}(k^*) = \frac{1}{2-\delta(1-\alpha_0)}$.

Proof. Appendix.

This result follows from the fact that as a small market becomes large, matching probabilities become non-manipulable. In the r^{th} replica, by forcing k^* of his matches to end without trade, a seller reduces the ratio of the matching probabilities to

$$\frac{\beta_{k^*}}{\alpha_{k^*}} = \frac{rn/(rn+k^*)}{rn/(rn+r\Delta+k^*)}.$$

As r becomes large this ratio approaches $\frac{\beta_0}{\alpha_0}$, the equilibrium-path ratio of matching probabilities.

5 Conclusion

We have shown that the equilibrium of a small market need not be near the equilibrium of its associated large market. This result is a consequence of the fact that the equilibrium of a small market depends on the matching process in a way that the equilibrium of a large market does not. In particular, the equilibrium of a small market depends on the matching probabilities that prevail both on and off the equilibrium path. Moreover, as the discount factor approaches one, the equilibrium depends *only* on off-the-equilibrium-path matching probabilities. (This contrasts sharply with the situation in Rubinstein and Wolinsky (1985), where the equilibrium characterization depends only on the steady state equilibrium matching probability of each type of agent.) Thus, the equilibrium of a small market is not near the equilibrium of its associated large market when equilibrium-path matching probabilities are not near the matching probabilities that prevail off-the-equilibrium-path.

Of course, only equilibrium-path matching probabilities are ever observed. Thus, two small markets with different matching processes but identical appearances (that is, having the same number of agents and the same equilibrium-path matching probabilities) may have very different equilibria. For this reason, when the matching process is unknown, it may be difficult to determine empirically whether or not behavior in a small market is consistent with the theoretical outcome.

6 Appendix: Proofs

Proof of Proposition 1: That $\lim_{\delta\to 1} V_S^{k^*}(k^*) = \frac{\alpha_{k^*}(1+B_{k^*})}{\alpha_{k^*}(1+B_{k^*})+\beta_{k^*}(1+A_{k^*})}$ and $\lim_{\delta\to 1} V_B^{k^*}(k^*) = \frac{\beta_{k^*}(1+A_{k^*})}{\alpha_{k^*}(1+B_{k^*})+\beta_{k^*}(1+A_{k^*})}$ follows from (2) and (3). For $i \geq 0$, let $P(i)$ be the two-part proposition "$\lim_{\delta\to 1} V_S^i(k^*) = \lim_{\delta\to 1} V_S^{k^*}(k^*)$ and $\lim_{\delta\to 1} V_B^i(k^*) = \lim_{\delta\to 1} V_B^{k^*}(k^*)$." $P(i)$ is trivially true for $i = k^*$.

We show if $P(i)$ is true for some $i \in \{1, \ldots, k^*\}$, then $P(i-1)$ is true. We have from (4) that

$$V_S^{i-1}(k^*) = \frac{\frac{1}{2}\alpha_{i-1}}{1-\delta(1-\alpha_{i-1})}\left(1 - \delta V_B^i(k^*) + \delta V_S^i(k^*)\right).$$

Taking limits gives

$$\lim_{\delta\to 1} V_S^{i-1}(k^*) = \frac{1}{2}\left(1 - \lim_{\delta\to 1} V_B^{k^*}(k^*) + \lim_{\delta\to 1} V_S^{k^*}(k^*)\right) = \lim_{\delta\to 1} V_S^{k^*}(k^*),$$

where the first equality follows from the fact that $P(i)$ is true and the second follows from the fact that $1 - \lim_{\delta\to 1} V_B^{k^*}(k^*) = \lim_{\delta\to 1} V_S^{k^*}(k^*)$. The symmetric argument gives $\lim_{\delta\to 1} V_B^{i-1}(k^*) = \lim_{\delta\to 1} V_B^{k^*}(k^*)$. Therefore, $P(i-1)$ is true. By induction $P(i)$ is for $0 \leq i \leq k^*$. In particular it is true for $i = 0$, which is the result. □

Proof of Corollary 1: We have

$$\lim_{k^*\to\infty}\lim_{\delta\to 1} V_S^0(k^*) = \lim_{k^*\to\infty} \frac{1}{1 + \frac{\beta_{k^*}(1+A_{k^*})}{\alpha_{k^*}(1+B_{k^*})}} = \frac{1}{2},$$

where the first equality follows from Proposition 1 and the second equality follows from the fact that $\lim_{k^*\to\infty} \frac{\beta_{k^*}(1+A_{k^*})}{\alpha_{k^*}(1+B_{k^*})} = 1$. Thus, there is a K_S such that $k^* \geq K_S$ implies $\lim_{\delta\to 1} V_S^0(k^*) > \epsilon + \frac{\alpha_0}{1+\alpha_0}$. Hence, $k^* \geq K_S$ implies $\lim_{\delta\to 1}\left(V_S^0(k^*) - \frac{\alpha_0}{2-\delta(1-\alpha_0)}\right) > \epsilon$.

We also have

$$\lim_{k^*\to\infty}\lim_{\delta\to 1} V_B^0(k^*) = \lim_{k^*\to\infty} \frac{1}{1 + \frac{\alpha_{k^*}(1+B_{k^*})}{\beta_{k^*}(1+A_{k^*})}} = \frac{1}{2}.$$

Note that $\epsilon < \frac{1}{2} - \frac{\alpha_0}{1+\alpha_0}$ implies $\frac{1}{1+\alpha_0} - \epsilon > \frac{1}{2}$. Thus, there is a K_B such that $k^* \geq K_B$ implies $\lim_{\delta\to 1} V_B^0(k^*) < \frac{1}{1+\alpha_0} - \epsilon$. Hence, $k^* \geq K_B$ implies $\lim_{\delta\to 1}\left(\frac{1}{2-\delta(1-\alpha_0)} - V_B^0(k^*)\right) > \epsilon$. Choosing $K \geq \max\{K_S, K_B\}$, the result is obtained. □

Proof of Proposition 2: Let α_k^r and β_k^r denote the matching probabilities of sellers and buyers, respectively, in the r^{th} replica when $rn + k$ buyers are in the market. Then

$$\alpha_k^r = \begin{cases} \frac{rn}{rn+r\Delta+k} & \text{if } k \leq k^* \\ \frac{rn+k-k^*}{rn+r\Delta+k} & \text{if } k > k^*, \end{cases}$$

and for each $k \geq 0$, we have $\lim_{r\to\infty} \alpha_k^r = \alpha_0$. Similarly, for each $k \geq 0$ we have $\lim_{r\to\infty} \beta_k^r = 1$. That $\lim_{r\to\infty} V_S^{k^*,r}(k^*) = \frac{\alpha_0}{2-\delta(1-\alpha_0)}$ and $\lim_{r\to\infty} V_B^{k^*,r}(k^*) = \frac{1}{2-\delta(1-\alpha_0)}$ follows from (2) and (3), where α_k and β_k are replaced by α_k^r and β_k^r in these equations.

For $i \geq 0$, let $P(i)$ be the two-part proposition "$\lim_{r\to\infty} V_S^{i,r}(k^*) = \lim_{r\to\infty} V_S^{k^*,r}(k^*)$ and $\lim_{r\to\infty} V_B^{i,r}(k^*) = \lim_{r\to\infty} V_B^{k^*,r}(k^*)$." $P(i)$ is trivially true for $i = k^*$.

We show if $P(i)$ is true for some $i \in \{1,\ldots,k^*\}$, then $P(i-1)$ is true. We have from (4) that

$$V_S^{i-1,r}(k^*) = \frac{\frac{1}{2}\alpha_{i-1}^r}{1-\delta(1-\alpha_{i-1}^r)}\left(1-\delta V_B^{i,r}(k^*) + \delta V_S^{i,r}(k^*)\right).$$

Taking limits gives

$$\lim_{r\to\infty} V_S^{i-1,r}(k^*) = \frac{\frac{1}{2}\alpha_0}{1-\delta(1-\alpha_0)}\left(1-\frac{\delta}{2-\delta(1-\alpha_0)}+\frac{\delta\alpha_0}{2-\delta(1-\alpha_0)}\right)$$
$$= \frac{\alpha_0}{2-\delta(1-\alpha_0)},$$

where the first equality follows from $\lim_{r\to\infty} \alpha_{i-1}^r = \alpha_0$ and $P(i)$ is true. The symmetric argument gives $\lim_{r\to\infty} V_B^{i-1,r}(k^*) = \lim_{r\to\infty} V_B^{k^*,r}(k^*)$. Therefore, $P(i-1)$ is true. By induction $P(i)$ is true for $0 \leq i \leq k^*$. In particular it is true for $i = 0$, which is the result. □

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