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Walrasian equilibrium in matching models

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Abstract

We analyze trading in a model in which the agents and their preferences are the same as in the main models of matching and bargaining, but in which trade is centralized rather than decentralized. We characterize equilibrium when trade is centralized and, by comparing our results with results from the matching literature, we show conditions under which decentralized trading processes reproduce the allocations of our centralized one. We establish that the competitive price as defined in the matching literature (i.e., relative to the stocks, flows, or totals) coincides, in the appropriate setting, with the equilibrium price in our model. © 1998 Elsevier Science B.V.

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1. Introduction

A fundamental question posed in the matching and bargaining literature is whether decentralized trade leads to competitive outcomes as trading frictions vanish. Here we consider a model in which the agents and their preferences are the same as in the main models of matching and bargaining, but we depart from these models by studying centralized rather than decentralized trade. We characterize equilibrium when trade is centralized and, by comparing our results with results from the matching literature, we show when decentralized trading processes reproduce the allocations of our centralized one.

Our model of centralized trade provides a benchmark for the competitive outcome. Indeed, since in our model equilibrium prices are market clearing, we refer to its equilibria as “Walrasian” equilibria. In the matching literature, different benchmarks have been used to judge whether decentralized trade leads to competitive outcomes (i.e., the outcomes produced when trade is centralized). In models where an infinite measure

of agents enters the market over its lifetime, Rubinstein and Wolinsky [6] define the competitive price relative to the *stocks* of buyers and sellers in the market, while Gale [1] defines it relative to the *flows* of agents entering the market. When a finite measure of agents enters the market, the competitive price has been defined relative to the *totals*.¹ Here we show that the competitive price relative to the stocks, flows, and totals coincides, in the appropriate setting, with the equilibrium price in our model of centralized trade.

We begin by describing the model of [6], henceforth RW, since the current paper is largely inspired by their work. In their model each seller is endowed with a single unit of an indivisible good and has a reservation price of zero, while each buyer demands a single unit of the good with a reservation price of one. Agents discount future gains. The trading process operates as follows. At each date every agent in the market is potentially matched with an agent of the opposite type and, in each match, one agent is randomly selected to propose a price. If the proposer's partner accepts the offered price then the agents trade at that price and exit the market. Otherwise, at the next date, agents are again randomly matched. Each agent who exits is replaced by an agent of the same type so the stock of each type is stationary.

The main result of RW is that in this game's unique equilibrium, if there is an initial excess of sellers, then sellers obtain a positive price even though there is an excess of sellers to buyers in the market at every date. Moreover, sellers obtain a positive price even as the discount factor δ converges to one (i.e., the market becomes frictionless). It has been debated whether or not the market equilibrium obtained as δ approaches one is indeed non-competitive as originally claimed in RW. This debate hinges on what is the appropriate benchmark to determine whether a market outcome is competitive. RW takes the competitive price as that given by the intersection of the demand and the supply curves drawn relative to the *stocks* of buyers and sellers in the market at each date. Therefore, since there is an excess of sellers in the market at each date, the competitive price (determined in this fashion) is zero at every date. If one accepts this definition of the competitive price, then the result that sellers obtain a positive price, even as δ approaches one, is striking: Although individual agents are negligible in the market and trading frictions are vanishingly small, the market equilibrium is not competitive.

Gale [1] defines the competitive price relative to the *flows* of agents entering the market. A price is "flow market-clearing" if it is given by the intersection of the supply and demand curves drawn relative to the flows of agents entering the market at each date. Given that trading agents are replaced in the RW market, there are equal inflows of sellers and buyers at each date regardless of the price prevailing in the market. Thus, any price between zero and one is competitive in Gale's flow market-clearing sense. Consequently according to Gale the market equilibrium of RW merely selects one of the many competitive prices.²

Here we interpret the models of RW and Gale as models of time-differentiated commodities markets, and we analyze these markets when trade is centralized. In our

¹See, for example, [2,7–9].

²See chapter 7 (section 5) of [3] for a discussion of defining the competitive price relative to the stocks and relative to the flows. Also see [1].

model at each date there is a spot market in which agents can trade the indivisible good without search. A Walrasian equilibrium is then an *infinite* sequence of prices, one for each date, such that all markets clear.

In order to facilitate the comparison of the RW equilibrium to the Walrasian equilibrium of the time-differentiated commodities market, in Section 2 we consider a model with homogeneous types and non-stationary entry. We show that if at every date the total measure of sellers having entered the market is greater than the total measure of buyers having entered, and if δ is less than one, then there is a unique Walrasian price sequence and the price of the good is zero at every date. Thus, for such markets the Walrasian price coincides with RW's competitive price relative to the stocks. We conclude that the RW equilibrium does not approach the Walrasian equilibrium of our model as δ approaches one.

Matching models where only a finite measure of agents enters the market over its lifetime may also be interpreted as models of time-differentiated commodities markets, since agents may trade at different dates. Rubinstein and Wolinsky [7] considers a matching model where S sellers and B buyers enter the market at date zero and there is no further entry. The competitive price relative to the total is zero if $S > B$ and is one if $S < B$. Results in Section 2 show that when this market is viewed as a time-differentiated commodities market, then the Walrasian price sequence is unique and the price of the good is the same as the competitive price relative to the totals.

In Section 3 we study the Walrasian equilibria of markets with heterogeneous types and exogenously given stationary inflows of agents. Gale ([1], section 6) considers a matching model of a market of this kind and shows that as δ approaches one the outcome obtained is competitive in his flow market-clearing sense. We show that for any δ less than one, the Walrasian price sequence is constant and equal to the flow market-clearing price if the flow market-clearing price is unique. Thus, the market outcome obtained in Gale approaches the Walrasian equilibrium of our model as δ approaches one.

The results of this paper also unify the different definitions of the competitive price (relative to the stocks, flows, or totals) found in the matching literature. Each of these definitions gives as competitive the same price as does the Walrasian equilibrium of the appropriate time-differentiated commodities market. The advantage of a unified approach to the competitive price is that it allows a meaningful comparison of results obtained in different models. If, for example, in one model the outcome is competitive using one benchmark, while in another model, using a different benchmark, the outcome is not, then we cannot determine whether the results differ due to some fundamental difference in the models (e.g., differences in the entry process or in the traders' information) or whether the results differ simply due to the use of different benchmarks.

2. The time-differentiated commodities model

In order to introduce the time-differentiated commodity approach we consider a market based on the one in RW. In that model each seller is endowed with a unit of indivisible good unit for which he has a reservation price of zero. Each buyer demands a

unit of indivisible good with a limit price of one. Agents discount future gains using discount factor δ . Therefore, a buyer who t periods after entering the market obtains a unit of the good at price p obtains utility $\delta^t(1 - p)$ while a seller who t periods after entering the market sells a unit at the same price obtains utility $\delta^t p$. Although each agent exits the market once he trades, the stock of each type of agent in the market does not change since each agent exiting is replaced in the market by an agent of the same type. RW showed in a matching and bargaining model of this market that when there is an excess of sellers over buyers, say, then sellers obtain a positive price even as the discount factor approaches one (i.e., market frictions vanish). We refer to this result as the “RW result.”

We consider a market in which there are exogenous inflows of entering agents rather than a market where trading agents are replaced. We do so since, when trading agents are replaced, the set of agents entering the market is endogenous. In that case it is not obvious how one applies the Arrow-Debreu notion of competitive equilibrium which takes the agents, their preferences, and their endowments as primitives. The replacement assumption is not essential to the RW result. The same result can be obtained in a model with an excess of sellers over buyers entering at the first date, and equal and *exogenously* given numbers of buyers and sellers entering the market at each subsequent date.³

In the differentiated commodities model agents are identified with points on the line. Let μ denote Lebesgue measure on the line \mathbb{R} .⁴ Time is discrete and is indexed by $t \in \{1, 2, \dots\}$. We use the indices S and B to refer to sellers and buyers respectively. For each t , let J_t^i denote the set of type i agents, $i \in \{S, B\}$, entering the market by date t (that is, at or prior to date t). Define $J_0^S = J_0^B = \emptyset$. It will be important to distinguish the set of type i agents entering *by* date t from the set of type i agents entering *at* date t . This latter set is given by $E_t^i = J_t^i \setminus J_{t-1}^i$, $i \in \{S, B\}$. The set of agents entering over the market’s lifetime is given by $A = \cup_{t=1}^{\infty} (E_t^S \cup E_t^B) \subseteq \mathbb{R}$. Let $\nu_t^i = \mu(E_t^i)$, $i \in \{S, B\}$, be the measure of type i agents entering at date t . Let $T: A \rightarrow \{S, B\}$ give the type of an agent as a function of his identity. Throughout we assume that the set of agents entering the market by date t has finite measure.

In this section we also assume that at each date a greater measure of sellers than buyers has entered the market, i.e., $0 < \sum_{h=1}^t \nu_h^B < \sum_{h=1}^t \nu_h^S < \infty$ for each t , and we abbreviate this assumption as “ $S > B$.” (The case where at each date a greater measure of buyers than sellers has entered the market is symmetric.) The case of primary interest is where **entry is infinite**, i.e., where $\lim_{t \rightarrow \infty} \sum_{h=1}^t \nu_h^B = \lim_{t \rightarrow \infty} \sum_{h=1}^t \nu_h^S = \infty$, but it will also be important to obtain results for the case in which **entry is finite**, i.e., $\lim_{t \rightarrow \infty} \sum_{h=1}^t \nu_h^B < \infty$ and $\lim_{t \rightarrow \infty} \sum_{h=1}^t \nu_h^S < \infty$.

In the time-differentiated commodities market, each seller entering the market is endowed with a single unit of the indivisible good at the date he enters. Each buyer entering the market is endowed with a unit of the divisible good at the date he enters. Both goods are costlessly storable.

At each date there is a spot market for the indivisible good. Let $p_t \in \mathbb{R}^+$ denote the price of a unit of indivisible good at date t in terms of divisible good at date t , and let \mathbf{p}

³See [10] for example.

⁴The results are unchanged if agents are identified with integers and μ is the counting measure.

denote a price sequence $\mathbf{p} = (p_1, p_2, \dots)$. Agents are (weakly) impatient, discounting future gains using discount factor $\delta \leq 1$. A buyer who enters at date t and trades at date $\tau \in \{t, t + 1, \dots\}$, given prices \mathbf{p} , has utility $u_t(\tau, \mathbf{p}; B) = \delta^{\tau-t}(1 - p_\tau)$. A seller who enters at date t and trades at date $\tau \in \{t, t + 1, \dots\}$, given prices \mathbf{p} , has utility $u_t(\tau, \mathbf{p}; S) = \delta^{\tau-t}p_\tau$. The utility of never trading is zero. Each seller therefore has a reservation price of zero for his unit of the indivisible good, and each buyer demands a unit of the indivisible good with a limit price of one. Consequently, p_τ and $1 - p_\tau$ are the (undiscounted) utilities of a seller and buyer who trade at date τ .

Since each agent has an interest in transacting only a single unit of the indivisible good, the object of choice for an agent is a date at which to trade. Moreover, since an agent can trade a unit of the indivisible good only following his entry into the market, the choice set for an agent entering at date t is $X_t = \{t, t + 1, \dots\} \cup \{\infty\}$, where the element ∞ denotes never trading. An allocation then is an assignment of a trading date to each agent. An **assignment** is a function $x: A \rightarrow X_1$ such that $x(E_t^S \cup E_t^B) \subseteq X_t$ for each t . We assume x is measurable.⁵

A Walrasian equilibrium is a sequence of spot market prices and an assignment such that (i) the assignment is market-clearing, and (ii) the trading date assigned to each agent maximizes the agent’s utility given the price sequence.

Definition: A *Walrasian equilibrium* is a pair (\mathbf{p}, x) satisfying, for each $t \geq 1$:

(i) Market-Clearing

$$\mu(\{j \in J_t^B : x(j) = t\}) = \mu(\{j \in J_t^S : x(j) = t\})$$

(ii) Utility Maximization

$$u_t(x(j), \mathbf{p}; T(j)) = \max_{\tau \in X_t} u_t(\tau, \mathbf{p}; T(j)) \quad \forall j \in E_t^S \cup E_t^B.$$

The market-clearing condition states that the measure of sellers trading at date t equals the measure of buyers trading at date t . Since the indivisible good is storable, the measure of trades at date t is not necessarily the same as the measure of sellers entering at date t . Market-clearing does not require that the measure of sellers who never trade equal the measure of buyers who never trade. The utility maximization condition states that for each agent the date at which he trades must yield at least as much utility as any other feasible trading date. We say \mathbf{p} is **Walrasian** if there exists an assignment x such that (\mathbf{p}, x) is a Walrasian equilibrium.

In the matching literature it has been conventional to ignore time-differentiation of commodities when defining the competitive price. When total entry is finite, the competitive price has been taken to be the one given by the intersection of the demand

⁵For each t , $\{j: x(j) = t\}$ belongs to $B(\mathbb{R})$ the Borel sets on \mathbb{R} .

and the supply curves drawn relative to the total measure of agents entering the market. (See, for example, [7]. Peters [4] and [5] also study a market with one-time entry under different trading rules.) More precisely, when entry is finite the **competitive price relative to the totals** is one if $\lim_{t \rightarrow \infty} \sum_{h=1}^t \nu_h^B > \lim_{t \rightarrow \infty} \sum_{h=1}^t \nu_h^S$ and is zero if $\lim_{t \rightarrow \infty} \sum_{h=1}^t \nu_h^B < \lim_{t \rightarrow \infty} \sum_{h=1}^t \nu_h^S$. When entry is infinite, ignoring the time-differentiation of commodities is problematic since supply and demand are both infinite. In this case we use the following definition, which is a generalization of one used in RW. The **competitive price relative to the stocks** is one if $0 < \sum_{h=1}^t \nu_h^S < \sum_{h=1}^t \nu_h^B < \infty$ for each t , and is zero if $0 < \sum_{h=1}^t \nu_h^B < \sum_{h=1}^t \nu_h^S < \infty$ for each t (i.e., if $S > B$). In the remainder of this section we discuss the relation of these definitions of the competitive price to the Walrasian equilibrium price.

Theorem 1, the main result of this section, establishes that if $S > B$ and either (i) the discount factor is less than one, or (ii) the discount factor is equal to one, entry is finite and the competitive price relative to the totals is zero, then there is a unique Walrasian price sequence and the price of the good is zero at every date.

Theorem 1: *Let $S > B$ and either (i) $\delta < 1$, or (ii) $\delta = 1$, entry is finite, and the competitive price relative to the totals is zero. If \mathbf{p} is Walrasian then $\mathbf{p} = (0, 0, \dots)$.*

Since a market in which an excess of sellers enters at the first date and an equal and exogenously given number of both types of agent enters at each subsequent date satisfies $S > B$, the RW result obtained in a matching model of this market establishes that prices are not Walrasian as frictions vanish.

Before proving Theorem 1, we first state a lemma which, if $p_t > 0$ for some t , gives a lower bound on the rate of growth of prices along some subsequence of dates. Moreover, at each date in the subsequence a positive measure of sellers trades. This result is then used to prove Theorem 1.

Lemma 1: *Let $S > B$ and $0 < \delta \leq 1$. If \mathbf{p} is Walrasian and $p_t > 0$, then for every $n \geq t$ there exists an $m > n$, such that $p_t \leq \delta^{m-t} p_m$ and $\mu(\{j \in J_n^S : x(j) = m\}) > 0$.*

Proof: Appendix A.

The intuition underlying Lemma 1 is straightforward. If the price at date t is positive then all sellers entering by date t obtain a positive utility trading at date t and so utility maximization implies that all such sellers trade eventually (the utility of never trading is zero). This, together with two additional facts, (i) $\sum_{h=1}^t \nu_h^B < \sum_{h=1}^t \nu_h^S$ and (ii) the market for the good clears at each date $k \leq t$, imply that a positive measure of sellers entering by date t trades at some date $m > t$, $m \neq \infty$. For trading at date m to be optimal it is necessary that $p_t \leq p_m \delta^{m-t}$ and so the lemma is true for $n = t$. An induction argument on n completes the lemma. Theorem 1 now follows easily.

Proof of Theorem 1: Let $S > B$ and $\delta < 1$. Suppose contrary to the theorem that $p_t > 0$ for some t . Since \mathbf{p} is Walrasian, then by Lemma 1 for any $n \geq t$, there exists an $m > n$

such that $p_t \leq p_m \delta^{m-t}$ and $\mu(\{j \in J_n^S : x(j) = m\}) > 0$. Since $\delta < 1$ this implies for n sufficiently large that there exists an m such that $p_m > 1$ and $\mu(\{j \in J_n^S : x(j) = m\}) > 0$. But $p_m > 1$ and utility maximization implies that $\mu(\{j \in J_m^B : x(j) = m\}) = 0$ since each buyer obtains a higher utility by never trading than by trading at a price greater than one. Thus market-clearing at date m is contradicted.

Let entry be finite, the competitive price relative to the totals be zero, and let $\delta = 1$. We now show if \mathbf{p} is Walrasian then $\mathbf{p} = (0, 0, \dots)$. Since the competitive price relative to the totals is zero, we have $\lim_{t \rightarrow \infty} \sum_{h=1}^t \nu_h^B < \lim_{t \rightarrow \infty} \sum_{h=1}^t \nu_h^S < \infty$. Market-clearing implies there is a date n and seller $j \in E_n^S$ such that j never trades. Now suppose for some t that $p_t > 0$. Then by Lemma 1, there is an $m > n$ such that $p_m \geq p_t$ and so $p_m \geq p_t > 0$. This, together with $x(j) = \infty$, contradicts utility maximization. \square

The equivalence (when $S > B$, entry is infinite, and δ is less than one) of the Walrasian equilibrium price and the competitive price relative to the stocks breaks down when $\delta = 1$. We say a price sequence \mathbf{p} is **constant** if $p_t = p_{t+1} \forall t$.

Theorem 2: *Let $S > B$, $\delta = 1$, and entry be infinite. Then \mathbf{p} is Walrasian if and only if $0 \leq p_1 = p_2 = \dots \leq 1$.*

Proof: Appendix A.

As Theorem 2 shows, the same indeterminacy that arises in matching models with $\delta = 1$ and infinite entry (i.e., the non uniqueness of equilibria) also appears in our model of centralized trade.

A Walrasian equilibrium exists under the assumptions of Theorem 1. In particular, the price sequence $\mathbf{p} = (0, 0, \dots)$ and any market-clearing assignment in which each buyer trades at the date he enters is a Walrasian equilibrium. (Such an assignment exists since $S > B$.) Under the assumptions of Theorem 2, any constant price sequence between zero and one combined with any market-clearing assignment in which every buyer trades at the date he enters and each seller trades eventually is a Walrasian equilibrium. Thus, we have the following theorem.

Theorem 3: *Let $S > B$ and $0 < \delta \leq 1$. Then there is a Walrasian equilibrium.*

We conclude this section by noting that in matching models each agent typically trades only following delay, while if $S > B$ and $\delta < 1$, then in a Walrasian equilibrium each buyer must trade at the date he enters. Thus, it is natural to ask whether the market equilibrium of a matching model can ever reproduce the Walrasian equilibrium assignment. To address this issue we write the discount factor as $\delta = e^{-r\Delta}$, where r is the instantaneous discount rate and Δ is, for every t , the length of time elapsing between date t and $t + 1$. One can then interpret letting the discount factor approach one as either letting the instantaneous discount rate approach zero, or letting the length of time separating dates approach zero.

With either interpretation, to determine whether the limit of equilibria of a matching

model, obtained as δ approaches one, is Walrasian, it is sufficient to show that trade occurs at the limiting Walrasian equilibrium prices. To see this, notice that if (in a matching model) an agent is matched at each date with probability α , then his expected delay before matching (and trading) is

$$(1 - \alpha)\alpha\Delta + (1 - \alpha)^2\alpha2\Delta + \dots = (1 - \alpha)\frac{\Delta}{\alpha}.$$

Thus, as Δ approaches zero, in the limit the expected delay is zero, and hence an agent obtains his Walrasian equilibrium assignment if and only if the bargained price in the matching model is the Walrasian price.⁶

Alternatively, we can take δ approaching one to mean that the instantaneous discount rate approaches zero. In this case, although expected delay is positive, this delay is insignificant in the limit since, when $r=0$, indivisible goods at different dates are perfect substitutes. Thus, as r approaches zero, in the limit only the price at which trade occurs is significant.

3. Stationary entry

In this section we derive the Walrasian equilibrium of a market with heterogeneous types and exogenously given stationary inflows of agents. In particular, we consider the market of ([1], section 6). In this market each seller is one of K possible types, $0 \leq s_1 < s_2 < \dots < s_K$, where a seller's type is his reservation price and each buyer is one of L possible types, $b_1 > b_2 > \dots > b_L \geq 0$, where a buyer's type is his limit price. Sellers are endowed with a single unit of indivisible good and buyers demand a single unit at their limit price. Exchange of a unit of the good by a type s_k seller and a type b_l buyer generates a surplus of $b_l - s_k$. Agents discount future gains. As in the preceding section, the identity of an agent is indexed by a point on the line. Let $T:A \rightarrow \{s_1, \dots, s_K\} \cup \{b_1, \dots, b_L\}$ give the type of each agent as a function of his identity.

In this section we consider the case where the distribution of entering types is stationary. Thus, we write $\nu^S(s_k)$ and $\nu^B(b_l)$ for the measure of sellers of type s_k and the measure of buyers of type b_l , respectively, entering at every date. It will be convenient to assume that $\nu^S(s_k) > 0$ for each s_k and $\nu^B(b_l) > 0$ for each b_l .

The utility of a buyer of type b_l who enters at date t and trades at date $\tau \in \{t, t + 1, \dots\}$, given prices \mathbf{p} , equals $\delta^{\tau-t}(b_l - p_\tau)$ and is denoted by $u_t(\tau, \mathbf{p}; b_l)$. Similarly, the utility of a seller of type s_k who enters at date t and trades at date $\tau \in \{t, t + 1, \dots\}$, given prices \mathbf{p} , equals $\delta^{\tau-t}(p_\tau - s_k)$ and is denoted by $u_t(\tau, \mathbf{p}; s_k)$. The utility of never trading for agent j of type $T(j)$, denoted by $u_t(\infty, \mathbf{p}; T(j))$, is zero for all t . A Walrasian equilibrium is defined as in the preceding section.

A price p^F is market-clearing relative to ν^S and ν^B if it is given by the intersection of the supply and demand curves drawn relative to the distributions of types given by ν^S and ν^B .

⁶Of course, the expected number of periods that elapse before an agent trades is $(1 - \alpha)/\alpha$.

Definition: p^F is market-clearing relative to (ν^S, ν^B) if:

$$\sum_{b_l: b_l > p^F} \nu^B(b_l) \leq \sum_{s_k: s_k \leq p^F} \nu^S(s_k) \leq \sum_{b_l: b_l \geq p^F} \nu^B(b_l) \tag{1}$$

and $p^F = \min\{b_l: b_l \geq p^F\}$ if the last inequality is strict, or

$$\sum_{s_k: s_k < p^F} \nu^S(s_k) \leq \sum_{b_l: b_l \geq p^F} \nu^B(b_l) \leq \sum_{s_k: s_k \leq p^F} \nu^S(s_k) \tag{2}$$

and $p^F = \max\{s_k: s_k \leq p^F\}$ if the last inequality is strict.

When entry is stationary Gale refers to such a price p^F as a “flow market-clearing price” (FMCP). Since we consider stationary entry in this section, we adopt this terminology as well and say p^F is a FMCP if p^F is market-clearing relative to (ν^S, ν^B) . A typical FMCP p^F when, for example, there are two buyer types and a single seller type is illustrated below in Fig. 1.

The main result of this section is that if the FMCP is unique and if $\delta < 1$, then the Walrasian price sequence is constant and equal to the FMCP at each date. This result is surprising given the entirely different motivation of these benchmarks. Gale [1] argues the FMCP is appealing as it is the only price which is consistent with a stationary equilibrium in his matching model of this market if agents enter whenever they can profitably trade. If the bargained price exceeds the FMCP then the stocks of agents in the market will not be stationary; the market will tend to accumulate sellers. In contrast, the

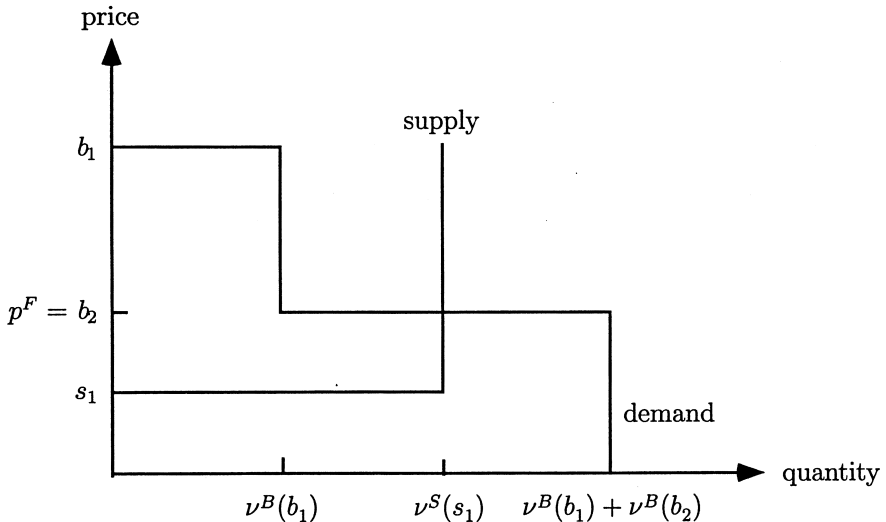


Fig. 1. p^F is market-clearing relative to (ν^S, ν^B) .

Walrasian price sequence follows from utility maximization and market-clearing. The precise statement of the result is as follows.

Theorem 4: *Let p^F be the unique FMCP and $0 < \delta < 1$. Then \mathbf{p} is Walrasian if and only if $\mathbf{p} = (p^F, p^F, \dots)$.*

Before proving Theorem 4 we state a result which places a lower bound on the rate of growth (decline) of Walrasian prices along a subsequence of dates if $p_t > p^F$ ($p_t < p^F$) and p^F is the unique FMCP. This result is used in the proof of Theorem 4 to show that, in fact, in any Walrasian equilibrium $p_t = p^F$ for all t .

Lemma 2: *Let p^F be the unique FMCP and $0 < \delta < 1$. Let \mathbf{p} be Walrasian and assume that there exists a date t such that $p_t \neq p^F$. Let t^* be the first such date. Then for any $n \geq t^*$ there exists an $m > n$, such that (i) $p_t - p^F \leq \delta^{m-t}(p_m - p^F) \forall t \in \{t^*, \dots, m-1\}$ if $p_{t^*} > p^F$, and (ii) $p^F - p_t \leq \delta^{m-t}(p^F - p_m) \forall t \in \{t^*, \dots, m-1\}$ if $p_{t^*} < p^F$.*

Proof: Appendix A.

The proof of Theorem 4 follows easily given Lemma 2.

Proof of Theorem 4: Proof of (\Rightarrow): Suppose to the contrary that $p_t \neq p^F$ for some t . Let date t^* be the first such date. We consider the case where $p_{t^*} > p^F$. (The case where $p_{t^*} < p^F$ is symmetric.) By Lemma 2, for any $n \geq t^*$ there exists an $m > n$ such that $p_{t^*} - p^F \leq \delta^{m-t^*}(p_m - p^F)$. Since $\delta < 1$, this implies for n sufficiently large that $p_m > b_1$. That the FMCP is unique implies that $b_1 \geq s_1$, and thus $p_m > s_1$. Utility maximization implies that there is a date $k \geq m$ such that $\mu(\{j \in E_m^S : T(j) = s_1 \text{ and } x(j) = k\}) > 0$ and $\delta^{k-m}(p_k - s_1) \geq p_m - s_1$. But this contradicts market-clearing at date k as every buyer obtains a higher utility by never trading than by trading at date k .

Proof of (\Leftarrow): Trivial. \square

Theorem 4 shows that for the market of Gale with stationary entry, the FMCP and the Walrasian price sequence coincide when $\delta < 1$ and the FMCP is unique. This relationship fails to hold when $\delta = 1$. We will argue informally that if $\delta = 1$, then any constant price sequence between b_1 and s_1 is Walrasian. For example, suppose the set of agents entering were as illustrated in Fig. 1. Then $\mathbf{p} = (b_1, b_1, \dots)$ is Walrasian where the assignment is constructed as follows. No type b_2 buyers trade. At each date every buyer of type b_1 trades as he enters and an equal measure of sellers of type s_1 trades. Of course, since $\nu^S(s_1) > \nu^B(b_1)$, market-clearing requires that not every type s_1 seller trades at the date he enters. Nonetheless, since an infinite measure of type b_1 buyers eventually enters, we can construct an assignment such that every type s_1 seller eventually trades. Such an assignment is consistent with utility maximization given that an agent's utility is independent of the date he trades when $\delta = 1$ and the price sequence is constant. The price b_1 is not, however, a FMCP.

4. Concluding remarks

The present paper analyzed a model in which the agents and their preferences are the same as in the main models of matching and bargaining, but in which trade is centralized rather than decentralized. Since equilibrium prices in our model are market-clearing, we refer to its equilibria as “Walrasian.”

There are two main results. Theorem 1 applies to markets with homogenous buyers and sellers. It shows that if the discount factor is less than one and at each date more sellers than buyers have entered the market, then the Walrasian equilibrium price of the good is zero at each date. (Hence the Walrasian equilibrium price at each date coincides with RW’s competitive price relative to the *stocks* when entry is infinite, and coincides with the competitive price relative to the *totals* when entry is finite.) In contrast, RW finds for a market fitting this framework, but in which trade is decentralized, that the market price is strictly positive in the limit as the discount factor approaches one. We conclude that the RW result is not Walrasian as frictions vanish.

Theorem 4, the second main result, applies to markets with heterogenous types of buyers and sellers and stationary entry. It shows that if there is a unique “flow market clearing price” (in the sense of Gale) and the discount factor is less than one, then the Walrasian equilibrium price of the good at each date coincides with the flow market clearing price. For such markets, Gale [1] shows for his model of decentralized trade that, in the limit as the discount factor approach one, the market price is the flow market clearing price. Hence, whether trade is centralized or decentralized, the equilibrium price is the same in either case in the limit as frictions vanish.

Together Theorems 1 and 4 unify the three definitions of the competitive price – relative to the *stocks*, *totals*, and *flows* – used in the matching literature. Each coincides with the Walrasian equilibrium price in the setting in which it is applied.

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Appendix A

Proof of Lemma 1: The lemma is established with an induction argument. Suppose \mathbf{p} is Walrasian and $p_t > 0$ for some t . Let $P(n)$ be the proposition “there exists an $m > n$ such that $p_t \leq \delta^{m-t} p_m$ and $\mu(\{j \in J_n^S : x(j) = m\}) > 0$.” We begin by showing that $P(n)$ is true for $n = t$.

Suppose to the contrary that for each $m > t$, either $p_t > \delta^{m-t} p_m$ or $\mu(\{j \in J_t^S : x(j) = m\}) = 0$. If $p_t > \delta^{m-t} p_m$ then $\delta^{t-k} p_t > \delta^{m-k} p_m$ and so any seller entering at date k , $k \leq t$, obtains a higher utility trading at t than trading at date m , and therefore $\mu(\{j \in E_k^S : x(j) = m\}) = 0$. Since $J_t^S = \cup_{t \leq k \leq t} E_k^S$, this implies $\mu(\{j \in J_t^S : x(j) = m\}) = 0$. Moreover, given

$p_t > 0$ then utility maximization implies every seller entering at date $k \leq t$ eventually trades. Thus we have

$$\mu(\{j \in J_t^S : x(j) \leq t\}) = \sum_{h=1}^t \nu_h^S > \sum_{h=1}^t \nu_h^B \geq \mu(\{j \in J_t^B : x(j) \leq t\}),$$

where the strict inequality follows from the entry assumption that $\sum_{h=1}^t \nu_h^S > \sum_{h=1}^t \nu_h^B \forall t$. But this contradicts market-clearing since $\mu(\{j \in J_k^S : x(j) = k\}) = \mu(\{j \in J_k^B : x(j) = k\}) \forall k$ implies, summing over $k \leq t$, that $\mu(\{j \in J_t^S : x(j) \leq t\}) = \mu(\{j \in J_t^B : x(j) \leq t\})$.

We now assume $P(i)$ is true for $i \geq t$ and show $P(i + 1)$ is true. That $P(i)$ is true means that there exists an $m > i$ such that $p_i \leq \delta^{m-t} p_m$ and $\mu(\{j \in J_i^S : x(j) = m\}) > 0$. Suppose $P(i + 1)$ is false and thus for each $s > i + 1$, either $p_i > \delta^{s-t} p_s$ or $\mu(\{j \in J_{i+1}^S : x(j) = s\}) = 0$. Clearly, it must be the case that $m = i + 1$. (If $m > i + 1$ then choosing $s = m$, we have $p_i \leq \delta^{s-t} p_s$ and $\mu(\{j \in J_i^S : x(j) = s\}) > 0$ since $P(i)$ is true. Thus, $\mu(\{j \in J_{i+1}^S : x(j) = m\}) > 0$ since $J_i^S \subset J_{i+1}^S$, contradicting that $P(i + 1)$ is false.) For those s where $p_i > \delta^{s-t} p_s$ we have $\delta^{i+1-t} p_{i+1} > \delta^{s-t} p_s$ since $\delta^{i+1-t} p_{i+1} \geq p_i$.

Thus for each $s > i + 1$, either $\delta^{i+1} p_{i+1} > \delta^s p_s$ or $\mu(\{j \in J_{i+1}^S : x(j) = s\}) = 0$. For those $s > i + 1$ such that $\delta^{i+1} p_{i+1} > \delta^s p_s$, every seller entering by date $i + 1$ obtains a higher utility by trading at date $i + 1$ than by trading at date s . Thus, for all $s > i + 1$ we have $\mu(\{j \in J_{i+1}^S : x(j) = s\}) = 0$. Further, $p_{i+1} > 0$ implies any seller entering by date $i + 1$ eventually trades. We have shown $x(j) \leq i + 1 \forall j \in J_{i+1}^S$ but, following the same argument as above, this contradicts market-clearing. \square

Proof of Lemma 2: Note that p^F is the unique FMCP if and only if

$$\sum_{b_l : b_l > p^F} \nu^B(b_l) < \sum_{s_k : s_k \leq p^F} \nu^S(s_k) \tag{3}$$

and

$$\sum_{s_k : s_k < p^F} \nu^S(s_k) < \sum_{b_l : b_l \geq p^F} \nu^B(b_l). \tag{4}$$

Suppose \mathbf{p} is Walrasian and t^* is the first date such that $p_t \neq p^F$. We consider the case where $p_{t^*} > p^F$. The case when $p_{t^*} < p^F$ is symmetric. Let $P(n)$ be the proposition “there exists an $m > n$, such that $p_t - p^F \leq \delta^{m-t} (p_m - p^F) \forall t \in \{t^*, \dots, m - 1\}$.” To show that $P(t^*)$ is true we need to show that there exists an $m > t^*$, such that $p_t - p^F \leq \delta^{m-t} (p_m - p^F) \forall t \in \{t^*, \dots, m - 1\}$. First we show that $p_{t^*} > p^F$ and $p_t = p^F \forall t < t^*$ implies

$$\mu(\{j \in J_{t^*}^B : x(j) = t^*\}) \leq \sum_{b_l : b_l > p^F} \nu^B(b_l). \tag{5}$$

Since $p_{t^*} > p^F$ any buyer entering by date t^* , and of type $b_l \leq p^F$, obtains a higher utility by never trading than by trading at t^* . Buyers of type $b_l > p^F$ entering prior to date t^* obtain a higher utility by trading at the date they enter than by trading at t^* since $p_t < p_{t^*} \forall t < t^*$. Therefore, at most those buyers of type $b_l > p^F$ entering at date t^* trade at t^* .

We now show that there exists a $j' \in J_{t^*}^S$ such that $x(j') > t^*$ and $T(j') \leq p^F$. Suppose

to the contrary that $\forall j \in J_{t^*}^S$ either $x(j) \leq t^*$ or $T(j) > p^F$. Then at least those sellers of type $s_k \leq p^F$ entering at date t^* trade at t^* , and so

$$\mu(\{j \in J_{t^*}^S : x(j) = t^*\}) \geq \sum_{s_k : s_k \leq p^F} \nu^S(s_k). \tag{6}$$

Since p^F is the unique FMCP, Eq. (3), Eq. (5), and Eq. (6) imply

$$\mu(\{j \in J_{t^*}^S : x(j) = t^*\}) \geq \sum_{s_k : s_k \leq p^F} \nu^S(s_k) > \sum_{b_l : b_l > p^F} \nu^B(b_l) \geq \mu(\{j \in J_{t^*}^B : x(j) = t^*\}).$$

But this contradicts market-clearing at date t^* . Utility maximization and $p_{t^*} > p^F$ imply that $x(j') < \infty$ since a seller of type $T(j') \leq p^F$ obtains a positive utility by trading at date t^* but obtains only a utility of zero by never trading. Thus, we have shown that there exists a $j' \in J_{t^*}^S$ such that $t^* < x(j') < \infty$ and $T(j') \leq p^F$. Choose $m = x(j')$.

Next we establish that $p_t - p^F \leq \delta^{m-t}(p_m - p^F) \forall t \in \{t^*, \dots, m-1\}$. Suppose to the contrary for some $t \in \{t^*, \dots, m-1\}$, that

$$p_t - p^F > \delta^{m-t}(p_m - p^F).$$

Adding $p^F - s_k$ to each side and then multiplying both sides by δ^{t-t^*} gives

$$\delta^{t-t^*}(p_t - s_k) > \delta^{m-t^*}(p_m - p^F) + \delta^{t-t^*}(p^F - s_k).$$

Since $\delta^{m-t^*} < \delta^{t-t^*}$ this implies, for $s_k \leq p^F$, that

$$\delta^{t-t^*}(p_t - s_k) > \delta^{m-t^*}(p_m - s_k).$$

In other words, all sellers of type $s_k \leq p^F$ entering at date t^* obtain a higher utility trading at date t than trading at date m . It is easy to see that this inequality also implies all sellers of type $s_k \leq p^F$ entering prior to t^* obtain a higher utility trading at t than trading at m . But then the existence of a $j' \in J_{t^*}^S$ such that $x(j') = m$ and $T(j') \leq p^F$ contradicts utility maximization.

We now assume $P(i)$ is true for $i \geq t^*$ and show that $P(i+1)$ is true. Given $P(i)$ is true there exists an $m > i$ such that

$$p_t - p^F \leq \delta^{m-t}(p_m - p^F) \forall t \in \{t^*, \dots, m-1\}. \tag{7}$$

We now show that Eq. (7) and $p_{t^*} > p^F$ implies

$$\mu(\{j \in J_m^B : x(j) = m\}) \leq \sum_{b_l : b_l > p^F} \nu^B(b_l).$$

Since $p_{t^*} > p^F$ then Eq. (7) implies $p_m > p^F$. That $p_m > p^F$ implies $x(j) \neq m \forall j \in J_m^B$ such that $T(j) \leq p^F$. We now show that $x(j) \neq m \forall j \in J_{m-1}^B$ such that $T(j) > p^F$. Multiplying Eq. (7) by -1 and adding $b_l - p^F$ to both sides of the result gives

$$b_l - p_t \geq \delta^{m-t}(p^F - p_m) + b_l - p^F \forall t \in \{t^*, \dots, m-1\}.$$

If $b_i - p^F > 0$ then $b_i - p_t > \delta^{m-t}(b_i - p_m) \forall t \in \{t^*, \dots, m-1\}$. Therefore, all buyers of type $b_i > p^F$ entering at date $t \in \{t^*, \dots, m-1\}$ obtain a higher utility trading at date t than trading at date m . For $t < t^*$ we have $p_t = p^F$ while $p_m > p^F$ so buyers of type $b_i > p^F$ entering prior to t^* also obtain a higher utility trading at the date they enter than trading at date m . Therefore, at most those buyers of type $b_i > p^F$ entering at date m trade at m .

We now show that there exists a $j' \in J_m^S$ such that $x(j') > m$ and $T(j') \leq p^F$. Suppose to the contrary that $\forall j \in J_m^S$ either $x(j) \leq m$ or $T(j) > p^F$. Then at least those sellers of type $s_k \leq p^F$ and entering at date m trade at m , and so

$$\mu(\{j \in J_m^S : x(j) = m\}) \geq \sum_{s_k : s_k \leq p^F} \nu^S(s_k) > \sum_{b_i : b_i > p^F} \nu^B(b_i) \geq \mu(\{j \in J_m^B : x(j) = m\}),$$

where the strict inequality follows from Eq. (3) and that p^F is the unique FMCP. This contradicts market-clearing at date m . Utility maximization and $p_m > p^F$ implies $x(j') < \infty$ since a seller of type $T(j) \leq p^F$ obtains a strictly positive utility by trading at date m but obtains only a utility of zero by never trading. Thus, we have shown that there exists a $j' \in J_m^S$ such that $m < x(j') < \infty$ and $T(j') \leq p^F$. That $x(j') > m$ and $m > i$ implies $x(j') > i + 1$.

It must also be the case that $p_t - p^F \leq \delta^{x(j')-t}(p_{x(j')} - p^F) \forall t \in \{m, \dots, x(j') - 1\}$, since otherwise, using the same argument as above, a seller of type $s_k \leq p^F$ who enters by date m could not optimally trade at date $x(j')$. We have for $t = m$, in particular, that $p_m - p^F \leq \delta^{x(j')-m}(p_{x(j')} - p^F)$. Substituting for $p_m - p^F$ in Eq. (7) yields $p_t - p^F \leq \delta^{x(j')-t}(p_{x(j')} - p^F) \forall t \in \{t^*, \dots, m-1\}$. This combined with $p_t - p^F \leq \delta^{x(j')-t}(p_{x(j')} - p^F) \forall t \in \{m, \dots, x(j') - 1\}$ is the desired result. \square

Proof of Theorem 2: Proof of (\Rightarrow) : Let (\mathbf{p}, x) be a Walrasian equilibrium. If $x(j_s) = x(j_b) = \infty$ for both some $j_s \in J_1^S$ and some $j_b \in J_1^B$ this contradicts utility maximization as either $u_1(1, \mathbf{p}; S) > 0$ or $u_1(1, \mathbf{p}; B) > 0$. Therefore, without loss of generality assume that $\mu(\{j \in J_1^B : x(j) = k\}) > 0$ for some $k < \infty$.

For trading at date k to be optimal for a buyer entering at date one it is necessary that $p_k \leq 1$ and

$$p_k \leq p_s \forall s \geq 1. \tag{8}$$

Market-clearing at date k requires that a positive measure of sellers who enter by date k also trades at date k . For a seller to optimally trade at date k requires that $p_k \geq p_s \forall s \geq k$. Define η to be the smallest positive integer such that

$$p_k \geq p_s \forall s \geq \eta. \tag{9}$$

We know that such an integer exists and is not greater than k since $p_k \geq p_s \forall s \geq k$. Together Eq. (8) and Eq. (9) imply that $p_\eta = p_{\eta+1} = \dots$, and thus \mathbf{p} is constant from date η .

We now show that $\eta = 1$. Suppose to the contrary that $\eta > 1$. Given the definition of η , we have $p_{\eta-1} > p_k$. Since $p_{\eta-1} > p_\eta = p_{\eta+1} = \dots$ then sellers entering prior to date η can not optimally trade at date η or later. Nor is it optimal for these sellers never to trade since $p_{\eta-1} > p_k \geq 0$.

Consider sellers entering at date one. We have shown that all such sellers must optimally trade prior to date η . Thus there exists a date $\gamma \leq \eta - 1$ where a positive measure of such sellers optimally trades and where $p_\gamma \geq p_{\eta-1}$. Market-clearing at date γ requires that a positive measure of buyers entering by date γ also trades in γ . But this contradicts utility maximization since $p_\gamma \geq p_{\eta-1} > p_k$ implies that $u_i(\gamma, \mathbf{p}; B) < u_i(k, \mathbf{p}; B) \forall t \leq \gamma$. Therefore $\eta = 1$ and \mathbf{p} is constant. That \mathbf{p} is constant and $0 \leq p_k \leq 1$ implies $0 \leq p_1 \leq 1$.

Proof of (\Leftarrow): Trivial. \square

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