Dual Auctions for Assigning Winners and Compensating Losers*

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April 7, 2023

Abstract

We study the bargaining problem of allocating homogeneous goods or chores when participants have equal claim to a unit of the good or equal obligation to undertake a chore. We propose two sequential auctions for solving problems of this type: a sequential ascending clock "goods" auction and a sequential descending clock "chore" auction, which are duals of one another. Either auction can be used for allocating goods or chores by suitably defining a good or a chore. The auctions are budget balanced, ex-post efficient and, when bidders are risk neutral, payoff equivalent. We characterize equilibrium bidding under constant absolute risk aversion and show that equilibrium converges to maxmin perfect bidding in the limit as bidders become infinitely risk averse. Connecting these results to cooperative game theory, we show that under maxmin perfect bidding the ascending clock goods auction gives each bidder his normative Shapley value allocation, while the descending clock chore auction gives each bidder his strategic Shapley value allocation. These two Shapley value allocations have different fairness interpretations, and thus the choice of the auction format determines which fair allocation results.

*We are grateful to Rabah Amir, John Ledyard, and Greg Leo for helpful comments. We are grateful to two anonymous referees and the associate editor for useful comments.

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Keywords: sequential auctions, Bayes Nash equilibrium, Shapley value, risk aversion, maxmin

JEL Codes: C71, C72, C78, D44

1 Introduction

We study the bargaining problem of allocating K identical units of a good, or K identical chores, to N individuals, where $K < N$ and each individual has an equal claim to a unit of the good or an equal obligation to undertake a chore. Problems of this kind arise frequently. For example, N fishermen may have the right to fish but, to reduce overfishing, the rights of $N - K$ are to be withdrawn. The problem then is to allocate the remaining fishing rights to the K fishermen who value them most highly, with those fishermen who retain the right compensating those who forfeit it. Likewise, if there are K identical chores that must be completed, the problem is to assign chores to the K individuals who have the lowest cost for undertaking them, with the $N - K$ individuals excused from the chore compensating the rest. These types of problems have often been solved by means of lotteries, beauty contests, queues, and taking turns (in repeated interactions), but these methods do not allocate goods or chores efficiently, or are subject to rent seeking.[1]

We propose two dynamic auctions for solving problems of this type. They have a number of desirable properties: They are simple and always budget balanced. They treat the participants, henceforth bidders, symmetrically. Whether bidders are risk neutral or risk averse, goods are allocated in equilibrium to the bidders with the highest values for consuming them and chores are assigned to the bidders with the lowest cost for undertaking them. Participation is individually rational: a bidder who participates obtains an equilibrium payoff that exceeds the payoff he would obtain were the goods or chores assigned randomly.

Further, when bidders follow their maxmin perfect bidding strategies, then each auction generates an allocation that is fair in the sense that each bidder obtains his Shapley value allocation of an appropriately defined cooperative game. There are two natural approaches to defining the cooperative game: under the strategic approach, the cooperative game is defined by what a coalition can guarantee itself; under the normative approach the cooperative game is defined by what a coalition can obtain in the best case. Each of these cooperative games, respectively, has its own Shapley value – a strategic Shapley value and a normative Shapley value. We show that the

[1] See Boyce (1994) for historical examples of the use of lotteries. See the FCC Report to Congress on Spectrum Auctions (1997) for the use of lotteries in allocating radio frequency. See Holt and Sherman (1982) for a strategic model of queues, and Leo (2017) for a model of taking turns.

chore auction generates the strategic Shapley value allocation under maxmin perfect bidding, while the goods auction generates the normative Shapley value allocation under maxmin perfect bidding. Hence, while the auctions are payoff equivalent under risk-neutral bidding, they satisfy different notations of fairness under maxmin perfect bidding.[2]

We first describe the descending clock chore auction for allocating K chores to N bidders: The auction takes place over $N-K$ rounds. At each round, the price, starting from the highest possible cost of completing a chore, descends continuously. A bidder may drop out at any point. A bidder who drops out is excused from undertaking the chore, but must pay compensation equal to the price at which he drops out, with the compensation to be shared equally among the K bidders who ultimately undertake a chore. Each time a bidder drops out, the price is reset to the highest possible cost and the process repeats. The auction ends when $N-K$ bidders have dropped out. The K bidders remaining each undertake a chore and each receives $1/K$-th of the total compensation promised by the excused bidders.

The structure of the ascending clock goods auction, that assigns K goods to N bidders, is the same except that at each round the price starts at zero and ascends. A bidder who drops out surrenders his claim to a unit of the good and in return is promised compensation equal to his dropout price from the eventual winners of a unit of the good. The auction ends when $N-K$ bidders have dropped out. The K bidders remaining each obtain a unit of the good and pay $1/K$-th of the total compensation promised to the bidders who surrendered their claim. Although these auctions concern the allocation of indivisible goods or chores, the rules are inspired by the Dubins and Spanier (1961) moving knife algorithm for the fair division of a divisible "cake," in which a participant receives compensation (cake) in return for surrendering his claim to the remaining cake.

The chore auction and goods auction are different mechanisms, but either can be used whether the problem is to allocate chores or goods. Consider, for example, the problem of assigning K units of a good to N bidders. It can be solved via an ascending clock goods auction with $N-K$ rounds. At each round the auction identifies a "loser" – the bidder who drops out and surrenders his claim to a unit of the good. Alternatively, we can assign a unit of the good to each of the N bidders

[2] We provide a general payoff equivalence theorem for all symmetric, efficient, and budget balanced mechanisms.

(ignoring that we only have K units) and define the chore to be the surrender of a unit of the good. The allocation problem can then be solved by a chore auction with K rounds. At each round the auction identifies a "winner" – the bidder who drops out is excused from the chore and keeps their unit of the good.

Although solving the same problem, the two auctions generate different outcomes: In the goods auction, the $N-K$ "losing" bidders receive idiosyncratic compensations for surrendering their claim to the good, while the "winning" bidders all pay the same compensation. In the chore auction, the K "winning" bidders pay idiosyncratic compensations to keep their good, while the "losing" bidders all receive the same compensation for surrendering their claim.

For both auctions we characterize necessary and sufficient conditions for bid functions to form a symmetric equilibrium in increasing and differentiable strategies when bidders have independent private costs or values. We provide closed-form solutions for the (unique) symmetric equilibrium when bidders are risk neutral and when they have constant absolute risk aversion (CARA). Since the auctions are dynamic, an equilibrium is characterized by a sequence of bid functions, where the t-th bid function identifies a bidder's dropout price in round t as a function of his cost (or value) and the prices at which bidders have dropped out in prior rounds.

In both auctions, bidders drop out earlier as they become more risk averse. In the chore auction this means bidders pay more compensation to be excused from the chore, whereas in the goods auction they accept less compensation in return for surrendering their claim to the good. For fixed K and N, as the CARA index of risk aversion goes to infinity, the equilibrium bidding strategies of the two auctions converge to the same limit bidding strategy, which is linear in costs or values. This limit bidding strategy has a decision-theoretic interpretation, which we describe next.

In the actual application of any allocation mechanism, the participants may be concerned with their worst-case outcome. For each auction, we identify a bidder's maxmin payoff as the maximum payoff that he can guarantee himself at the outset of the auction, regardless of the behavior of the other bidders. We show that the maxmin payoff in the chore auction of a bidder whose cost is x is $-Kx/N$, and the maxmin payoff in the goods auction for a bidder whose value is x is Kx/N.

A maxmin strategy is a strategy that guarantees a bidder at least his maxmin payoff. There are many such strategies in both auctions, and we focus on a natural refinement. A maxmin strategy is "perfect" if it maximizes the payoff that a bidder can guarantee himself starting from any history of play. We show that there is a unique maxmin perfect strategy for both the chore and the goods auctions. Furthermore, the maxmin perfect strategy is the same for both auctions and it coincides with the (symmetric) equilibrium bidding strategy as bidders become infinitely risk averse. Consequently, the equilibrium allocation of the chore auction approaches the strategic Shapley value allocation as bidders become infinitely risk averse, while the equilibrium allocation of the goods auction approaches the normative Shapley value allocation as bidders become infinitely risk averse. While infinite risk aversion is implausible as a description of real-world bidding, maxmin preferences are an interesting theoretical benchmark for behavior. These results contribute to the Nash program by providing games whose non-cooperative solutions coincide with solution concepts from cooperative game theory.[3]

Various criteria have been proposed for selecting between auctions that would be payoff equivalent were bidders risk neutral, e.g., bidders concerned with the privacy of their information may prefer an ascending bid auction to a sealed bid auction; in a second price sealed bid auction, equilibrium is in dominant strategies, but not in a first price sealed bid auction; the English ascending clock auction is obviously strategy proof, whereas the second price sealed bid auction is not (Li (2017)); a first price sealed bid auction raises more revenue than a second price auction when bidders are in fact risk averse, and so on. Our results suggest fairness considerations as a novel criteria for choosing between auctions in a bargaining context: the allocations generated by the chore and the goods auctions satisfy different notions of fairness as bidders become infinitely risk averse.

Related Literature

The problem of assigning a single unit of a good to one of two players with equal claims is the well-known problem of dissolving a partnership. The most commonly used mechanism for dissolving two-person partnerships is the "Texas-Shootout," where one partner proposes a price and the other partner is compelled to either buy his partner's share or sell his own share at that price. (This is simply the classic "divide and choose" procedure, studied in the cake cutting literature, where the indivisible good is made divisible by using money transfers.)[4] de Frutos and Kittsteiner

[3] The Nash program is the research agenda concerned with providing non-cooperative foundations for cooperative game theory. See Serrano (2008, 2021) for surveys.

[4] There are many connections between the literature on dissolving partnerships and the cake

(2008) study the Texas-Shootout when the proposer is determined endogenously via an auction. McAfee (1992) and de Frutos (2000) characterize equilibrium bidding for the Winner's Bid and the Loser's Bid auction for two-person partnerships. Wasser (2013) studies a family of auctions for dissolving partnerships in which the distributions of the bidders' values and the ownership shares are asymmetric. Morgan (2004) and Brooks, Landeo, and Spier (2010) study dissolving partnerships in common value settings.

For partnerships with more than two players, but only a single unit of the good, Cramton, Gibbons, and Klemperer (1987) provide necessary and sufficient conditions for a partnership to be efficiently dissolvable and, when it is, they identify a (static) auction that dissolves it. They show that equal share partnerships are always dissolvable by simple auctions.[5] Van Essen and Wooders (2016) propose a dynamic auction for dissolving such partnerships and characterize equilibrium bidding when bidders are risk neutral or CARA risk averse. Van Essen and Wooders (2021) studies the problem of allocating N positions to N bidders, and relates maxmin perfect bidding to the normative Shapley value of that bargaining problem. The present paper generalizes and extends these papers in several ways: It introduces the descending clock chore auction. It deals with the case of multiple units of a good or multiple chores, and shows the duality of the ascending clock goods auction and the descending clock chore auction. Most important, it establishes the relationship between the chore auction and the strategic Shapley value and the dual relationship between the goods auction and the normative Shapley value. Thus, this work demonstrates different auctions can be used to achieve different notions of fairness. By contrast, other auctions that have been studied for solving this bargaining problem, e.g., the $k + 1$ auction studied in Cramton, Gibbons, and Klemperer (1987), do not generate Shapley value allocations.[6] In the Discussion section, we show that a uniform auction

cutting literature. Classic papers on cake cutting include Steinhaus (1948) and Dubins and Spanier (1961). Brams and Taylor (1996) surveys this literature. Su (1999) discusses how an envy free cake-cutting algorithm can be applied to dividing a chore or sharing a cost.

[5] Cramton, Gall, Sujarittanonta, and Wilson (2013) characterizes the equilibrium of first and second price sealed bid auctions of a single item in which the auction revenue is divided equally among the bidders, for both private and affiliated values, with risk-neutral bidders. It proposes these auctions for the allocation of Internet domain names.

[6] In the $k+1$ auction, bidders not allocated the item each receive the same compensation. This is inconsistent with the normative Shapley allocation, which requires that they be receive compensation

does not generate either normative or strategic Shapley value allocations. Hence, the uniform price auction is not appropriate if one is interested in allocations that are fair in the Shapley sense.

The goods auction studied here can be viewed as a mechanism for reorganizations of partnerships that reduce the number of partners from N to K, for any $K < N$. In this case, $N - K$ partners must be compensated for surrendering their share of the partnership. Our results show that the goods auction efficiently reorganizes partnerships when bidders are either risk neutral or risk averse. Alternatively, the chore auction can be applied instead and, when K is small, it reorganizes a partnership with an auction that takes fewer rounds.

In both the present paper and the papers above, the solution concept is Bayes Nash equilibrium. If the solution concept is dominant strategy incentive compatibility, it is well known from Green and Laffont (1977) and Walker (1980) that there is no mechanism for our setting that is efficient and budget balanced. Long, Mishra, and Sharma (2017) relaxes efficiency and provides a mechanism for the single unit problem that is dominant strategy incentive compatible, budget balanced, but not efficient. It is nearly efficient when the number of bidders is large. Long (2016) extends this mechanism to the general multi-unit case.

Maxmin has long been important in both decision theory and game theory, going back to the 1940's. Models incorporating ambiguity aversion are more recent. For example, Gilboa and Schmeidler's (1989) well-known notion of maxmin expected utility includes maxmin preferences as a special case. Ambiguity aversion has appeared in both the theoretical and empirical auction literature. Salo and Weber (1995) uses a model of ambiguity aversion to explain overbidding in first price sealed bid auctions. Levin and Ozdenoren (2004) studies buyer and seller responses to ambiguity in the number of bidders in the first and second price sealed bid auctions. Bose, Ozdenoren and Pape (2006) applies a mechanism design approach to characterize the seller's optimal auction when bidders are ambiguity averse. Chen, Katuscak, and Ozdenoren (2007) conducts a laboratory experiment of the first and second price sealed bid auctions with independent private values, where the distribution of bidder valuations may be unknown. Stong (2018) explores the consequences of ambiguity in all-pay auctions.[7]

based on their value.

[7] See Pourbabaee (2022) and Sung (2022) for other recent applications of ambiguity aversion.

Finally, our paper contributes to a literature on multi-unit sequential auctions with single-unit demands and bidder risk aversion. Recent contributions include Mezzetti (2011) and Hu and Zou (2015) who provide conditions for the sequence of prices received by a seller to be increasing or decreasing. In our setting we study the allocation of homogenous chores and goods when the bidders have equal obligations or claims. Unlike these papers, in our context there is no seller.

2 The Model

There are N bidders and $K < N$ identical chores or $K < N$ identical goods. The bidders' costs and values for the chore and the good, respectively, are independently and identically distributed according to cumulative distribution function F with support $[0, \bar{x}]$, where $\bar{x} < \infty$ and $f \equiv F'$ is continuous and positive on $[0, \bar{x}]$. Bidders have a common utility function u, where $u' > 0$ and $u'' \leq 0$.

Let $X_1, \ldots, X_N$ be N independent draws from F. When the X_i's are costs, it is convenient to order them from highest to lowest. Let $Y_1^{(N)}, \ldots, Y_N^{(N)}$ be a rearrangement of the X_i's such that $Y_1^{(N)} \geq Y_2^{(N)} \geq \ldots \geq Y_N^{(N)}$. The joint density of $Y_1^{(N)}, \ldots, Y_N^{(N)}$ is

$$g_{1,\ldots,N}^{(N)}(y_1, \ldots, y_N) = N! \prod_{i=1}^{N} f(y_i)$$

if $y_1 \geq y_2 \geq \ldots \geq y_N$ and zero otherwise. The conditional density of $Y_t^{(N)}$ given $Y_1^{(N)} = y_1, \ldots, Y_{t-1}^{(N)} = y_{t-1}$ is

$$g_t^{(N)}(y_t | y_1, \ldots, y_{t-1}) = g_t^{(N)}(y_t | y_{t-1}) = (N - (t-1)) f(y_t) \frac{F(y_t)^{N-t}}{F(y_{t-1})^{N-(t-1)}}$$

if $y_1 \geq \ldots \geq y_{t-1}$ and is zero otherwise.

When the X_i's are values, conversely, it is convenient to order them from lowest to highest. Let $Z_1^{(N)}, \ldots, Z_N^{(N)}$ be a rearrangement of the X_i's such that $Z_1^{(N)} \leq Z_2^{(N)} \leq \ldots \leq Z_N^{(N)}$. The joint density of $Z_1^{(N)}, \ldots, Z_N^{(N)}$ is

$$h_{1,\ldots,N}^{(N)}(z_1, \ldots, z_N) = N! \prod_{i=1}^{N} f(z_i)$$

if $z_1 \leq z_2 \leq \ldots \leq z_N$ and zero otherwise. The conditional density of $Z_t^{(N)}$ given $Z_1^{(N)} = z_1, \ldots, Z_{t-1}^{(N)} = z_{t-1}$ is

$$h_t^{(N)}(z_t | z_1, \ldots, z_{t-1}) = h_t^{(N)}(z_t | z_{t-1}) = (N - (t-1)) f(z_t) \frac{[1 - F(z_t)]^{N-t}}{[1 - F(z_{t-1})]^{N-(t-1)}}$$

if $0 \leq z_1 \leq \ldots \leq z_{t+1}$ and is zero otherwise.

With these orderings, in equilibrium (as we will show) bidders drop in order of their index, smallest to largest.

The Chore Auction

The chore auction selects K of N bidders to undertake K identical chores or, equivalently, it selects K bidders to undertake a single chore that requires K bidders to complete. At each round there is a descending clock auction in which the price starts at $\bar{x}$ and decreases continuously. Bidders may drop out at any point. A bidder who at round t drops out at p_t is excused from the chore, pays p_t in compensation, and obtains a payoff of $u(-p_t)$. A new round then begins and this process repeats until exactly K bidders remain (i.e., for $N-K$ rounds). Each of these bidders undertakes a chore and receives an equal share of the total compensation, i.e., $\frac{1}{K}\Sigma_{j=1}^{N-K} p_j$, paid by the bidders who dropped. The payoff of a bidder with cost x who undertakes a chore is $u(\frac{1}{K}\Sigma_{j=1}^{N-K} p_j - x)$.

The Goods Auction

The goods auction selects K of N bidders to receive one of K identical items. At each round there is an ascending clock auction in which the price starts at zero and increases continuously. Bidders may drop out at any point. A bidder who at round t drops out at p_t surrenders his claim to an item, receives p_t in compensation, and obtains a payoff of $u(p_t)$. A new round then begins and this process repeats until exactly K bidders remain (i.e., for $N-K$ rounds). Each of these bidders receives an item and pays an equal share of the total compensation, i.e., $\frac{1}{K}\Sigma_{j=1}^{N-K} p_j$, promised to the bidders who dropped. The payoff of a bidder with value x who receives an item is $u(x - \frac{1}{K}\Sigma_{j=1}^{N-K} p_j)$.

For both auctions, a strategy for a bidder is a list of $N-K$ functions which identifies a bidder's drop out price at each round of the auction. We write $\mathbf{p}_t$ for $(p_1, \ldots, p_t)$ and take $\mathbf{p}_0 = 0$. For the chore auction we denote a strategy by $\delta = (\delta_1, \ldots, \delta_{N-K})$, where $\delta_t(x; \mathbf{p}_{t-1})$ gives the dropout price in the t-th round of a bidder with cost x when $t-1$ bidders have dropped out at prices $\mathbf{p}_{t-1}$. Likewise, for the goods auction we denote a strategy by $\beta = \left(\beta_1, \ldots, \beta_{N-K}\right)$, where $\beta_t(x; \mathbf{p}_{t-1})$ gives the dropout price in the t-th round of a bidder with value x when $t-1$ bidders have dropped out at prices $\mathbf{p}_{t-1}$.

3 Equilibrium

Proposition 1(i) identifies necessary conditions for δ to be a symmetric Bayes Nash equilibrium of the chore auction in increasing and differentiable strategies. Proposition 1(ii) establishes that the necessary conditions are also sufficient. The analogous result for the goods auction is provided in the Appendix as Proposition 1$'$.

Proposition 1: *(i) Any symmetric equilibrium δ of the chore auction in increasing and differentiable bidding strategies satisfies the following system of differential equations:*

$$u'(-\delta_{N-K}(x;\mathbf{p}_{N-K-1}))\delta'_{N-K}(x;\mathbf{p}_{N-K-1})$$

$$= \left[u(-\delta_{N-K}(x;\mathbf{p}_{N-K-1})) - u\left(\frac{1}{K}\left[\delta_{N-K}(x;\mathbf{p}_{N-K-1}) + \sum_{i=1}^{N-K-1} p_i\right] - x\right)\right]\bar{\lambda}^N_{N-K}(x),$$

and, for $t \in \{1,\ldots,N-K-1\}$, that

$$\begin{aligned} & u'\left(-\delta_t(x;\mathbf{p}_{t-1})\right)\delta'_t(x;\mathbf{p}_{t-1}) \\ = \ & [u\left(-\delta_t(x;\mathbf{p}_{t-1})\right) - u(-\delta_{t+1}(x;\mathbf{p}_{t-1},\delta_t(x;\mathbf{p}_{t-1})))]\bar{\lambda}^N_t(x), \end{aligned}$$

where

$$\bar{\lambda}^N_t(x) = (N-t)\frac{f(x)}{F(x)}.$$

(ii) If $\delta = (\delta_1,\ldots,\delta_{N-K})$ is a solution to the system of differential equations in (i), then it is an equilibrium.

These differential equations have a simple interpretation. At any round, the marginal benefit to a bidder from remaining in the chore auction a moment longer is that he pays less compensation if no other bidder drops out in the interim. At the last round, the marginal cost of remaining a moment longer is that a rival bidder drops out and the bidder must undertake the chore. At earlier rounds, the marginal cost of remaining is that a rival bidder drops out, in which case the bidder continues into the next round and pays the compensation that his rival would have paid. The differential equations state that, at each round, marginal benefit and cost are equalized at equilibrium.

We will provide closed-form expressions for the unique symmetric equilibrium when the bidders are either risk neutral or CARA risk averse with utility function

$$u^\alpha(x) = \frac{1-e^{-\alpha x}}{\alpha},$$

where α is the index of risk aversion. Bidders are risk neutral in the limit as α approaches zero. When bidders have index of risk aversion α, we denote the equilibrium bid functions for the chore and the goods auctions by δ_t^α and β_t^α, respectively.

3.1 Risk Neutral Bidders

Proposition 2 characterizes equilibrium bidding in the chore and the goods auctions when bidders are risk neutral.

Proposition 2: *Suppose that bidders are risk neutral.*
(P 2.1) *The unique symmetric equilibrium in increasing and differentiable strategies for the chore auction is, for $t = 1, \ldots, N-K$,*

$$\delta_t^0(x; \mathbf{p}_{t-1}) = \frac{K}{N-t+1}\left(E\left[Y_{N-K}^{(N)} | Y_{t-1}^{(N)} > x > Y_t^{(N)}\right] - \frac{1}{K}\sum_{i=1}^{t-1} p_i\right).$$

(P 2.2) *The unique symmetric equilibrium in increasing and differentiable strategies for the goods auction is, for $t = 1, \ldots, N-K$,*

$$\beta_t^0(x; \mathbf{p}_{t-1}) = \frac{K}{N-t+1}\left(E\left[Z_{N-K}^{(N)} | Z_t^{(N)} > x > Z_{t-1}^{(N)}\right] - \frac{1}{K}\sum_{i=1}^{t-1} p_i\right).$$

The goods auction is the "dual" of the chore auction: it is an ascending clock auction rather than a descending clock auction, and bidders receive compensation to surrender their claim to the good rather than pay compensation to be excused from the chore. The dual nature of the auctions is apparent from the equilibrium bid functions, which have the same structure when expressed using the highest and lowest order statistics. In the bid function for the chore auction, the term $E[Y_{N-K}^{(N)} | Y_{t-1}^{(N)} > x > Y_t^{(N)}]$ is the expected cost of the lowest cost bidder that will be excused from the chore, conditional on the bidder's own cost being between the $t-1$-st and t-th highest cost. Likewise for the goods auction.

The bid functions are increasing in x for both auctions. Since the chore auction is a descending clock auction, in equilibrium the bidders with the $N-K$ highest costs drop and pay compensation to the bidders with the K lowest costs, who each undertake a chore. Since the goods auction is an ascending clock auction, in equilibrium the bidders with the $N-K$ lowest values drop and receive compensation from the bidders with the K highest values, who each receive an item.

Example 1: Suppose $N = 4$, $K = 2$, bidders are risk neutral, and values are distributed $U[0,1]$. In the chore auction, the equilibrium drop price in round 1 is

$$\delta_1^0(x) = \frac{3}{10}x$$

and in round 2 is

$$\delta_2^0(x; p_1) = \frac{1}{2}x - \frac{1}{3}p_1.$$

Figure 1 below shows these bid functions. The round 2 bid function is shown when the realized round 1 compensation is $p_1 = 1/10$, which reveals that the highest cost of a bidder is 1/3.

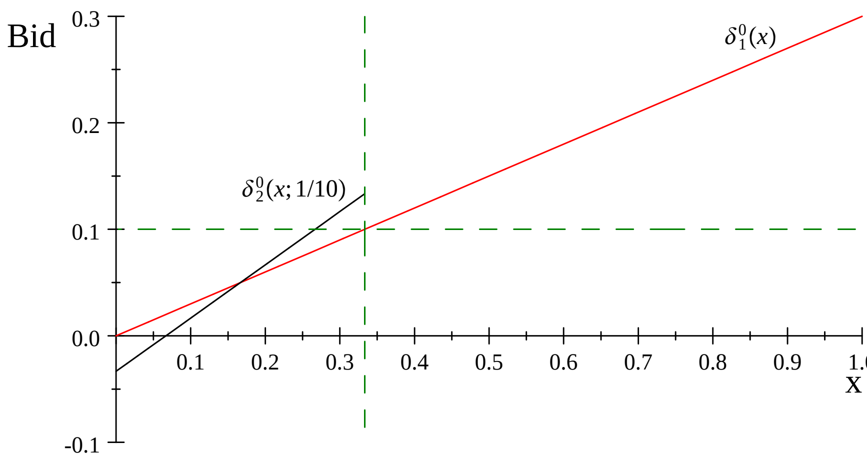

Fig. 1: Equilibrium bids by round, for $N = 4$, $K = 2$, and $U[0,1]$.

The figure illustrates several interesting features of equilibrium. First, bid functions "jump up" whenever a bidder drops out. In this example, a bidder with cost 1/3 drops at the first round at the price $\delta_1^0(1/3) = 1/10$. At the next round, a bidder with the same cost would drop at a higher price of $\delta_2^0(1/3; 1/10) = 2/15$. This is a general feature of equilibrium: more precisely $\delta_2^0(x; \delta_1^0(x)) \geq \delta_1^0(x)$ for any x. To see this, observe that if $\delta_2^0(x; \delta_1^0(x)) < \delta_1^0(x)$ then it would not be optimal for a bidder with value x to obey equilibrium and drop at round 1 at $\delta_1^0(x)$ since by waiting for a moment longer either (i) he drops at round 1 and pays lower compensation, or (ii) a rival bidder with value x' (with $x' < x$) drops in the interim. In the later case, the bidder, by bidding at round 2 as though his own value were x' pays $\delta_2^0(x'; \delta_1^0(x'))$ compensation. By continuity of the bidding strategies we have $\delta_2^0(x'; \delta_1^0(x')) < \delta_1^0(x')$ for x' close to x. Also, since δ_1^0 is increasing we have that $\delta_1^0(x') < \delta_1^0(x)$. Thus

$\delta_2^0(x'; \delta_1^0(x')) < \delta_1^0(x)$ and so he pays less compensation in the later case as well. In either case the bidder pays less compensation, which is a contradiction.

Figure 1 also illustrates, since $\delta_2^0(x; p_1) < 0$ for some x, that a bidder may accept zero or negative compensation to undertake the chore. While this may at first seem surprising, it is intuitive given that the compensations promised by bidders who have dropped at earlier rounds may be large enough to make undertaking the chore attractive. In this example, when $p_1 = 1/10$, then round 2 bids are negative for costs less than 1/15. Suppose that the second highest cost is 1/15. Then the bidder with this cost drops (and is excused from the chore) when compensation reaches zero. Each remaining bidder undertakes a chore, obtains compensation of $p_1/2 = 1/20$ and obtains a positive payoff when their cost is below 1/20.

Payoff Equivalence

Since the chore and the goods auctions are both efficient and can be used to solve the same allocation problem, it is natural to question whether bidders would have a preference for one mechanism over the other. Following a standard argument, Proposition 3 shows that risk neutral bidders obtain the same expected payoff in every efficient symmetric equilibrium of any symmetric budget balanced mechanism. We state the proposition for goods rather than chores.

Proposition 3: *Suppose that bidders are risk neutral. Let ξ be a symmetric budget-balanced mechanism and let β be a symmetric equilibrium of ξ in which K units of a good are allocated to the bidders with the K highest values. The expected utility of a bidder with value x is*

$$\frac{K}{N} E\left[Z_{N-K}^{(N)}\right] + \int_0^x H_{N-K}^{(N-1)}(z)dz,$$

and is independent of ξ, where $H_{N-K}^{(N-1)}(z)$ is the probability that the $N-K$-th lowest of $N-1$ values is less than z.

The problem of allocating K chores among N bidders can be solved by either a chore auction (with $N-K$ rounds and one bidder excused from the chore at each round) or a goods auction with $N-K$ goods (with K rounds and one bidder assigned a chore at each round).[8] Since the requirements of Proposition 3 are met, i.e., the

[8]In the later case, the "good" is to be excused from the chore and thus a bidder who surrenders his claim to the good must undertake the chore.

chore and goods auctions are symmetric and budget balanced, then the auctions are payoff equivalent at efficient and symmetric equilibria.

Corollary 1: *Suppose there are N bidders. The chore auction with K chores is payoff equivalent to the goods auction with $N-K$ goods.*

3.2 Risk Averse Bidders

In this section we characterize equilibrium for CARA bidders. The CARA assumption is appropriate when bidders' preferences over lotteries do not exhibit wealth effects, and is common in the literature. The results in this section provide a bridge between risk neutral equilibrium bidding and maxmin perfect bidding (discussed in the next section). We first provide a closed form expression for the symmetric equilibrium. We then identify the limit bidding strategies as bidders become infinitely risk averse. The next section establishes that the limit bidding strategies are "maxmin perfect."

Proposition 4 characterizes equilibrium in the chore and goods auctions when bidders have CARA preferences.

Proposition 4: *Suppose that bidders are CARA risk averse with index of risk aversion $\alpha > 0$.*
(P 4.1): *The unique symmetric equilibrium for the chore auction in increasing and differentiable strategies is given, for $t = 1, \ldots, N-K$, by*

$$\delta_t^\alpha(x; \mathbf{p}_{t-1}) = \frac{N-t}{(N-t+1)\,\alpha} \ln\left(S_t^\alpha(x)\right) - \frac{1}{N-t+1} \sum_{i=1}^{t-1} p_i,$$

where

$$S_{N-K}^\alpha(x) = E\left[e^{\alpha Y_{N-K}^{(N)}} | Y_{N-K-1}^{(N)} > x > Y_{N-K}^{(N)}\right]$$

and, for $t < N-K$, $S_t^\alpha(x)$ is defined recursively as

$$S_t^\alpha(x) = E\left[\left(S_{t+1}^\alpha(Y_t^{(N)})\right)^{\frac{N-t-1}{N-t}} | Y_{t-1}^{(N)} > x > Y_t^{(N)}\right].$$

(P 4.2): *The unique symmetric equilibrium for the goods auction in increasing and differentiable strategies is given, for $t = 1, \ldots, N-K$, by*

$$\beta_t^\alpha(x; \mathbf{p}_{t-1}) = -\frac{N-t}{(N-t+1)\,\alpha} \ln\left(D_t^\alpha(x)\right) - \frac{1}{N-t+1} \sum_{i=1}^{t-1} p_i$$

where

$$D^{\alpha}_{N-K}(x) = E\left[e^{-\alpha Z^{(N)}_{N-K}} | Z^{(N)}_{N-K} > x > Z^{(N)}_{N-K-1}\right]$$

and, for $t < N-K$, $D^{\alpha}_t(x)$ is defined recursively as

$$D^{\alpha}_t(x) = E\left[\left(D^{\alpha}_{t+1}(Z^{(N)}_t)\right)^{\frac{N-t-1}{N-t}} | Z^{(N)}_t > x > Z^{(N)}_{t-1}\right].$$

While the chore and goods auctions are payoff equivalent when bidders are risk neutral, they are not payoff equivalent when bidders have CARA preferences.[9]

Bounds and Comparative Statics

Proposition 5 establishes that risk aversion causes a bidder to demand less compensation in the goods auction, forgoing some expected return in favor of a reduced riskiness in their payoff. Likewise, in the chore auction, risk aversion causes a bidder to offer more compensation to be excused from the chore. Proposition 5 also establishes bounds on the equilibrium bid functions.

Proposition 5: *For each $\alpha > 0$ and $t = 1, \ldots, N-K$, the bid functions $\delta^{\alpha}_t(x; \mathbf{p}_{t-1})$ and $\beta^{\alpha}_t(x; \mathbf{p}_{t-1})$ satisfy*

$$\delta^0_t(x; \mathbf{p}_{t-1}) < \delta^{\alpha}_t(x; \mathbf{p}_{t-1}) < \gamma_t(x; \mathbf{p}_{t-1}) < \beta^{\alpha}_t(x; \mathbf{p}_{t-1}) < \beta^0_t(x; \mathbf{p}_{t-1}) \ \forall x \in (0, \bar{x})$$

where

$$\gamma_t(x; \mathbf{p}_{t-1}) \equiv \frac{K}{N-t+1}(x - \frac{1}{K}\sum_{i=1}^{t-1} p_i).$$

The function γ_t provides an upper bound for bids in the chore auction and a lower bound for bids in the goods auction, and it has a natural interpretation. Consider the goods auction. The total surplus (as viewed by a bidder with value x) at round t available to the bidders who remain in the auction is $Kx - \sum_{i=1}^{t-1} p_i$. The bound γ_t is an equal share of this surplus divided among the $N-t+1$ remaining bidders. In the goods auction, since $\beta^{\alpha}_t(x; \mathbf{p}_{t-1}) > \gamma_t(x; \mathbf{p}_{t-1})$, a bidder demands compensation of at least this amount. A similar interpretation applies for the chore auction.

[9] Theorem 6 of McAfee (1992) shows that one auction does not interim dominate another for the two bidder case. In particular, under some regularity conditions, low types prefer the chore auction (what McAfee calls the winner's bid auction) and high types prefer the goods auction (what McAfee calls the loser's bid auction).

Figure 2 below illustrates Proposition 5 when values are distributed $U[0,1]$ and $K = 3$ and $N = 4$. The equilibrium bid functions for the chore auction are in red (a solid line labelled $\delta_1^0(x)$ for $\alpha = 0$, and a dashed line labelled $\delta_1^{10}(x)$ for $\alpha = 10$). The analogous bid functions for the goods auction are shown in green and are labelled $\beta_1^0(x)$ and $\beta_1^{10}(x)$. The bound $\gamma_1(x) = 3x/4$ is in black and lies in the center.

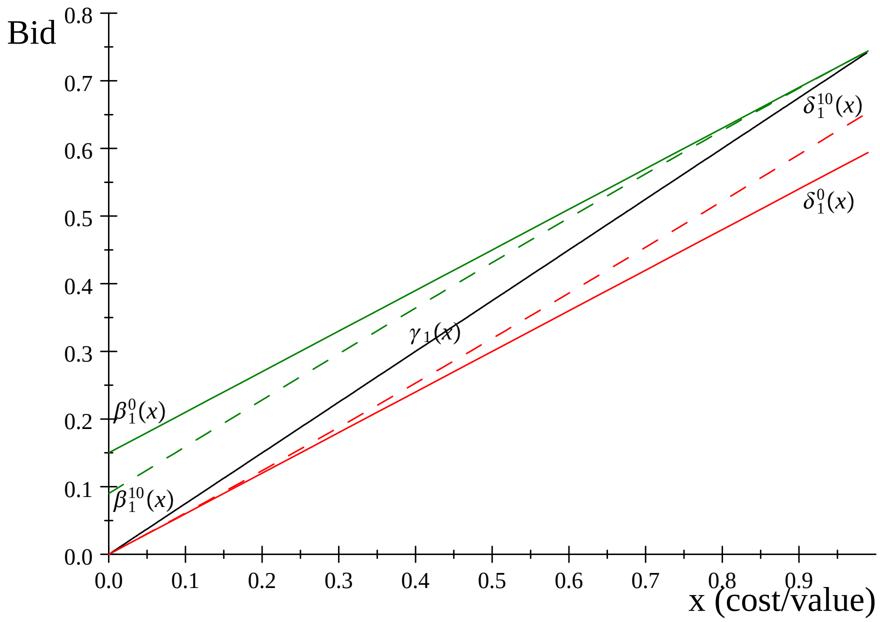

Fig. 2: Equilibrium bidding – $\alpha = 0,$ and $\alpha = 10,$ and maxmin perfect

Proposition 6 establishes that in the chore auction bidders drop out earlier and pay more compensation to be excused from the chore as they are more risk averse. In the goods auction, bidders also drop out earlier but receive less compensation for surrendering their claim to a unit of the good as they are more risk averse. Although the problems of allocating K chores or K goods are not equivalent, remarkably the chore and the goods auctions have the same equilibrium bid function in the limit as bidders become infinitely risk averse. (Recall that the problem of allocating K goods can be equivalently framed as a problem of allocating $N - K$ chores.)

Proposition 6: *For each t, the bid function $\delta_t^\alpha(x; \mathbf{p}_{t-1})$ is increasing in α, the bid function $\beta_t^\alpha(x; \mathbf{p}_{t-1})$ is decreasing in α, and both have $\gamma_t(x; \mathbf{p}_{t-1})$ as their limit as α approaches infinity, i.e.,*

$$\lim_{\alpha \to \infty} \delta_t^\alpha(x; \mathbf{p}_{t-1}) = \lim_{\alpha \to \infty} \beta_t^\alpha(x; \mathbf{p}_{t-1}) = \gamma_t(x; \mathbf{p}_{t-1}) \ \forall x.$$

The function γ_t will play an important role in the next section.

4 Maxmin and Maxmin Perfect Strategies

In this section we take a decision-theoretic approach to bidding in these auctions, identifying the strategy a bidder should follow if he seeks to maximize his minimum payoff. Maxmin has a long history as a decision making criteria under ambiguity (i.e., when probabilities are unknown). It is closely connected to "robust" decision making and "robust" optimization. Maxmin is a special case of maxmin expected utility (MEU), as axiomatized by Gilboa and Schmeidler (1989).

For an active bidder at round t, let $v_t(\mathbf{x}, \boldsymbol{\delta}, \mathbf{p}_{t-1})$ be the bidder's payoff in the chore auction when $\mathbf{x} = (x_i, x_{-i})$ is the profile of values, $\boldsymbol{\delta} = (\delta^i, \delta^{-i})$ is the profile of strategies, and $\mathbf{p}_{t-1}$ is the sequence of dropout prices.

Definition: A strategy δ^i *guarantees* bidder i with value x_i a payoff of $\bar{v}_t$ at round t, given $\mathbf{p}_{t-1}$, if $v_t((x_i, x_{-i}), (\delta^i, \delta^{-i}), \mathbf{p}_{t-1}) \geq \bar{v}_t \ \forall x_{-i}, \delta^{-i}$.

Let $\bar{v}_t(x_i, \mathbf{p}_{t-1})$ be the largest payoff that bidder i with value x_i can guarantee in round t given $\mathbf{p}_{t-1}$.[10] Then $\bar{v}_1(x_i, \mathbf{p}_0)$ is the largest payoff that bidder i with value x_i can guarantee at the start of the auction.

Definition: A strategy $\bar{\delta}^i$ is a *maxmin strategy for bidder* i if $\bar{\delta}^i$ guarantees $\bar{v}_1(x_i, \mathbf{p}_0)$ for each $x_i \in [0, \bar{x}]$.

Proposition 7, which follows, shows that $\bar{v}_1(x_i, \mathbf{p}_0) = -Kx_i/N$ for the chore auction. It is easy to see that the strategy which calls for a bidder to drop out whenever the bid falls to Kx_i/N is a maxmin strategy: If a bidder drops, then he pays $-Kx_i/N$. If he never drops, it is because $N-K$ rivals dropped at bids (and pay compensation) greater than Kx_i/N. The total compensation is thus at least $(N-K)Kx_i/N$, of which bidder i receives $1/K$-th. Hence he receives compensation of at least $(N-K)x_i/N$ and his payoff is at least $(N-K)x_i/N - x_i = -Kx_i/N$. By the same reasoning, the same strategy is a maxmin strategy for the goods auction.

[10] Proposition 7 will establish that $\bar{v}_t(x_i, \mathbf{p}_{t-1})$ is well defined.

This strategy is simple in the sense that it does not depend on the prices at which rivals dropped at prior rounds. However, as the auction progresses, a bidder may be able to guarantee himself more than $-Kx_i/N$, e.g., if at round 1 a rival bidder drops at a bid above Kx_i/N. A maxmin perfect strategy maximizes a bidder's minimum payoff at every point along every path of play.

Definition: A strategy $\bar{\delta}^i$ is a *maxmin perfect strategy for bidder* i if $\bar{\delta}^i$ guarantees $\bar{v}_t(x_i, \mathbf{p}_{t-1})$ for each t, $x_i \in [0, \bar{x}]$, and $\mathbf{p}_{t-1}$.

Proposition 7 identifies the unique maxmin perfect strategy for the chore auction and shows that the same strategy is also the unique maxmin perfect strategy for the goods auction when N and K are the same in both cases.[11] The maxmin perfect strategy is, furthermore, identified by Proposition 6 as the limit of the equilibrium bid functions as bidders become infinitely risk averse.

Proposition 7: *In both the chore auction with K chores and the goods auction with K goods, the strategy γ^i, given by*

$$\gamma_t^i(x_i; \mathbf{p}_{t-1}) = (Kx_i - \sum_{m=1}^{t-1} p_m)/(N - t + 1)$$

for each $t \in \{1, \ldots, N-K\}$, and every $x_i \in [0, \bar{x}]$ and $\mathbf{p}_{t-1}$ is the unique maxmin perfect strategy. In the chore auction

$$\bar{v}_t(x_i, \mathbf{p}_{t-1}) = -\gamma_t^i(x_i; \mathbf{p}_{t-1}),$$

with $\bar{v}_1(x_i, \mathbf{p}_0) = -Kx_i/N$. In the goods auction

$$\bar{v}_t(x_i, \mathbf{p}_{t-1}) = \gamma_t^i(x_i; \mathbf{p}_{t-1}),$$

with $\bar{v}_1(x_i, \mathbf{p}_0) = Kx_i/N$.

The intuition for why the two auctions have the same maxmin perfect strategy is clear. Consider round $N-K$ in the chore auction following drop out prices $\mathbf{p}_{N-K-1}$. A bidder with value x_i whose strategy calls for him to drop at price p either drops

[11] For fixed N and K, the two auctions solve different problems. As noted in the Introduction, allocating K chores can be reframed as allocating $N-K$ goods.

at price p (and obtains $-p$) or a rival bidder drops at a higher price in which case he obtains a payoff of at least

$$\frac{1}{K}p + \frac{1}{K}\sum_{m=1}^{N-K-1} p_m - x_i.$$

The bidder maximizes his minimum payoff by choosing p to equate these two payoffs. Solving for p yields $\gamma^i_{N-K}(x_i; \mathbf{p}_{N-K-1})$.

In the goods auction, by contrast, a bidder whose strategy calls for him to drop at price p either drops at price p (and obtains p) or a rival bidder drops at a lower price in which case he wins the item and he obtains a payoff of at least

$$x_i - \frac{1}{K}p - \frac{1}{K}\sum_{m=1}^{N-K-1} p_m.$$

Again, the bidder maximizes his minimum payoff by choosing p to equate these two payoffs. It is immediate that the same p maximizes the bidder's minimum payoff in both the chore auction and the goods auction in the last round. An induction argument establishes the proposition.

An implication of Proposition 7 is that participation in the chore auction is individually rational if the alternative is the random assignment of chores. To see this, note that a bidder's equilibrium payoff must be at least the certain payoff he can guarantee himself in round 1, i.e., at least $u(-Kx_i/N)$. By the concavity of u and since $u(0) = 0$, we have

$$u\left(\frac{K}{N}(-x_i)\right) = u\left(\frac{K}{N}(-x_i) + (1 - \frac{K}{N})(0)\right) \geq \frac{K}{N}u(-x_i) + (1 - \frac{K}{N})u(0).$$

Hence a bidder's equilibrium payoff is at least $Ku(-x_i)/N$, the expected payoff obtained if chores are allocated randomly. The symmetric argument establishes the individual rationality of the goods auction.

5 Relating Non-cooperative and Cooperative Solutions

An alternative approach to solving the bargaining problem studied in this paper is provided by cooperative game theory. In a cooperative game, the "value" of a collection of players, i.e., a coalition, is the surplus that its members can generate. A "solution" to a cooperative game identifies a payoff to each player and his allocation, when the players can negotiate among themselves.

In this section we relate the equilibrium allocations of the goods auction and the chore auction to the Shapley value allocations of two cooperative games.[12] A cooperative game is defined by a set of players $I = \{1, \ldots, N\}$ and a characteristic function $v : 2^I \to \mathbb{R}^+$, where $v(S)$ is the value of coalition S. We consider two characteristic functions. For the *pessimistic characteristic function*, denoted by $\underline{v}$, the value of coalition S is

$$\underline{v}(S) = \begin{cases} 0 & \text{if } K \leq N - |S| \\ \sum_{m=N-K+1}^{|S|} z_m^{(S)} & \text{if } K > N - |S|, \end{cases}$$

where $z_1^{(S)} \leq \ldots \leq z_{|S|}^{(S)}$. For the *optimistic characteristic function*, denoted by $\bar{v}$, the value of coalition S is

$$\overline{v}(S) = \begin{cases} \sum_{m=1}^{K} y_m^{(S)} & \text{if } K < |S| \\ \sum_{m=1}^{|S|} y_m^{(S)} & \text{if } K \geq |S|, \end{cases}$$

where $y_1^{(S)} \geq \ldots \geq y_{|S|}^{(S)}$.

The interpretation is as follows: For the pessimistic characteristic function $\underline{v}$, a coalition S receives units only after each member of the complementary coalition receives a unit. If the complementary coalition has K or more members, then S receives no units and $\underline{v}(S) = 0$. If the complementary coalition has fewer than K members then S receives $K - (N - |S|)$ units. For the optimistic characteristic function $\overline{v}$, by contrast, coalition S receives units first, i.e., it receives $\min\{|S|, K\}$ units. In both cases, units allocated to S are allocated efficiently among its members, i.e., to the members of S with the highest values. Clearly, $\underline{v}(S) \leq \overline{v}(S)$ for any S and $\underline{v}(N) = \overline{v}(N)$.

A solution in cooperative game theory is function ϕ that maps the characteristic function v of an N-player cooperative game to a payoff profile $\phi(v) = (\phi_1(v), \ldots, \phi_N(v))$. The leading solution concept in cooperative game theory is the Shapley value. Shapley (1951) shows that the Shapley value is the unique solution that satisfies axioms

[12] Introductions to cooperative game theory can be found in Moulin (1995) and Chakravarty, Mitra, and Sarkar (2015).

of efficiency, symmetry, additivity, and null player. The Shapley value, and the associated allocation, is often taken as the benchmark for a fair allocation (e.g., Myerson (1977), Roth (1988), Moulin (1992)).

In a cooperative game with characteristic function v, the Shapley value of a player i is

$$\phi_i(v) = \sum_{S \subseteq I} \frac{(|S|-1)!(N-|S|)!}{N!} \left[v(S) - v(S \backslash \{i\}) \right].$$

A player's Shapley value can be interpreted as his expected marginal contribution when the grand coalition is formed by adding players one after another in a random order.

Define $\underline{\phi}_i = \phi_i(\underline{v})$ and $\bar{\phi}_i = \phi_i(\bar{v})$. We refer to $\underline{\phi}_i$ and $\bar{\phi}_i$, respectively, as the *strategic* and the *normative* Shapley values of player i. We refer to the *strategic Shapley value allocation* as (i) the efficient assignment of the K items among the N players and (ii) the associated transfers which give each player his strategic Shapley payoff. The *normative Shapley value* allocation is defined similarly.

The Core and the Anti-core

The strategic and normative Shapley value allocations belong, respectively, to the core and anti-core of two appropriately defined cooperative games, a relationship which aids their interpretation. Given a cooperative game (I, v), the payoff vector $(\pi_1, \ldots, \pi_N)$ is in the *core-v* if (i) $\sum_{i=1}^N \pi_i = v(I)$ and (ii) for every $S \in 2^N$ we have that $\sum_{i \in S} \pi_i \geq v(S)$. The payoff vector $(\pi_1, \ldots, \pi_N)$ is in the *anti-core-v* if (i) $\sum_{i=1}^N \pi_i = v(I)$ and (ii) for every $S \in 2^N$ we have that $\sum_{i \in S} \pi_i \leq v(S)$.[13]

Moulin (1995) provides interpretations. The core-$\underline{v}$ has a strategic interpretation: If a payoff vector $(\pi_1, \ldots, \pi_N)$ is not in the core-$\underline{v}$, then there is a coalition S such that $\sum_{i \in S} \pi_i < \underline{v}(S)$. Such a payoff vector would be blocked by the members of S since the coalition can guarantee its members $\underline{v}(S)$ even when it receives units only after every member of the complementary coalition has received a unit. By contrast, the anti-core-$\bar{v}$ has a normative interpretation: If a payoff vector $(\pi_1, \ldots, \pi_N)$ is not in the anti-core-$\bar{v}$, then there is a coalition S such that $\sum_{i \in S} \pi_i > \bar{v}(S)$. Such a payoff vector is unfair as the members of S not only receive units before members of the complementary coalition, but they also receive a subsidy from the complementary coalition. The complementary coalition would object.

[13] Moulin (1995) calls the anti-core the "core_". See pp. 404-406.

Write $\underline{\phi} = (\phi_1(\underline{v}), \ldots, \phi_N(\underline{v}))$ and $\bar{\phi} = (\phi_1(\bar{v}), \ldots, \phi_N(\bar{v}))$. Note that $\underline{\phi}$ and $\bar{\phi}$ are, respectively, the barycenter of the core-$\underline{v}$ and anti-core-$\bar{v}$, as can be seen in the figures below.[14]

THE CORE AND ANTI-CORE WHEN N=3

To illustrate these concepts, consider allocating a single item to one of three players, A, B, and C, whose values are $x_A = 3/4$, $x_B = 1/2$, and $x_C = 1/4$. Table 1 provides the constraints that define the anti-core-$\bar{v}$ and core-$\underline{v}$.

S	Anti-core-$\bar{v}$	Core-$\underline{v}$
$\{A\}$	$\pi_A \leq 3/4$	$\pi_A \geq 0$
$\{B\}$	$\pi_B \leq 1/2$	$\pi_B \geq 0$
$\{C\}$	$\pi_C \leq 1/4$	$\pi_C \geq 0$
$\{A, B\}$	$\pi_A + \pi_B \leq 3/4$	$\pi_A + \pi_B \geq 0$
$\{A, C\}$	$\pi_A + \pi_C \leq 3/4$	$\pi_A + \pi_C \geq 0$
$\{B, C\}$	$\pi_B + \pi_C \leq 1/2$	$\pi_B + \pi_C \geq 0$
$\{A, B, C\}$	$\pi_A + \pi_B + \pi_C = 3/4$	$\pi_A + \pi_B + \pi_C = 3/4$

Table 1: Anti-core-$\bar{v}$ and core-$\underline{v}$ when $K = 1$

Figure 3 shows the anti-core-$\bar{v}$ and core-$\underline{v}$. The normative Shapley value is $(\bar{\phi}_A, \bar{\phi}_B, \bar{\phi}_C) =$

[14] This follows from Shapley (1971, Theorem 7) and the concavity (convexity) of the cooperative game with charateristic function $\bar{v}$ ($\underline{v}$).

(11/24, 5/24, 2/24), and the strategic Shapley value is $(\underline{\phi}_A, \underline{\phi}_B, \underline{\phi}_C) = (1/4, 1/4, 1/4)$.

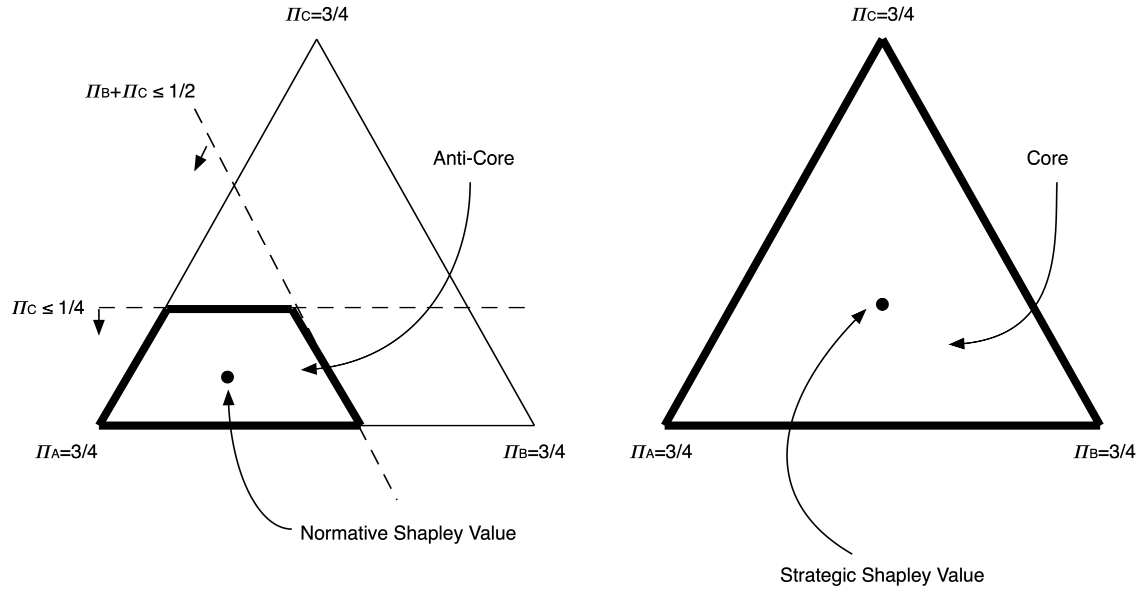

Figure 3: Anti-Core and Core, $N = 3$ and $K = 1$

Table 2 provides the constraints that define the anti-core-$\bar{v}$ and core-$\underline{v}$ when two units are allocated to players A, B, and C.

S	Anti-Core-$\bar{v}$	Core-$\underline{v}$
$\{A\}$	$\pi_A \leq 3/4$	$\pi_A \geq 0$
$\{B\}$	$\pi_B \leq 1/2$	$\pi_B \geq 0$
$\{C\}$	$\pi_C \leq 1/4$	$\pi_C \geq 0$
$\{A, B\}$	$\pi_A + \pi_B \leq 5/4$	$\pi_A + \pi_B \geq 3/4$
$\{A, C\}$	$\pi_A + \pi_C \leq 1$	$\pi_A + \pi_C \geq 3/4$
$\{C, B\}$	$\pi_B + \pi_C \leq 3/4$	$\pi_B + \pi_C \geq 1/2$
$\{A, B, C\}$	$\pi_A + \pi_B + \pi_C = 5/4$	$\pi_A + \pi_B + \pi_C = 5/4$

Table 2: Anti-core-$\bar{v}$ and Core-$\underline{v}$ when $K = 2$

Figure 4 shows the anti-core-$\bar{v}$ and core-$\underline{v}$. The normative Shapley value is $(\bar{\phi}_A, \bar{\phi}_B, \bar{\phi}_C) =$

(8/12, 5/12, 2/12), and the strategic Shapley value is $(\underline{\phi}_A, \underline{\phi}_B, \underline{\phi}_C) = (4/8, 3/8, 3/8)$.

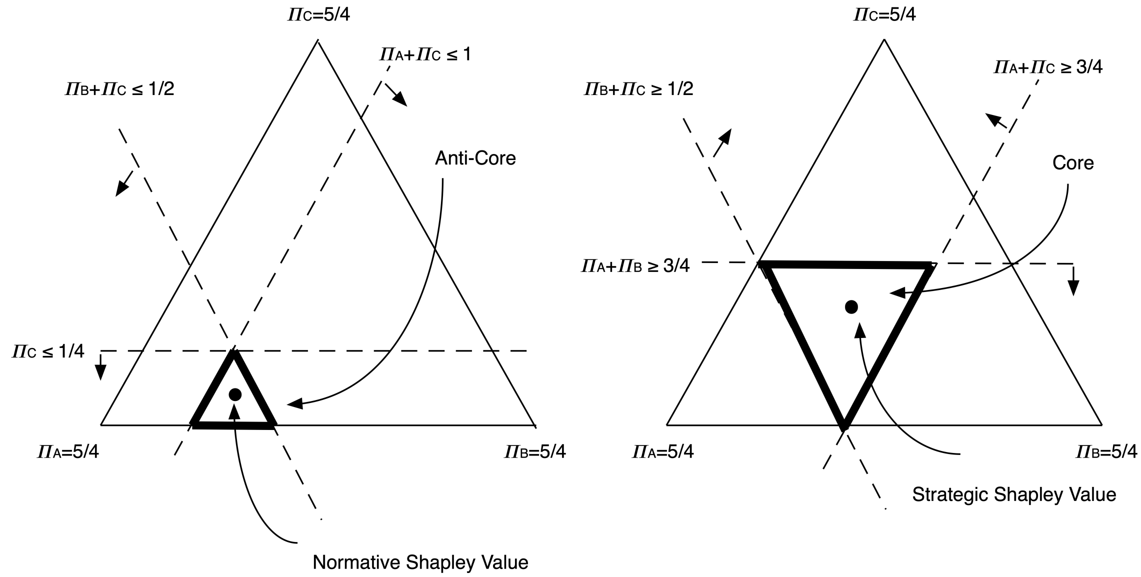

Figure 4: Anti-Core-$\bar{v}$ and Core-$\underline{v}$, $N = 3$ and $K = 2$

Maxmin Perfect Bidding

We now relate the equilibrium allocations that result under maxmin perfect bidding in the goods auction and chore auction to, respectively, the normative and strategic Shapley value allocations. We compare the goods auction with K goods, which completes in $N - K$ rounds, to the chore auction which allocates K goods. The chore auction has $\bar{K} = N - K$ chores and completes in K rounds. For example, if $N = 3$ and $K = 1$, then the goods auction runs for two rounds with, at each round, a bidder surrendering his claim for the good; the last remaining bidder obtains the good and pays compensation to the bidders who dropped out. In the chore auction that allocates one good there are two chores (where a chore is to give up a unit of the good). The chore auction runs for one round where the bidder who drops is excused from the chore and thus keeps the unit of the good.

Proposition 8: *(i) Suppose that K units of a good are allocated to N bidders via a chore auction (with $\bar{K}$ chores). If each bidder follows their maxmin perfect bidding strategy, then each bidder obtains his strategic Shapley value allocation associated with $\underline{\phi}$.*
(ii) Suppose that K units of a good are allocated to N bidders via a goods auction (with K goods). If each bidder follows his maxmin perfect bidding strategy, then each

bidder obtains his normative Shapley value allocation associated with $\bar{\phi}$.

Example 2: We illustrate Proposition 8 when allocating a single unit of the good among three bidders (A, B, and C) whose values are $x_A = 3/4$, $x_B = 1/2$, and $x_C = 1/4$. The maxmin perfect bid functions are given in Proposition 7.

Goods Auction (with one good): At round 1, Bidder C drops out at $p_1 = \frac{1}{3}x_C$ and receives compensation of 1/12. At round 2, Bidder B drops out at $p_2 = \frac{1}{2}x_B - \frac{1}{2}p_1$ and receives compensation of 5/24. Bidder A wins the good, pays compensation $p_1 + p_2 = 7/24$, and obtains a payoff of $3/4 - 7/24 = 11/24$. This allocation is the normative Shapley value allocation.

Chore Auction (with two chores): At round 1, Bidder A drops out at $p_1 = \frac{2}{3}x_A$, is excused from the chore (i.e., wins the unit of the good), pays $p_1 = 1/2$ compensation, and obtains a payoff of $3/4 - 1/2 = 1/4$. Bidders B and C each receive compensation of $p_1/2$ and obtain payoffs of 1/4. This is the strategic Shapley value allocation.

By Propositions 6 and 7, the equilibrium bid function in the goods auction and the chore auction both approach the maxmin perfect bid function, as bidders become infinitely risk averse. By Proposition 8, maxmin perfect bidding yields the normative Shapley value allocation in the goods auction and the strategic Shapley value in the chore auction. Thus we have the following corollary.

Corollary 2: *Suppose that bidders are CARA risk averse with index of risk aversion $\alpha > 0$. Then (i) the equilibrium allocation of the chore auction converges to the strategic Shapley value allocation as α approaches ∞; (ii) the equilibrium allocation of the goods auction converges to the normative Shapley value allocation as α approaches ∞.*

6 Discussion

This section discusses several aspects of our results. First, we show that a uniform price auction cannot generate either normative or strategic Shapley value allocations, in contrast to the goods and chore auctions. Second, we show that the normative and strategic Shapley value allocations can be interpreted as the outcome of sequential demands for equal shares of residual surpluses. The normative Shapley value allocation is obtained when demands are made sequentially by bidders ordered from the lowest to highest value, while the strategic Shapley value allocation is obtained when bidders are ordered from highest to lowest values. Finally, we illustrate the rate of convergence to the Shapley value allocation as a function of the degree of risk aversion. The discussion follows a running example in which there are 4 bidders and 2 items (i.e., $N = 4$ and $K = 2$), and the bidders have values $x_A > x_B > x_C > x_D$.

Uniform Price Auction and Shapley Value Allocation

We have shown that the chore and goods auctions are efficient, but there are other auctions which solve the same allocation problem efficiently. Consider, for example, a uniform price auction to allocate K units of a good, in which the price ascends continuously from zero. As the prices ascends, bidders exit the auction, and the auction ends when exactly K bidders remain. If the auction ends at price p, then every bidder who exits receives compensation p and obtains (the same) payoff of p; each of the remaining bidders receives a unit of the good and pays an equal share of the total compensation $(N - K)p$ owed to the bidders who exited. When bidders are risk neutral, this auction is payoff equivalent to both the goods auction and the chore auction, but the uniform price auction cannot generate Shapley value allocations, as we now show.

We first show that the uniform price auction cannot generate the *strategic* Shapley value allocation. The strategic Shapley values are given by

$$\underline{\phi}_A = \frac{1}{2}x_A$$

and

$$\underline{\phi}_B = \underline{\phi}_C = \underline{\phi}_D = \frac{1}{6}x_A + \frac{1}{3}x_B.$$

For the uniform price auction to generate the Shapley allocation requires that A receive an item and pay $p = \frac{1}{2}x_A$, for a payoff of $x_A - p = \underline{\phi}_A$. It requires that B receive an item and pay $p = \frac{2}{3}x_B - \frac{1}{6}x_A$, for a payoff of $x_B - p = \underline{\phi}_B$. However, generically A and B cannot pay the same price since $\frac{1}{2}x_A \neq \frac{2}{3}x_B - \frac{1}{6}x_A$.

Next, we show that the uniform price auction cannot generate the *normative*

Shapley value allocation. The normative Shapley values are given by

$$\begin{aligned}\overline{\phi}_A &= x_A - \frac{1}{3}x_C - \frac{1}{6}x_D \\ \overline{\phi}_B &= x_B - \frac{1}{3}x_C - \frac{1}{6}x_D \\ \overline{\phi}_C &= \frac{2}{3}x_C - \frac{1}{6}x_D \\ \overline{\phi}_D &= \frac{1}{2}x_D.\end{aligned}$$

For the uniform price auction to generate the normative Shapley allocation requires that neither C nor D receive an item and they receive different compensations since $\frac{2}{3}x_C - \frac{1}{6}x_D \neq \frac{1}{2}x_D$. This is not possible since they both receive the same compensation in the uniform price auction.

Shapley Values as Fair Shares of Residual Surpluses

The strategic and normative Shapley values can be interpreted as equal or "fair" shares of residual surpluses when units are allocated efficiently, as we now show. To compute the strategic Shapley value, we proceed sequentially from the bidder with the highest value to the bidder with the lowest value. Bidder A's strategic Shapley value is an equal share of the total surplus as he values it, i.e.,

$$\underline{\phi}_A = \frac{1}{N}(Kx_A) = \frac{1}{2}x_A.$$

Efficiency requires Bidder A receive an item. To obtain his Shapley payoff requires he pay $\frac{1}{2}x_A$.[15]

After A receives his allocation, the total residual surplus as B values it is equal to A's payment plus B's value for the remaining unit, i.e., $\frac{1}{2}x_A + x_B$. Bidder B's strategic Shapley value is an equal share of this surplus, i.e.,

$$\underline{\phi}_B = \frac{1}{N-1}(\frac{1}{2}x_A + x_B) = \frac{1}{6}x_A + \frac{1}{3}x_B.$$

Efficiency requires Bidder B receive an item. To obtain his Shapley payoff requires he pay $x_B - \underline{\phi}_B$ (i.e., $\frac{2}{3}x_B - \frac{1}{6}x_A$).[16] The total surplus remaining to be divided is the

[15] Observe that Bidder A's payment of $\frac{1}{2}x_A$ equals his round 1 maxmin perfect bid $\gamma_1^A(x_A)$, which in turn equals his payment in the chore auction under maxmin perfect bidding.

[16] Bidder B's payment of $\frac{2}{3}x_B - \frac{1}{6}x_A$ equals his round 2 maxmin perfect bid (and payment) of $\gamma_2^B(x_{B;}\frac{1}{2}x_A)$ in the chore auction, given Bidder A's round 1 maxmin perfect bid.

sum of A's and B's payments, i.e., $\frac{1}{3}x_A + \frac{2}{3}x_B$. Bidder C's strategic Shapley value is an equal share of this surplus, i.e.,

$$\underline{\phi}_C = \frac{1}{N-2}(\frac{2}{3}x_B + \frac{1}{3}x_A) = \frac{1}{6}x_A + \frac{1}{3}x_B.$$

After C receives his allocation, Bidder D receives the whole of the remaining surplus and hence

$$\underline{\phi}_D = \frac{1}{6}x_A + \frac{1}{3}x_B.$$

In this example, and in general, all bidders not receiving a unit receive the same payment and thus have the same strategic Shapley value.

Normative Shapley values can be computed in exactly the same fashion, except that we proceed sequentially from lowest to highest values. Bidder D's normative Shapley value is an equal share of the total surplus as he values it, i.e.,

$$\overline{\phi}_D = \frac{1}{N}(Kx_D) = \frac{1}{2}x_D.$$

Bidder D does not receive an item but receives a payment of $\overline{\phi}_D$.[17] The total residual surplus as C values it is $Kx_C - \overline{\phi}_D$. Bidder C's normative Shapley value is an equal share of this surplus, i.e.,

$$\overline{\phi}_C = \frac{1}{N-1}(Kx_C - \overline{\phi}_D) = \frac{2}{3}x_C - \frac{1}{6}x_D.$$

Bidder C does not receive an item but receives a payment of $\overline{\phi}_C$. The total residual surplus as B values it is $Kx_B - \overline{\phi}_C - \overline{\phi}_D$. Bidder B's normative Shapley value is

$$\overline{\phi}_B = \frac{1}{N-2}(Kx_B - \overline{\phi}_C - \overline{\phi}_D) = x_B - \frac{1}{3}x_C - \frac{1}{6}x_D.$$

Bidder B receives an item and pays $x_B - \overline{\phi}_B$. Bidder A receives the entire remaining surplus of

$$\underline{\phi}_A = x_A + (x_B - \overline{\phi}_B) - \overline{\phi}_C - \overline{\phi}_D = x_A - \frac{1}{3}x_C - \frac{1}{6}x_D.$$

Notice that A and B make equal payments of $\frac{1}{3}x_C + \frac{1}{6}x_D$. In general, all bidders receiving an item make the same payment.

Convergence Of Equilibrium to Shapley Allocation

[17] In the goods auction, bids are amounts received. Bidder D receives a payment $\frac{1}{2}x_D$, which equals his maximin perfect bid $\gamma_1^D(x_D)$.

By Corollary 2, the equilibrium allocation (of items and transfers) in the chore (goods) auction approaches the strategic (normative) Shapley value allocation. We illustrate this result with the 4 bidder and 2 item example above. To illustrate the equilibrium transfers we assume that $x_A = \frac{4}{5}$, $x_B = \frac{3}{5}$, $x_C = \frac{2}{5}$, and $x_D = \frac{1}{5}$.

Note that in both the goods auction and the chore auction, in equilibrium, A and B each receive an item, while C and D do not. The auctions allocate items efficiently, and according to the Shapley allocation, regardless of the bidders' risk atttitudes.

We first show that equilibrium transfers in the chore auction converge to the strategic Shapley value transfers as the index of risk aversion α grows large. In round 1 of the auction, bidder A pays $\delta_1^\alpha(\frac{4}{5})$, and in round 2 bidder B pays $\delta_2^\alpha(\frac{3}{5}; \delta_1^\alpha(\frac{4}{5}))$. Bidders C and D each receive a transfer of $\left[\delta_1^\alpha(\frac{4}{5}) + \delta_2^\alpha(\frac{3}{5}; \delta_1^\alpha(\frac{4}{5}))\right]/2$. Figure 5 shows each bidder's equilibrium transfer as a function of α. The dashed lines show the strategic Shapley value transfers for A, B, C, and D of, respectively, $-\frac{2}{5}$, $-\frac{8}{30}$, $\frac{1}{3}$, and $\frac{1}{3}$. It is evident that as α grows large, the players' equilibrium transfers approach their Shapley transfers and hence, since items are allocated as in the strategic Shapley allocation, the equilibrium allocation approaches the strategic Shapley value allocation.

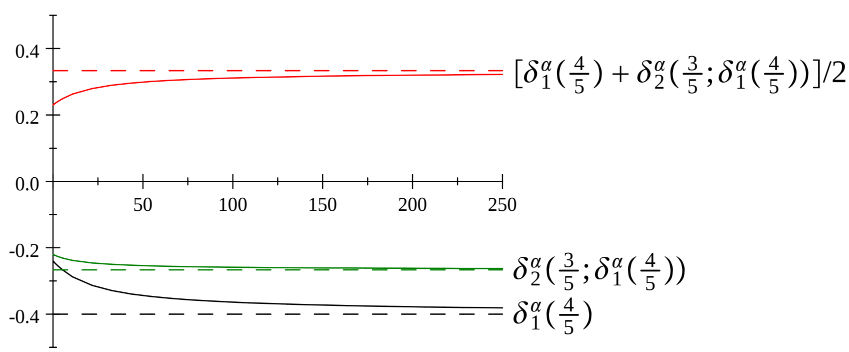

Figure 5: Equilibrium Transfers, Chore Auction

Equilibrium transfers in the goods auction likewise converge to the normative Shapley value transfers as the index of risk aversion α grows large. In round 1 of the auction, bidder D receives compensation $\beta_1^\alpha(\frac{1}{5})$ and in round 2 bidder C receives compensation $\beta_2^\alpha\left(\frac{2}{5}; \beta_1^\alpha(\frac{1}{5})\right)$. Bidders A and B each pay $\left[\beta_1^\alpha(\frac{1}{5}) + \beta_2^\alpha\left(\frac{2}{5}; \beta_1^\alpha(\frac{1}{5})\right)\right]/2$. Figure 6 shows equilibrum transfers (solid lines) and normative Shapley value transfers (dashed lines).[18]

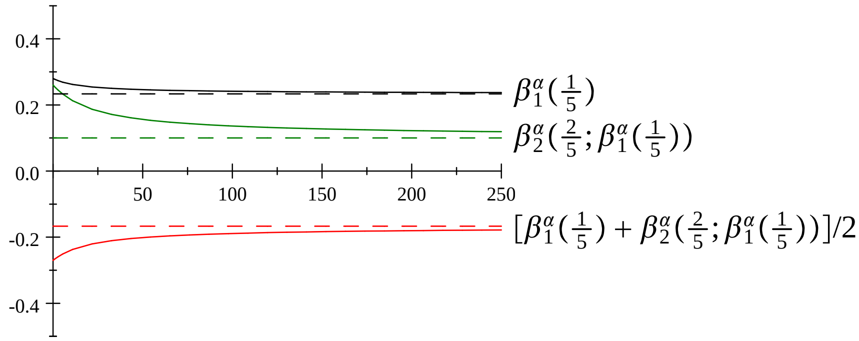

Figure 6: Equilibrium Transfers, Goods Auction

The speed at which the transfers converge is an open question and will likely depend on the primitives of the problem – such as number of bidders and the distribution of values of the problem.

7 Appendix

This appendix contains the statement of Proposition 1′, which provides necessary and sufficient conditions for the goods auction and is the analog to Proposition 1 in the body of the paper for the chore auction. It also contains the proofs for our results on the chore auction. The proofs for the goods auction are symmetric to the proofs for the chore auction, and are relegated to the Supplemental Appendix.[19]

Proposition 1′: *(i) Any symmetric equilibrium β of the goods auction in increasing and differentiable bidding strategies, satisfies the following system of differential equations:*

$$\begin{aligned}&u'(\beta_{N-K}(x;\mathbf{p}_{N-K-1}))\beta'_{N-K}(x;\mathbf{p}_{N-K-1})\\ &=[u(\beta_{N-K}(x;\mathbf{p}_{N-K-1}))-u(x-\frac{1}{K}(\beta_{N-K}(x;\mathbf{p}_{N-K-1})+\sum\nolimits_{i=1}^{N-K-1}p_i))]\lambda^N_{N-K}(x),\end{aligned}$$

and, for $t\in\{1,\ldots,N-K-1\}$, that

$$\begin{aligned}&u'\left(\beta_t(x;\mathbf{p}_{t-1})\right)\beta'_t(x;\mathbf{p}_{t-1})\\ =\;&[u\left(\beta_t(x;\mathbf{p}_{t-1})\right)-u(\beta_{t+1}(x;\mathbf{p}_{t-1},\beta_t(x;\mathbf{p}_{t-1})))]\lambda^N_t(x),\end{aligned}$$

[18]The normative Shapley value transfers for bidders D, C, B, and A are $\frac{1}{10}$, $\frac{7}{30}$, $-\frac{5}{30}$, and $-\frac{5}{30}$, respectively.

[19]See http://www.johnwooders.com/papers/DualAuctionsSupplementalAppendix.pdf.

where

$$\lambda_t^N(x) = (N-t)\frac{f(x)}{1-F(x)}.$$

(ii) If $\beta = (\beta_1, \ldots, \beta_{N-K})$ *is a solution to the system of differential equations in* (i)*, then it is an equilibrium.*

The proof of Proposition 1 follows.

Proof of Proposition 1: Let $\delta = (\delta_1, \ldots, \delta_{N-K})$ be a symmetric equilibrium in increasing and differentiable strategies. For each $t \leq N-K$, let $\pi_t(\hat{x}, x|\mathbf{y}_{t-1})$ be the expected payoff to a bidder with value x who in round t deviates from equilibrium and bids as though his value is $\hat{x}$ (i.e., he bids $\delta_t(\hat{x}|\mathbf{y}_{t-1})$), when $\mathbf{y}_{t-1}$ is the profile of values of the $t-1$ bidders to drop in prior rounds. In this case we will sometimes say the bidder "bids $\hat{x}$". Since equilibrium is in increasing strategies, at any round t the sequence of dropout prices $(p_1, \ldots, p_{t-1})$ reveals the $t-1$ highest values $\mathbf{y}_{t-1} = (y_1, \ldots, y_{t-1})$, and hence we can condition π_t and δ_t on $\mathbf{y}_{t-1}$ rather than $\mathbf{p}_{t-1}$.

Let

$$\Pi_t(x|\mathbf{y}_{t-1}) = \pi_t(x, x|\mathbf{y}_{t-1})$$

be the bidder's equilibrium payoff in round t. We establish the following two-part claim is true, which proves Proposition 1. For each $t \in \{1, \ldots, N-K\}$ we have

(a) For each $\mathbf{y}_{t-1}$:

- (a.i) δ_t satisfies the differential equation given in Proposition 1(i).
- (a.ii) if $x \leq y_{t-1}$ then $x \in \arg\max_{\hat{x}} \pi_t(\hat{x}, x|\mathbf{y}_{t-1})$, i.e., it is optimal for each bidder to follow δ_t in round t; if $x > y_{t-1}$ then $y_{t-1} \in \arg\max_{\hat{x}} \pi_t(\hat{x}, x|\mathbf{y}_{t-1})$.

(b) For each $\mathbf{y}_{t-1}$:

$$\frac{d\Pi_t(x|\mathbf{y}_{t-1})}{dx} \leq 0\,.$$

The proof is by induction. We first show the claim is true for round $N-K$. Let $\mathbf{y}_{N-K-1}$ be arbitrary and consider an active bidder whose value is x but who bids as though it is $\hat{x} \leq y_{N-K-1}$. There are two cases to consider: (i) $x \leq y_{N-K-1}$ and (ii) $x > y_{N-K-1}$.

Case (i): $x \leq y_{N-K-1}$. With a bid of $\hat{x} \leq y_{N-K-1}$, if $\hat{x} < y_{N-K}$ then a rival bidder drops out first at the price $\delta_{N-K}(y_{N-K}|\mathbf{y}_{N-K-1})$, the bidder undertakes the chore, and he receives compensation of

$$\frac{1}{K}\left(\delta_{N-K}(y_{N-K}|\mathbf{y}_{N-K-1}) + \sum_{i=1}^{N-K-1} p_i\right).$$

If $\hat{x} > y_{N-K}$ then the bidder drops before any rival and he pays compensation $\delta_{N-K}(\hat{x}|\mathbf{y}_{N-K-1})$. Hence $\pi_{N-K}(\hat{x}, x|\mathbf{y}_{N-K-1}) =$

$$\int_0^{\hat{x}} u(-\delta_{N-K}(\hat{x}|\mathbf{y}_{N-K-1}))g_{N-K}^{(N-1)}(y_{N-K}|y_{N-K-1})dy_{N-K}$$
$$+ \int_{\hat{x}}^{y_{N-K-1}} u\left(\frac{1}{K}\left(\delta_{N-K}(y_{N-K}|\mathbf{y}_{N-K-1}) + \sum_{i=1}^{N-K-1} p_i\right) - x\right) g_{N-K}^{(N-1)}(y_{N-K}|y_{N-K-1})dy_{N-K}.$$

Differentiating with respect to $\hat{x}$ yields $\partial\pi_{N-K}(\hat{x}, x|\mathbf{y}_{N-K-1})/\partial\hat{x} =$

$$\begin{aligned} &-u'(-\delta_{N-K}(\hat{x}|\mathbf{y}_{N-K-1}))\delta'_{N-K}(\hat{x}|\mathbf{y}_{N-K-1})G_{N-K}^{(N-1)}(\hat{x}|y_{N-K-1}) \\ &+u(-\delta_{N-K}(\hat{x}|\mathbf{y}_{N-K-1}))g_{N-K}^{(N-1)}(\hat{x}|y_{N-K-1}) \\ &-u\left(\frac{1}{K}\left(\delta_{N-K}(\hat{x}|\mathbf{y}_{N-K-1}) + \sum_{i=1}^{N-K-1} p_i\right) - x\right) g_{N-K}^{(N-1)}(\hat{x}|y_{N-K-1}). \end{aligned} \tag{1}$$

A necessary condition for δ to be an equilibrium is that $\partial\pi_{N-K}(\hat{x}, x|\mathbf{z}_{K+1})/\partial\hat{x}|_{\hat{x}=x} = 0$, i.e.,

$$\begin{aligned} &u'(-\delta_{N-K}(x|\mathbf{y}_{N-K-1}))\delta'_{N-K}(x|\mathbf{y}_{N-K-1}) \\ &= [u(-\delta_{N-K}(x|\mathbf{y}_{N-K-1})) - u\left(\frac{1}{K}(\delta_{N-K}(x|\mathbf{y}_{N-K-1}) + \sum_{i=1}^{N-K-1} p_i) - x\right)]\bar{\lambda}_{N-K}^{N}(x). \end{aligned}$$

where

$$\frac{g_{N-K}^{(N-1)}(x|y_{N-K-1})}{G_{N-K}^{(N-1)}(x|y_{N-K-1})} = K\frac{f(x)}{F(x)} = \bar{\lambda}_{N-K}^{N}(x).$$

Alternatively, since types can be inferred from dropout prices, we can write the necessary condition as

$$\begin{aligned} &u'(-\delta_{N-K}(x;\mathbf{p}_{N-K-1}))\delta'_{N-K}(x;\mathbf{p}_{N-K-1}) \\ &= [u(-\delta_{N-K}(x;\mathbf{p}_{N-K-1})) - u\left(\frac{1}{K}(\delta_{N-K}(x;\mathbf{p}_{N-K-1}) + \sum_{i=1}^{N-K-1} p_i) - x\right)]\bar{\lambda}_{N-K}^{N}(x), \end{aligned}$$

which establishes (a.i) for $t = N - K$.

The necessary condition holds for all x and, in particular, it holds for $x = \hat{x}$, i.e.,

$$\begin{aligned} & u'(-\delta_{N-K}(\hat{x}|y_{N-K-1}))\delta'_{N-K}(\hat{x}|y_{N-K-1}) \\ & = [u(-\delta_{N-K}(\hat{x}|y_{N-K-1})) - u(\tfrac{1}{K}(\delta_{N-K}(\hat{x}|y_{N-K-1}) + \textstyle\sum_{i=1}^{N-K-1} p_i) - \hat{x})]\bar{\lambda}^N_{N-K}(\hat{x}). \end{aligned} \tag{2}$$

Substituting (2) into (1) and simplifying yields

$$\frac{\partial \pi_{N-K}(\hat{x}, x|\mathbf{y}_{N-K-1})}{\partial \hat{x}} = \begin{bmatrix} u(\frac{1}{K}(\delta_{N-K}(\hat{x}|\mathbf{y}_{N-K-1}) + \sum_{i=1}^{N-K-1} p_i) - \hat{x}) \\ -u(\frac{1}{K}(\delta_{N-K}(\hat{x}|\mathbf{y}_{N-K-1}) + \sum_{i=1}^{N-K-1} p_i) - x) \end{bmatrix} g^{(N-1)}_{N-K}(\hat{x}|y_{N-K-1}).$$

Clearly, $\partial \pi_{N-K}(\hat{x}, x|\mathbf{y}_{N-K-1})/\partial \hat{x}|_{\hat{x}=x} = 0$. Moreover, for $\hat{x} \leq y_{N-K-1}$ we have

$$\frac{\partial^2 \pi_{N-K}(\hat{x}, x|\mathbf{y}_{N-K-1})}{\partial \hat{x} \partial x} = u' \left(\frac{1}{K}(\delta_{N-K}(\hat{x}|\mathbf{y}_{N-K-1}) + \sum_{i=1}^{N-K-1} p_i) - x \right) g^{(N-1)}_{N-K}(\hat{x}|y_{N-K-1}) \geq 0,$$

where the inequality holds since $u' > 0$ and $g^{(N-1)}_{N-K}(\hat{x}|y_{N-K-1}) \geq 0$. Hence, if $x \leq y_{N-K-1}$ then $x \in \arg\max_{\hat{x}} \pi_{N-K}(\hat{x}, x|\mathbf{y}_{N-K-1})$ by Lemma 0 of McAfee (1992).

Case (ii): $x > y_{N-K-1}$. It is clearly never optimal for a bidder to bid as though his type is greater than y_{N-K-1}, i.e., bid more than $\delta_{N-K}(y_{N-K-1}|\mathbf{y}_{N-K-1})$, since he pays less compensation with a bid of $\delta_{N-K}(y_{N-K-1}|\mathbf{y}_{N-K-1})$. (For either bid, he drops out for sure since the other remaining bidders all have costs below y_{N-K-1}.)

For $\hat{x} \leq y_{N-K-1}$ we have

$$\frac{\partial \pi_{N-K}(\hat{x}, x|\mathbf{y}_{N-K-1})}{\partial \hat{x}} = \begin{bmatrix} u(\frac{1}{K}(\delta_{N-K}(\hat{x}|\mathbf{y}_{N-K-1}) + \sum_{i=1}^{N-K-1} p_i) - \hat{x}) \\ -u(\frac{1}{K}(\delta_{N-K}(\hat{x}|\mathbf{y}_{N-K-1}) + \sum_{i=1}^{N-K-1} p_i) - x) \end{bmatrix} g^{(N-1)}_{N-K}(\hat{x}|y_{N-K-1}) > 0$$

and thus $y_{N-K-1} \in \arg\max_{\hat{x}} \pi_{N-K}(\hat{x}, x|\mathbf{y}_{N-K-1})$. Hence (a.ii) is true for $t = N - K$.

To prove (b), note that $d\Pi_{N-K}(x|\mathbf{y}_{N-K-1})/dx$ is

$$\begin{aligned} & \left.\frac{\partial \pi_{N-K}(\hat{x}, x|\mathbf{y}_{N-K-1})}{\partial \hat{x}}\right|_{\hat{x}=x} + \left.\frac{\partial \pi_{N-K}(\hat{x}, x|\mathbf{y}_{N-K-1})}{\partial x}\right|_{\hat{x}=x} \\ = & -\int_{\hat{x}}^{y_{N-K-1}} u' \left(\frac{1}{K}(\delta_{N-K}(y_{N-K}|\mathbf{y}_{N-K-1}) + \sum_{i=1}^{N-K-1} p_i) - x \right) g^{(N-1)}_{N-K}(y_{N-K}|y_{N-K-1}) dy_{N-K} \\ \leq & \; 0, \end{aligned}$$

where the second equality holds since $\partial \pi_{N-K}(\hat{x}, x|\mathbf{z}_{N-K-1})/\partial \hat{x}|_{\hat{x}=x} = 0$. Hence (b) holds for $t = N - K$.

Assume the claim is true for rounds $t+1$ through $N-K$. We show it is true for round t. Let $\mathbf{y}_{t-1}$ be arbitrary. If $x > y_{t-1}$ then, by the same argument as before, $y_{t-1} \in \arg\max_{\hat{x}} \pi_t(\hat{x}, x|\mathbf{y}_{t-1})$.

Suppose $x \leq y_{t-1}$. Consider an active bidder in the t-th round whose value is x and who bids as though his value is $\hat{x} \leq y_{t-1}$. We need to distinguish between two cases: (i) $\hat{x} \in [x, y_{t-1}]$ and (ii) $\hat{x} < x$, since his payoff function differs in each case. In what follows, we denote the payoff to a bid of $\hat{x}$ as $\pi_t^H(\hat{x}, x|\mathbf{y}_{t-1})$ if $\hat{x} \in [x, y_{t-1}]$ and as $\pi_t^L(\hat{x}, x|\mathbf{y}_{t-1})$ if $\hat{x} < x$.

Case (i): Suppose $\hat{x} \in [x, y_{t-1}]$. If $y_t \in [\hat{x}, y_{t-1}]$ the bidder continues to round $t+1$ where, by the induction hypothesis, he optimally bids x and he has an expected payoff of $\Pi_{t+1}(x|\mathbf{y}_{t-1}, y_t)$. If $y_t \leq \hat{x}$ he pays compensation of $\delta_t(\hat{x}|\mathbf{y}_{t-1})$. Hence his payoff is

$$\begin{aligned}\pi_t^H(\hat{x}, x|\mathbf{y}_{t-1}) = \quad & \textstyle\int_{\hat{x}}^{y_{t-1}} \Pi_{t+1}(x|\mathbf{y}_{t-1}, y_t) g_t^{(N-1)}(y_t|y_{t-1}) dy_t \\ & + \textstyle\int_0^{\hat{x}} u\left(-\delta_t(\hat{x}|\mathbf{y}_{t-1})\right) g_t^{(N-1)}(y_t|y_{t-1}) dy_t.\end{aligned}$$

Differentiating with respect to $\hat{x}$ yields $\partial \pi_t^H(\hat{x}, x|\mathbf{y}_{t-1})/\partial \hat{x} =$

$$\begin{aligned}& -\Pi_{t+1}(x|\mathbf{y}_{t-1}, \hat{x}) g_t^{(N-1)}(\hat{x}|y_{t-1}) + u\left(-\delta_t(\hat{x}|\mathbf{y}_{t-1})\right) g_t^{(N-1)}(\hat{x}|y_{t-1}) \\ & -u'\left(-\delta_t(\hat{x}|\mathbf{y}_{t-1})\right) \delta_t'(\hat{x}|\mathbf{y}_{t-1}) G_t^{(N-1)}(\hat{x}|y_{t-1}).\end{aligned}$$

Since

$$\Pi_{t+1}(x|\mathbf{y}_{t-1}, x) = u(-\delta_{t+1}(x|\mathbf{y}_{t-1}, x)),$$

and

$$\frac{g_t^{(N-1)}(x|y_{t-1})}{G_t^{(N-1)}(x|y_{t-1})} = (N-t)\frac{f(x)}{F(x)} = \bar{\lambda}_t^N(x),$$

the necessary condition for equilibrium that $\partial \pi_t^H(\hat{x}, x|\mathbf{y}_{t-1})/\partial \hat{x}|_{\hat{x}=x} \leq 0$ can be written as

$$\left[u\left(-\delta_t(x|\mathbf{y}_{t-1})\right) - u(-\delta_{t+1}(x|\mathbf{y}_{t-1}, x))\right] \bar{\lambda}_t^N(x) \leq u'\left(-\delta_t(x|\mathbf{y}_{t-1})\right) \delta_t'(x|\mathbf{y}_{t-1}). \qquad (3)$$

Also, for $\hat{x} \in [x, y_{t-1}]$ we have

$$\frac{\partial^2 \pi_t^H(\hat{x}, x|\mathbf{y}_{t-1})}{\partial x \partial \hat{x}} = -\frac{d\Pi_{t+1}(x|\mathbf{y}_{t-1}, \hat{x})}{dx} g_t^{(N-1)}(\hat{x}|y_{t-1}) \geq 0,$$

where the inequality follows since (b) is true for round $t+1$ by the induction hypothesis.

Case (ii): Suppose $\hat{x} < x$. If $y_t \in [x, y_{t-1}]$, then the bidder continues to round $t+1$ and, by the induction hypothesis, he bids x and obtains $\Pi_{t+1}(x|\mathbf{y}_{t-1}, y_t)$. If $y_t \in [\hat{x}, x]$, then he continues to round $t+1$ and, by the induction hypothesis, he bids y_t and pays compensation of $\delta_{t+1}(y_t|\mathbf{y}_{t-1}, y_t)$. If $y_t < \hat{x}$ then in round t he pays compensation of $\delta_t(\hat{x}|\mathbf{y}_{t-1})$. His payoff at round t is therefore

$$\begin{aligned}\pi_t^L(\hat{x}, x|\mathbf{y}_{t-1}) = \; & \int_x^{y_{t-1}} \Pi_{t+1}(x|\mathbf{y}_{t-1}, y_t) g_t^{(N-1)}(y_t|y_{t-1}) dy_t \\ & + \int_{\hat{x}}^{x} u(-\delta_{t+1}(y_t|\mathbf{y}_{t-1}, y_t)) g_t^{(N-1)}(y_t|y_{t-1}) dy_t, \\ & + \int_0^{\hat{x}} u\left(-\delta_t(\hat{x}|\mathbf{y}_{t-1})\right) g_t^{(N-1)}(y_t|y_{t-1}) dy_t.\end{aligned}$$

Differentiating with respect to $\hat{x}$ yields

$$\begin{aligned}\partial \pi_t^L(\hat{x}, x|\mathbf{y}_{t-1})/\partial \hat{x} \;=\; & -u(-\delta_{t+1}(\hat{x}|\mathbf{y}_{t-1}, \hat{x})) g_t^{(N-1)}(\hat{x}|y_{t-1}) + u\left(-\delta_t(\hat{x}|\mathbf{y}_{t-1})\right) g_t^{(N-1)}(y_t|y_{t-1}) \\ & -u'\left(-\delta_t(\hat{x}|\mathbf{y}_{t-1})\right) \delta_t'(\hat{x}|\mathbf{y}_{t-1}) G_t^{(N-1)}(\hat{x}|y_{t-1}).\end{aligned}$$

A necessary condition for equilibrium is that $\partial \pi_t^L(\hat{x}, x|\mathbf{y}_{t-1})/\partial \hat{x}|_{\hat{x}=x} \geq 0$, i.e.,

$$u'\left(-\delta_t(x|\mathbf{y}_{t-1})\right) \delta_t'(x|\mathbf{y}_{t-1}) \leq [u\left(-\delta_t(x|\mathbf{y}_{t-1})\right) - u(-\delta_{t+1}(x|\mathbf{y}_{t-1}, x))] \bar{\lambda}_t^N(x). \qquad (4)$$

Equations (3) and (4) imply that

$$u'\left(-\delta_t(x|\mathbf{y}_{t-1})\right) \delta_t'(x|\mathbf{y}_{t-1}) = [u\left(-\delta_t(x|\mathbf{y}_{t-1})\right) - u(-\delta_{t+1}(x|\mathbf{y}_{t-1}, x))] \bar{\lambda}_t^N(x). \qquad (5)$$

Hence (3) holds as an equality, i.e., $\partial \pi_t^H(\hat{x}, x|\mathbf{y}_{t-1})/\partial \hat{x}|_{\hat{x}=x} = 0$.

Since the bid functions are increasing, we can replace $\mathbf{y}_{t-1}$ with $\mathbf{p}_{t-1}$ and replace $\delta_{t+1}(x|\mathbf{y}_{t-1}, x)$ with $\delta_{t+1}(x; \mathbf{p}_{t-1}, \delta_t(x; \mathbf{p}_{t-1}))$, writing the first order condition as

$$\begin{aligned}& u'\left(-\delta_t(x; \mathbf{p}_{t-1})\right) \delta_t'(x; \mathbf{p}_{t-1}) \\ & = [u\left(-\delta_t(x; \mathbf{p}_{t-1})\right) - u(-\delta_{t+1}(x; \mathbf{p}_{t-1}, \delta_t(x; \mathbf{p}_{t-1})))] \bar{\lambda}_t^N(x),\end{aligned}$$

which establishes (a.i) for round t.

Equation (5) holds for all x and, in particular, it holds for $x = \hat{x}$, i.e.,

$$u'\left(-\delta_t(\hat{x}|\mathbf{y}_{t-1})\right) \delta_t'(\hat{x}|\mathbf{y}_{t-1}) = [u\left(-\delta_t(\hat{x}|\mathbf{y}_{t-1})\right) - u(-\delta_{t+1}(\hat{x}|\mathbf{y}_{t-1}, \hat{x}))] \bar{\lambda}_t^N(\hat{x}).$$

Substituting this expression into the expression for $\partial \pi_t^L(\hat{x}, x|\mathbf{y}_{t-1})/\partial \hat{x}$ yields

$$\frac{\partial \pi_t^L(\hat{x}, x|\mathbf{y}_{t-1})}{\partial \hat{x}} = 0 \text{ for } \hat{x} \leq x.$$

Furthermore,

$$\frac{\partial^2 \pi_t^L(\hat{x}, x|\mathbf{y}_{t-1})}{\partial \hat{x} \partial x} = 0 \text{ for } \hat{x} \geq x.$$

We have shown that

$$\left.\frac{\partial \pi_t^L(\hat{x}, x|\mathbf{y}_{t-1})}{\partial \hat{x}}\right|_{\hat{x}=x} = \left.\frac{\partial \pi_t^H(\hat{x}, x|\mathbf{y}_{t-1})}{\partial \hat{x}}\right|_{\hat{x}=x} = 0$$

and

$$\frac{\partial^2 \pi_t^H(\hat{x}, x|\mathbf{y}_{t-1})}{\partial \hat{x} \partial x} \geq 0 \text{ for } \hat{x} \in [x, y_{t-1}] \text{ and } \frac{\partial^2 \pi_t^L(\hat{x}, x|\mathbf{y}_{t-1})}{\partial \hat{x} \partial x} \geq 0 \text{ for } \hat{x} < x.$$

Hence (a.ii) is true for round t by Van Essen and Wooders (2016) extension of McAfee's (1992) Lemma 0.

To establish (b) is true for round t, observe that

$$\Pi_t(x|\mathbf{y}_{t-1}) = \begin{array}{l} \int_x^{y_{t-1}} \Pi_{t+1}(x|\mathbf{y}_{t-1}, y_t) g_t^{(N-1)}(y_t|y_{t-1}) dy_t \\ + \int_0^x u\left(-\delta_t(x|\mathbf{y}_{t-1})\right) g_t^{(N-1)}(y_t|y_{t-1}) dy_t. \end{array}$$

Differentiating and simplifying yields

$$\frac{d\Pi_t(x|\mathbf{y}_{t-1})}{dx} = \int_x^{y_{t-1}} \frac{d\Pi_{t+1}(x|\mathbf{y}_{t-1}, y_t)}{dx} g_t^{(N-1)}(y_t|y_{t-1}) dy_t \leq 0,$$

where the equality follows from $\Pi_{t+1}(x|\mathbf{y}_{t-1}, x) = u(-\delta_{t+1}(x|\mathbf{y}_{t-1}, x))$ and (5), and the inequality follows since $d\Pi_{t+1}(x|\mathbf{y}_{t-1}, y_t)/dx \leq 0$ by the induction hypothesis. □

Proof of Proposition 2.1: The proof is by induction. By Proposition 1(i), at round $N-K$ the differential equation for the equilibrium bid function is

$$\delta'_{N-K}(x; \mathbf{p}_{N-K-1}) = -\left[\frac{K+1}{K}\delta_{N-K}(x; \mathbf{p}_{N-K-1}) + \frac{1}{K}\sum_{i=1}^{N-K-1} p_i - x\right]\bar{\lambda}_{N-K}^N(x).$$

Multiplying both sides by $F(x)$ we obtain

$$\delta'_{N-K}(x; \mathbf{p}_{N-K-1})F(x) + (K+1)\delta_{N-K}(x; \mathbf{p}_{N-K-1})f(x) = \left[x - \frac{1}{K}\sum_{i=1}^{N-K-1} p_i\right] Kf(x),$$

i.e.,

$$\frac{d}{dx}\left(\delta_{N-K}(x; \mathbf{p}_{N-K-1})F(x)^{K+1}\right) = \left[x - \frac{1}{K}\sum_{i=1}^{N-K-1} p_i\right] Kf(x)F(x)^K.$$

By the Fundamental Theorem of Calculus we have

$$\delta_{N-K}(x;\mathbf{p}_{N-K-1})F(x)^{K+1} = \int_0^x \left[s - \frac{1}{K}\sum_{i=1}^{N-K-1} p_i\right] Kf(s)F(s)^K ds + C.$$

The LHS of this equation is zero when $x = 0$ (since $F(0) = 0$), which implies $C = 0$. Hence

$$\delta_{N-K}(x;\mathbf{p}_{N-K-1}) = \int_0^x \left[s - \frac{1}{K}\sum_{i=1}^{N-K-1} p_i\right] K\frac{F(s)^K}{F(x)^{K+1}} f(s)ds.$$

Since

$$\int_0^x s(K+1)\frac{F(s)^K}{F(x)^{K+1}} f(s)ds = E[Y_{N-K}^{(N)}|Y_{N-K-1}^{(N)} > x > Y_{N-K}^{(N)}],$$

then

$$\delta_{N-K}(x;\mathbf{p}_{N-K-1}) = \frac{K}{K+1}E[Y_{N-K}^{(N)}|Y_{N-K-1}^{(N)} > x > Y_{N-K}^{(N)}] - \frac{1}{K+1}\sum_{i=1}^{N-K-1} p_i,$$

which establishes the result for round $N-K$.

Assume in round t that

$$\delta_t(x;\mathbf{p}_{t-1}) = \frac{K}{N-t+1}E\left[Y_{N-K}^{(N)}|Y_{t-1}^{(N)} > x > Y_t^{(N)}\right] - \frac{1}{N-t+1}\sum_{i=1}^{t-1} p_i.$$

We need to show that $\delta_{t-1}(x;\mathbf{p}_{t-2})$ is as given in Proposition 2(i). The differential equation for round $t-1$ is

$$\delta'_{t-1}(x;\mathbf{p}_{t-2}) = [-\delta_{t-1}(x;\mathbf{p}_{t-2}) + \delta_t(x;\mathbf{p}_{t-2},\delta_{t-1}(x;\mathbf{p}_{t-2}))]\bar{\lambda}_{t-1}(x).$$

By the induction hypothesis

$$\begin{aligned}\delta_t(x;\mathbf{p}_{t-2},\delta_{t-1}(x;\mathbf{p}_{t-2})) \quad &= \quad \frac{K}{N-t+1}E\left[Y_{N-K}^{(N)}|Y_{t-1}^{(N)} > x > Y_t^{(N)}\right] \\ &\quad -\frac{1}{N-t+1}(\sum_{i=1}^{t-2} p_i + \delta_{t-1}(x;\mathbf{p}_{t-2})).\end{aligned}$$

Hence

$$\delta'_{t-1}(x;\mathbf{p}_{t-2}) = \left[\begin{array}{c} -\frac{N-t+2}{N-t+1}\delta_{t-1}(x;\mathbf{p}_{t-2}) + \frac{K}{N-t+1}E\left[Y_{N-K}^{(N)}|Y_{t-1}^{(N)} > x > Y_t^{(N)}\right] \\ -\frac{1}{N-t+1}\sum_{i=1}^{t-2} p_i \end{array}\right]\bar{\lambda}_{t-1}(x).$$

Multiplying both sides by $F(x)^{N-t+2}$ yields

$$\frac{d}{dx}\left(\delta_{t-1}(x;\mathbf{p}_{t-2})F(x)^{N-t+2}\right) = \left[KE\left[Y_{N-K}^{(N)}|Y_{t-1}^{(N)} > x > Y_t^{(N)}\right] - \sum_{i=1}^{t-2} p_i\right] F(x)^{N-t+1}f(x).$$

By the Fundamental Theorem of Calculus and since $F(0) = 0$ then

$$\delta_{t-1}(x;\mathbf{p}_{t-2})F(x)^{N-t+2} = \int_0^x \left[KE\left[Y_{N-K}^{(N)}|Y_{t-1}^{(N)} > s > Y_t^{(N)}\right] - \sum_{i=1}^{t-2} p_i\right] F(s)^{N-t+1}f(s)ds.$$

Hence

$$\delta_{t-1}(x;\mathbf{p}_{t-2}) = \int_0^x \left[KE\left[Y_{N-K}^{(N)}|Y_{t-1}^{(N)} > s > Y_t^{(N)}\right] - \sum_{i=1}^{t-2} p_i\right] f(s)\frac{F(s)^{N-t+1}}{F(x)^{N-t+2}}ds.$$

Since (to be established momentarily)

$$\begin{aligned} & \int_0^x E\left[Y_{N-K}^{(N)}|Y_{t-1}^{(N)} > s > Y_t^{(N)}\right](N-t+2)f(s)\frac{F(s)^{N-t+1}}{F(x)^{N-t+2}}ds \\ = \; & E\left[Y_{N-K}^{(N)}|Y_{t-2}^{(N)} > x > Y_{t-1}^{(N)}\right] \end{aligned}$$

then

$$\delta_{t-1}(x;\mathbf{p}_{t-2}) = \frac{K}{N-t+2}\left(E\left[Y_{N-K}^{(N)}|Y_{t-2}^{(N)} > x > Y_{t-1}^{(N)}\right] - \frac{1}{K}\sum_{i=1}^{t-2} p_i\right),$$

which completes the proof.

Finally, we establish the equality just used. We have

$$\begin{aligned} & \int_0^x E\left[Y_{N-K}^{(N)}|Y_{t-1}^{(N)} > s > Y_t^{(N)}\right](N-t+2)f(s)\frac{F(s)^{N-t+1}}{F(x)^{N-t+2}}ds \\ = & \int_0^x \left(\int_0^s r\frac{(N-t+1)!}{K!(N-t-K)!}\frac{F(r)^K[F(s)-F(r)]^{N-t-K}}{F(s)^{N-t+1}}f(r)dr\right)(N-t+2)f(s)\frac{F(s)^{N-t+1}}{F(x)^{N-t+2}}ds \\ = & \int_0^x r\frac{(N-t+1)!}{K!(N-t-K)!}\int_r^x \frac{F(r)^K[F(s)-F(r)]^{N-t-K}}{F(x)^{N-t+2}}(N-t+2)f(r)f(s)dsdr \\ = & \int_0^x r\frac{(N-t+2)!}{K!(N-t-K+1)!}\frac{F(r)^K[F(x)-F(r)]^{N-t-K+1}}{F(x)^{N-t+2}}f(r)dr \\ = & \; E\left[Y_{N-K}^{(N)}|Y_{t-2}^{(N)} > x > Y_{t-1}^{(N)}\right]. \; \square \end{aligned}$$

Proof of Proposition 3: See the Online Appendix.

Proof of Proposition 4.1: To save space we write δ_t rather than $\delta_t(x;\mathbf{p}_{t-1})$. At round $t = N-K$, the differential equation that characterizes equilibrium behavior is

$$\frac{d}{dx}\left(e^{\alpha\frac{K+1}{K}\delta_{N-K}}F(x)^{K+1}\right) = e^{-\alpha\left(\frac{1}{K}\sum_{i=1}^{N-K-1}p_i - x\right)}(K+1)F(x)^K f(x).$$

From the Fundamental Theorem of Calculus, we have

$$e^{\alpha\frac{K+1}{K}\delta_{N-K}}F(x)^{K+1} = \int_0^x e^{-\alpha\left(\frac{1}{K}\sum_{i=1}^{N-K-1}p_i - s\right)}(K+1)F(s)^K f(s)ds + C.$$

At $x = 0$, the LHS of the above equation is equal to zero and hence $C = 0$. So

$$e^{\alpha\frac{K+1}{K}\delta_{N-K}} = e^{-\alpha\left(\frac{1}{K}\sum_{i=1}^{N-K-1}p_i\right)}\frac{\int_0^x e^{\alpha z}(K+1)F(z)^K f(z)dz}{F(x)^{K+1}}.$$

Taking logs of both sides we have

$$\alpha\frac{K+1}{K}\delta_{N-K} = -\alpha\frac{1}{K}\left(\sum_{i=1}^{N-K-1}p_i\right) + \ln\left(\frac{\int_0^x e^{\alpha z}(K+1)F(z)^K f(z)dz}{F(x)^{K+1}}\right),$$

and hence

$$\begin{aligned}
\delta_{N-K}\left(x;\mathbf{p}_{N-K-1}\right) &= \frac{K}{(K+1)\alpha}\ln\left(\frac{\int_0^x e^{\alpha z}(K+1)F(z)^K f(z)dz}{F(x)^{K+1}}\right) - \frac{1}{K+1}\sum_{i=1}^{N-K-1}p_i \\
&= \frac{K}{(K+1)\alpha}\ln\left(E\left[e^{\alpha Y_{N-K}^{(N)}}|Y_{N-K-1}^{(N)} > x > Y_{N-K}^{(N)}\right]\right) - \frac{1}{K+1}\sum_{i=1}^{N-K-1}p_i \\
&= \frac{K}{(K+1)\alpha}\ln\left(S_{N-K}^{\alpha}(x)\right) - \frac{1}{K+1}\sum_{i=1}^{N-K-1}p_i.
\end{aligned}$$

Next, we solve for the round $t-1$ bid function. Assume that in round $t \leq N-K$, bidders follow the bid function

$$\delta_t\left(x;\mathbf{p}_{t-1}\right) = \frac{N-t}{(N-t+1)\alpha}\ln\left(S_t^{\alpha}(x)\right) - \frac{1}{N-t+1}\sum_{i=1}^{t-1}p_i.$$

Note that this implies that $\delta_t\left(x;\mathbf{p}_{t-2},\delta_{t-1}\left(x;\mathbf{p}_{t-2}\right)\right) =$

$$\frac{N-t}{(N-t+1)\alpha}\ln\left(S_t^{\alpha}(x)\right) - \frac{1}{N-t+1}\sum_{i=1}^{t-2}p_i - \frac{1}{N-t+1}\delta_{t-1}\left(x;\mathbf{p}_{t-2}\right).$$

After some manipulation, the differential equation for round $t-1$ from Proposition 1 can be written as

$$\frac{d}{dx}\left(e^{\alpha\frac{N-t+2}{N-t+1}\delta_{t-1}}F(x)^{N-t+2}\right) = e^{-\alpha\left(\frac{1}{N-t+1}\sum_{i=1}^{t-2}p_i\right)}S_t^{\alpha}(x)^{\frac{N-t}{N-t+1}}(N-t+2)F(x)^{N-t+1}f(x).$$

From the Fundamental Theorem of Calculus we have

$$e^{\alpha \frac{N-t+2}{N-t+1}\delta_{t-1}} F(x)^{N-t+2} = \int_0^x e^{-\alpha\left(\frac{1}{N-t+1}\sum_{i=1}^{t-2} p_i\right)} S_t^\alpha(s)^{\frac{N-t}{N-t+1}} (N-t+2) F(s)^{N-t+1} f(s) ds + C.$$

At $x = 0$, the LHS of the above equation is equal to zero and hence $C = 0$. Rearranging yields $\delta_{t-1}(x; \mathbf{p}_{t-2}) =$

$$\frac{N-t+1}{(N-t+2)\alpha} \ln\left(\frac{\int_0^x S_t^\alpha(s)^{\frac{N-t}{N-t+1}} (N-t+2) F(s)^{N-t+1} f(s) ds}{F(x)^{N-t+2}}\right) - \frac{1}{N-t+2}\sum_{i=1}^{t-2} p_i$$

$$= \frac{N-t+1}{(N-t+2)\alpha} \ln\left(S_{t-1}^\alpha(x)\right) - \frac{1}{N-t+2}\sum_{i=1}^{t-2} p_i,$$

where second equality holds since

$$S_{t-1}^\alpha(x) = E\left[S_t^\alpha(Y_{t-1}^{(N)})^{\frac{N-t}{N-t+1}} | Y_{t-2}^{(N)} > x > Y_{t-1}^{(N)}\right].\square$$

Proof of Proposition 5: We establish the inequalities for the chore auction here, and leave the proofs for the goods auction to the Supplemental Appendix.

We show that for each $t = 1, \ldots, N-K$ and $\mathbf{p}_{t-1}$ that $\delta_t^0(x; \mathbf{p}_{t-1}) < \delta_t^\alpha(x; \mathbf{p}_{t-1})$ for $x > 0$. The proof is by induction. For $t = N-K$, since e^x is a convex function, then by Jensen's Inequality, for $x > 0$ we have

$$e^{E[\alpha Y_{N-K}^{(N)} | Y_{N-K}^{(N)} < x < Y_{N-K-1}^{(N)}]} < E[e^{\alpha Y_{N-K}^{(N)}} | Y_{N-K-1}^{(N)} > x > Y_{N-K}^{(N)}].$$

Noting that the RHS is $S_{N-K}^\alpha(x)$, taking the log of both sides and multiplying through by $K/((K+1)\alpha)$ yields

$$\frac{K}{K+1} E[Y_{N-K}^{(N)} | Y_{N-K-1}^{(N)} > x > Y_{N-K}^{(N)}] < \frac{K}{(K+1)\alpha} \ln(S_{N-K}^\alpha(x)).$$

Adding $-\frac{1}{K+1}\sum_{i=1}^{N-K-1} p_i$ to both sides yields $\delta_{N-K}^0(x; \mathbf{p}_{N-K-1}) < \delta_{N-K}^\alpha(x; \mathbf{p}_{N-K-1})$ for $x > 0$.

For $t \leq N-K$, define

$$\Delta_t^0(x) = \frac{K}{N-t+1} E[Y_{N-K}^{(N)} | Y_{t-1}^{(N)} > x > Y_t^{(N)}],$$

and

$$\Delta_t^\alpha(x) = \frac{1}{\alpha} \ln\left(S_t^\alpha(x)^{\frac{N-t}{N-t+1}}\right),$$

where $S_t^\alpha(x)$ is defined in P4.1. We have that

$$e^{-\alpha\Delta_t^\alpha(x)} = S_t^\alpha(x)^{\frac{N-t}{N-t+1}}.$$

We established above that $\Delta_{N-K}^0(x) < \Delta_{N-K}^\alpha(x)$.

Assume for $t \leq N-K-1$ that $\Delta_{t+1}^0(x) < \Delta_{t+1}^\alpha(x)$ for $x > 0$. We show that $\Delta_t^0(x) < \Delta_t^\alpha(x)$ for $x > 0$. Since $\alpha\Delta_{t+1}^0(x) < \alpha\Delta_{t+1}^\alpha(x)$ and e^x is increasing, then

$$e^{\alpha\Delta_{t+1}^0(x)} < e^{\alpha\Delta_{t+1}^\alpha(x)} \text{ for } x > 0,$$

or

$$e^{\alpha\Delta_{t+1}^0(x)} < S_{t+1}^\alpha(x)^{\frac{N-t-1}{N-t}} \text{ for } x > 0.$$

Thus

$$E[e^{\alpha\Delta_{t+1}^0(Y_t^{(N)})}|Y_{t-1}^{(N)} > x > Y_t^{(N)}] < E[S_{t+1}^\alpha(Y_t^{(N)})^{\frac{N-t-1}{N-t}}|Y_{t-1}^{(N)} > x > Y_t^{(N)}] = S_t^\alpha(x).$$

Since e^x is convex, then

$$e^{E[\alpha\Delta_{t+1}^0(Y_t^{(N)})|Y_{t-1}^{(N)}>x>Y_t^{(N)}]} < E[e^{\alpha\Delta_{t+1}^0(Y_t^{(N)})}|Y_{t-1}^{(N)} > x > Y_t^{(N)}]$$

and hence

$$e^{E[\alpha\Delta_{t+1}^0(Y_t^{(N)})|Y_{t-1}^{(N)}>x>Y_t^{(N)}]} < S_t^\alpha(x).$$

Taking logs of both sides of this inequality yields

$$E[\alpha\Delta_{t+1}^0(Y_t^{(N)})|Y_{t-1}^{(N)} > x > Y_t^{(N)}] < \ln\left(S_t^\alpha(x)\right).$$

Multiplying both sides by $\frac{N-t}{(N-t+1)\alpha}$ yields

$$\begin{aligned} \int_0^x \Delta_{t+1}^0(s)\frac{F(s)^{N-t}f(s)}{F(x)^{N-t+1}}ds &= \frac{K}{N-t+1}E[Y_{N-K}^{(N)}|Y_{t-1}^{(N)} > x > Y_t^{(N)}] \\ &< \frac{N-t}{(N-t+1)\alpha}\ln\left(S_t^\alpha(x)\right). \end{aligned}$$

Adding $-\frac{1}{N-t+1}\sum_{i=1}^{t-1} p_i$ to both sides yields $\delta_t^0(x;\mathbf{p}_{t-1}) < \delta_t^\alpha(x;\mathbf{p}_{t-1})$ for $x > 0$.

We now show that for each $t = 1, \ldots, N-K$ and $\mathbf{p}_{t-1}$ that $\delta_t^\alpha(x;\mathbf{p}_{t-1}) < \gamma_t(x;\mathbf{p}_{t-1})$ for $x > 0$. The proof is by induction. We first show $\delta_{N-K}^\alpha(x;\mathbf{p}_{N-K-1}) < \gamma_{N-K}(x;\mathbf{p}_{N-K-1})$. Since $e^{\alpha s} < e^{\alpha x}$ for $0 < s < x$ then

$$S_{N-K}^\alpha(x) = E[e^{\alpha Y_{N-K}^{(N)}}|Y_{N-K-1}^{(N)} > x > Y_{N-K}^{(N)}] < e^{\alpha x}.$$

Taking logs of both sides and rearranging yields

$$\frac{K}{(K+1)\alpha}\ln(S^{\alpha}_{N-K}(x)) < \frac{K}{K+1}x.$$

Adding $-\frac{1}{K+1}\sum_{i=1}^{N-K-1}p_i$ to both sides yields $\delta^{\alpha}_{N-K}(x;\mathbf{p}_{t-1}) < \gamma_{N-K}(x;\mathbf{p}_{N-K-1})$ for $x>0$.

Assume for $t \le N-K-1$ that $\Delta^{\alpha}_{t+1}(x) < Kx/(N-t)$ for $x>0$. Since $\Delta^{\alpha}_{t+1}(x)$ is increasing, then for $s<x$ we have $\Delta^{\alpha}_{t+1}(s) < \Delta^{\alpha}_{t+1}(x) < Kx/(N-t)$ or $\alpha\Delta^{\alpha}_{t+1}(s) < \alpha\Delta^{\alpha}_{t+1}(x) < \alpha Kx/(N-t)$ and thus

$$e^{\alpha\Delta^{\alpha}_{t+1}(s)} = S^{\alpha}_{t+1}(s)^{\frac{N-t-1}{N-t}} < e^{\alpha\Delta^{\alpha}_{t+1}(x)} < e^{\alpha K\frac{x}{N-t}}.$$

Hence

$$E\left[\left(S^{\alpha}_{t+1}(Y^{(N)}_t)\right)^{\frac{N-t-1}{N-t}}|Y^{(N)}_{t-1} > x > Y^{(N)}_t\right] = S^{\alpha}_t(x) < e^{\alpha K\frac{x}{N-t}}.$$

Taking logs of both sides yields

$$\ln(S^{\alpha}_t(x)) < \alpha K\frac{x}{N-t},$$

and so

$$\frac{N-t}{(N-t+1)\alpha}\ln(S^{\alpha}_t(x)) < \frac{Kx}{N-t+1}.$$

Hence $\Delta^{\alpha}_t(x) < Kx/(N-t+1)$ for $x>0$. Adding $-\sum_{i=1}^{t-1}p_i$ to each side gives us $\delta^{\alpha}_t(x;\mathbf{p}_{t-1}) < \gamma_t(x;\mathbf{p}_{t-1})$ for $x>0$. □

Proof of Proposition 6: We establish the results for the chore auction here, and leave the proofs for the goods auction to the Supplemental Appendix. We first show that $\delta^{\alpha}_t(x;\mathbf{p}_{t-1})$ is increasing in α. The proof is by induction. Suppose $\tilde{\alpha} > \alpha$. Since the transformation $y = x^{\frac{\alpha}{\tilde{\alpha}}}$ is concave, then by Jensen's inequality we have that

$$\begin{aligned}\left(S^{\tilde{\alpha}}_{N-K}(x)\right)^{\frac{\alpha}{\tilde{\alpha}}} &= \left(E[e^{\tilde{\alpha}Y^{(N)}_{N-K}}|Y^{(N)}_{N-K-1} > x > Y^{(N)}_{N-K}]\right)^{\frac{\alpha}{\tilde{\alpha}}} \\ &> E[\left(e^{\tilde{\alpha}Y^{(N)}_{N-K}}\right)^{\frac{\alpha}{\tilde{\alpha}}}|Y^{(N)}_{N-K-1} > x > Y^{(N)}_{N-K}] \\ &= S^{\alpha}_{N-K}(x),\end{aligned}$$

for $x>0$. Taking logs and rearranging yields

$$\frac{K}{(K+1)\,\tilde{\alpha}}\ln S^{\tilde{\alpha}}_{N-K}(x) - \frac{1}{K+1}\sum_{i=1}^{N-K-1}p_i > \frac{K}{(K+1)\,\alpha}\ln S^{\alpha}_{N-K}(x) - \frac{1}{K+1}\sum_{i=1}^{N-K-1}p_i.$$

Hence $\delta^{\tilde{\alpha}}_{N-K}(x;\mathbf{p}_{N-K-1}) > \delta^{\alpha}_{N-K}(x;\mathbf{p}_{N-K-1})$.

Let

$$\Delta^{\alpha}_{t+1}(x) = \frac{N-t-1}{(N-t)\alpha}\ln(S^{\alpha}_{t+1}(x)).$$

Suppose $\delta^{\tilde{\alpha}}_{t+1}(x;\mathbf{p}_t) > \delta^{\alpha}_{t+1}(x;\mathbf{p}_t)$ and hence $\Delta^{\tilde{\alpha}}_{t+1}(x) > \Delta^{\alpha}_{t+1}(x)$. We show that $\delta^{\tilde{\alpha}}_{t}(x;\mathbf{p}_{t-1}) > \delta^{\alpha}_{t}(x;\mathbf{p}_{t-1})$. Jensen's inequality and $\Delta^{\tilde{\alpha}}_{t+1}(x) > \Delta^{\alpha}_{t+1}(x)$ imply

$$\begin{aligned}\left(E[e^{\tilde{\alpha}\Delta^{\tilde{\alpha}}_{t+1}(Y^{(N)}_t)}|Y^{(N)}_{t-1} > x > Y^{(N)}_t]\right)^{\frac{\alpha}{\tilde{\alpha}}} &> E[e^{\alpha\Delta^{\tilde{\alpha}}_{t+1}(Y^{(N)}_t)}|Y^{(N)}_{t-1} > x > Y^{(N)}_t] \\ &> E[e^{\alpha\Delta^{\alpha}_{t+1}(Y^{(N)}_t)}|Y^{(N)}_{t-1} > x > Y^{(N)}_t].\end{aligned}$$

Simple algebra yields

$$\begin{aligned}\Delta^{\tilde{\alpha}}_{t}(x) &= \frac{N-t}{(N-t+1)\tilde{\alpha}}\ln E[e^{\tilde{\alpha}\Delta^{\tilde{\alpha}}_{t+1}(Y^{(N)}_t)}|Y^{(N)}_{t-1} > x > Y^{(N)}_t] \\ &> \frac{N-t}{(N-t+1)\alpha}\ln E[e^{\alpha\Delta^{\alpha}_{t+1}(Y^{(N)}_t)}|Y^{(N)}_{t-1} > x > Y^{(N)}_t] \\ &= \Delta^{\tilde{\alpha}}_{t}(x),\end{aligned}$$

and therefore that $\delta^{\tilde{\alpha}}_{t}(x;\mathbf{p}_{t-1}) > \delta^{\alpha}_{t}(x;\mathbf{p}_{t-1})$.

Next we prove that $\lim_{\alpha\to\infty}\delta^{\alpha}_{t}(x;\mathbf{p}_{t-1}) = \gamma_t(x;\mathbf{p}_{t-1})$. The bid function $\delta^{\alpha}_{t}(x;\mathbf{p}_{t-1})$ can be written as

$$\delta^{\alpha}_{t}(x;\mathbf{p}_{k-1}) = \frac{1}{\alpha}\ln\left(S^{\alpha}_{t}(x)^{\frac{N-t}{N-t+1}}\right) - \sum_{i=1}^{t-1}\frac{1}{N-t+1}p_i.$$

By the definition of $S^{\alpha}_{t}(x)$ and by iteratively applying Jensen's inequality we obtain

$$S^{\alpha}_{t}(x)^{\frac{N-t}{N-t+1}} \geq E[e^{\frac{\alpha K}{N-t+1}Y^{(N)}_{N-K}}|Y^{(N)}_{t-1} > x > Y^{(N)}_t].$$

Thus we have

$$\frac{1}{\alpha}\ln(S^{\alpha}_{t}(x)^{\frac{N-t}{N-t+1}}) \geq \frac{1}{\alpha}\ln\left(E[e^{\frac{\alpha K}{N-t+1}Y^{(N)}_{N-K}}|Y^{(N)}_{t-1} > x > Y^{(N)}_t]\right).$$

The round t equilibrium bid function therefore is bounded below by

$$\delta^{\alpha}_{t}(x;\mathbf{p}_{t-1}) \geq \frac{1}{\alpha}\ln\left(E[e^{\frac{\alpha K}{N-t+1}Y^{(N)}_{N-K}}|Y^{(N)}_{t-1} > x > Y^{(N)}_t]\right) - \sum_{i=1}^{t-1}\frac{1}{N-t+1}p_i.$$

By Proposition 5 we have that

$$\gamma_t(x;\mathbf{p}_{t-1}) \geq \delta^{\alpha}_{t}(x;\mathbf{p}_{t-1}).$$

We complete the proof by establishing that $\lim_{\alpha\to\infty} \frac{1}{\alpha} \ln \left(E[e^{\frac{\alpha K}{N-t+1} Y_{N-K}^{(N)}} | Y_{t-1}^{(N)} > x > Y_t^{(N)}] \right) = \frac{K}{N-t+1} x$, i.e.,

$$\lim_{\alpha\to\infty} \frac{1}{\alpha} \ln \left(\int_0^x e^{\frac{\alpha K s}{N-t+1}} g(s) ds \right) = \frac{K}{N-t+1} x,$$

where

$$g(s) = \frac{(N-t+1)!}{(N-K-t)!K!} \frac{F(s)^K \left[F(x) - F(s)\right]^{N-K-t}}{F(x)^{N-t+1}} f(s).$$

The result then follows from the Squeeze Theorem.

We now establish the above limit. Applying l'Hopital's rule, this limit equals

$$\frac{K}{N-t+1} \lim_{\alpha\to\infty} \frac{\int_0^x s e^{\alpha s} g(s) ds}{\int_0^x e^{\alpha s} g(s) ds}.$$

We establish that that following limit holds for any $0 < g(s) < \infty$.

$$\lim_{\alpha\to\infty} \frac{\int_0^x s e^{\alpha s} g(s) ds}{\int_0^x e^{\alpha s} g(s) ds} = x$$

First, we have

$$\frac{\int_0^x s e^{\alpha s} g(s) ds}{\int_0^x e^{\alpha s} g(s) ds} \leq \frac{x \int_0^x e^{\alpha s} g(s) ds}{\int_0^x e^{\alpha s} g(s) ds} = x.$$

Second, for $\Delta > 0$ small we may write

$$\int_0^x s e^{\alpha s} g(s) ds = \int_{x-\Delta}^x s e^{\alpha s} g(s) ds + \int_0^{x-\Delta} s e^{\alpha s} g(s) ds,$$

so

$$
\begin{aligned}
\lim_{\alpha\to\infty} \frac{\int_0^x s e^{\alpha s} g(s)ds}{\int_0^x e^{\alpha s} g(s)ds} &\geq \lim_{\alpha\to\infty} \frac{(x-\Delta)\int_0^x e^{\alpha s} g(s)ds - \int_0^{x-\Delta} (x-\Delta-s)\, e^{\alpha s} g(s)ds}{\int_0^x e^{\alpha s} g(s)ds} \\
&= (x-\Delta) - \lim_{\alpha\to\infty} \frac{\int_0^{x-\Delta} (x-\Delta-s)\, e^{\alpha s} g(s)ds}{\int_0^x e^{\alpha s} g(s)ds} \\
&\geq (x-\Delta) - \lim_{\alpha\to\infty} \frac{e^{\alpha(x-\Delta)} \int_0^{x-\Delta} (x-\Delta-s)\, g(s)ds}{\int_{x-\frac{1}{2}\Delta}^{x} e^{\alpha s} g(s)ds} \\
&\geq (x-\Delta) - \lim_{\alpha\to\infty} \frac{e^{\alpha(x-\Delta)} \int_0^{x-\Delta} (x-\Delta-s)\, g(s)ds}{e^{\alpha(x-\frac{1}{2}\Delta)} \int_{x-\frac{1}{2}\Delta}^{x} g(s)ds} \\
&= (x-\Delta) - \frac{\int_0^{x-\Delta} (x-\Delta-s)\, g(s)ds}{\int_{x-\frac{1}{2}\Delta}^{x} g(s)ds} \lim_{\alpha\to\infty} e^{\alpha(-\frac{1}{2}\Delta)} \\
&= x-\Delta,
\end{aligned}
$$

where the last equality follows since $\lim_{\alpha\to\infty} e^{\alpha(-\frac{1}{2}\Delta)} = 0$. This inequality holds for any Δ positive (and small) and hence we have established the desired limit. The main result then follows from the Squeeze Theorem. □

Proof of Proposition 7: We construct γ recursively, showing that the strategy guarantees a payoff at round t in the chore auction of at least

$$\bar{v}_t(x_i, \mathbf{p}_{t-1}) = -\frac{K\left(x_i - \frac{1}{K}\sum_{m=1}^{t-1} p_m\right)}{N-t+1}$$

and guaranteeing at least $-\bar{v}_t(x_i, \mathbf{p}_{t-1})$ in the goods auction.

Consider round $N-K$. In the chore auction, a bidder with value x whose dropout price is b either (i) drops at b and obtains a payoff of $-b$, or (ii) a rival drops first at

$p_{N-K} \geq b$ and he obtains

$$\frac{1}{K}\sum_{m=1}^{N-K-1} p_m + \frac{1}{K}p_{N-K} - x.$$

The bidder maximizes his minimum payoff when b satisfies

$$-b = \frac{1}{K}\sum_{m=1}^{N-K-1} p_m + \frac{1}{K}b - x,$$

i.e., $b = \gamma_{N-K}(x; \mathbf{p}_{N-K-1})$. Hence at round $N-K$ the bidder guarantees himself a payoff of at least $-\gamma_{N-K}(x; \mathbf{p}_{N-K-1}) = \bar{v}_{N-K}(x; \mathbf{p}_{N-K-1})$.

The argument is the same for the goods auction, except that the signs of the payoffs are reversed. A bidder with value x whose dropout price is b either (i) drops at b and obtains a payoff of b, or (ii) a rival drops first at $p_{N-K} \leq b$ and he obtains

$$-\left(\frac{1}{K}\sum_{m=1}^{N-K-1} p_m + \frac{1}{K}p_{N-K} - x\right).$$

Hence the bidder maximizes his minimum payoff when $b = \gamma_{N-K}(x; \mathbf{p}_{N-K-1})$ and he guarantees himself a payoff of at least $\gamma_{N-K}(x; \mathbf{p}_{N-K-1}) = -\bar{v}_{N-K}(x; \mathbf{p}_{N-K-1})$. Since the argument is the same, hereafter we focus on the chore auction.

Suppose that at round $t+1$, given $\mathbf{p}_t$ a bidder with value x can guarantee himself $\bar{v}_{t+1}(x, \mathbf{p}_t)$. Consider round t. A bidder with value x whose dropout price is b either (i) drops at b and obtains a payoff of $-b$, or (ii) a rival drops first at $p_t \geq b$ and he obtains at least $\bar{v}_{t+1}(x, \mathbf{p}_t) \geq \bar{v}_{t+1}(x, (\mathbf{p}_{t-1}, b))$. His minimum payoff is maximized when $-b = \bar{v}_{t+1}(x, (\mathbf{p}_{t-1}, b))$, i.e., $b = \gamma_t(x; \mathbf{p}_{t-1})$. He obtains a payoff of at least $-\gamma_t(x; \mathbf{p}_{t-1}) = \bar{v}_t(x; \mathbf{p}_{t-1})$.

Next we show that $\bar{v}_t(x; \mathbf{p}_{t-1})$ is the largest payoff that a bidder can guarantee himself at round t given $\mathbf{p}_{t-1}$. Suppose to the contrary that he can guarantee himself $v'_t > \bar{v}_t(x; \mathbf{p}_{t-1})$. If all active bidders have the same value x then, since the game is symmetric, each bidder can guarantee himself v'_t and hence the total guaranteed payoff of the active bidders is at least

$$(N-t+1)v'_t > (N-t+1)\bar{v}_t(x; \mathbf{p}_{t-1}) = \sum_{m=1}^{t-1} p_m - Kx.$$

This is a contradiction since the right hand side is the total surplus that can be obtained by the active bidders at round t: In subsequent rounds, any additional compensation $p_t, \ldots, p_{N-K}$ that is received by a currently active bidder is also paid by a currently active bidder, and hence generates no additional surplus.

The proof that γ is the unique maxmin perfect strategy is straightforward so we only sketch it here. Suppose $\hat{\gamma} \neq \gamma$ is a maxmin perfect strategy. Then there is some x, t, and $\mathbf{p}_{t-1}$ such that $\hat{\gamma}_t(x; \mathbf{p}_{t-1}) \neq \gamma_t(x; \mathbf{p}_{t-1})$. Suppose $\hat{\gamma}_t(x; \mathbf{p}_{t-1}) > \gamma_t(x; \mathbf{p}_{t-1})$. If the bidder drops at round t he obtains a payoff of $-\hat{\gamma}_t(x; \mathbf{p}_{t-1}) < -\gamma_t(x; \mathbf{p}_{t-1}) = \bar{v}_t(x; \mathbf{p}_{t-1})$.

Suppose $\hat{\gamma}_t(x; \mathbf{p}_{t-1}) < \gamma_t(x; \mathbf{p}_{t-1})$ and rival bidder drops out $\hat{p}_t$ such that $\hat{\gamma}_t(x; \mathbf{p}_{t-1}) < \hat{p}_t < \gamma_t(x; \mathbf{p}_{t-1})$. One can show that the other bidders can hold him to a payoff of no more than $\bar{v}_{t+1}(x; (\mathbf{p}_{t-1}, \hat{p}_t))$. We have

$$\bar{v}_{t+1}(x; (\mathbf{p}_{t-1}, \hat{p}_t)) < \bar{v}_{t+1}(x; (\mathbf{p}_{t-1}, \gamma_t(x; \mathbf{p}_{t-1}))) = \bar{v}_t(x; \mathbf{p}_{t-1}),$$

where the inequality holds since $\bar{v}_{t+1}$ is increasing in p_t and the equality holds by construction. $\square$

We present below, without proof, the formulas for the strategic and normative Shapley values when K units of a good are allocated to N players. The strategic Shapley value of the player with the i-th lowest value $z_i^{(N)}$ is

$$\underline{\phi}_i = \begin{cases} \sum_{m=N-K+1}^{N} \frac{(N-K)}{(m-1)m} z_m^{(N)} & \text{if } i \leq N-K \\ \frac{1}{i}\left(i-(N-K)\right) z_i^{(N)} + \sum_{m=1}^{N-i} \frac{(N-K)}{(i+m-1)(i+m)} z_{i+m}^{(N)} & \text{if } i > N-K. \end{cases}$$

The normative Shapley value of the player with the i-th highest value $y_i^{(N)}$ is

$$\overline{\phi}_i = \begin{cases} y_i^{(N)} - \sum_{m=K+1-i}^{N-i} \frac{K}{(i+m-1)(i+m)} y_{i+m}^{(N)} & \text{if } i \leq K \\ \frac{K}{i} y_i^{(N)} - \sum_{m=1}^{N-i} \frac{K}{(i+m-1)(i+m)} y_{i+m}^{(N)} & \text{if } i > K. \end{cases}$$

Lemma 1: Suppose that each bidder in the chore auction follows the maxmin perfect bidding strategy given in Proposition 7. At round t, the bidders with the t highest values $z_{N-t+1}^{(N)}, \ldots, z_N^{(N)}$ have dropped. The total compensation pledged is

$$\sum_{i=1}^{t} p_t = (N-t)\bar{K} \left[\begin{array}{c} \frac{1}{(N-t)(N-t+1)} z_{N-t+1}^{(N)} + \frac{1}{(N-t+1)(N-t+2)} z_{N-t+2}^{(N)} \\ + \cdots + \frac{1}{(N-1)N} z_N^{(N)} \end{array} \right].$$

Proof: At round 1 the bidder with value $z_N^{(N)}$ drops at $p_1 = \frac{1}{N}\bar{K}z_N^{(N)}$, and total compensation pledged is p_1. At round 2, the bidder with value $z_{N-1}^{(N)}$ drops at $p_2 = \frac{1}{N-1}(\bar{K}z_{N-1}^{(N)} - \frac{1}{N}\bar{K}z_N^{(N)})$ and total compensation pledged is

$$\begin{aligned} p_1 + p_2 &= \frac{1}{N}\bar{K}z_N^{(N)} + \frac{1}{N-1}(\bar{K}z_{N-1}^{(N)} - \frac{1}{N}\bar{K}z_N^{(N)}) \\ &= \frac{1}{N-1}\bar{K}z_{N-1}^{(N)} + \frac{N-2}{N-1}\frac{1}{N}\bar{K}z_N^{(N)} \\ &= (N-2)\bar{K}\left(\frac{1}{(N-1)(N-2)}z_{N-1}^{(N)} + \frac{1}{(N-1)N}z_N^{(N)}\right). \end{aligned}$$

Assume the lemma is true at round $t-1$, i.e.,

$$\sum_{m=1}^{t-1} p_m = (N-t+1)\overline{K}\left(\begin{array}{c} \frac{1}{(N-t+1)(N-t+2)}z_{N-t+2}^{(N)} + \frac{1}{(N-t+2)(N-t+3)}z_{N-t+3}^{(N)} \\ +\cdots+\frac{1}{(N-1)N}z_N^{(N)} \end{array}\right).$$

We show it is true for round t. At round t the bidder with value $z_{N-t+1}^{(N)}$ drops at price

$$p_t = \frac{1}{N-t+1}\left(\bar{K}z_{N-t+1}^{(N)} - \sum_{m=1}^{t-1} p_m\right).$$

Total compensation pledged at round t is

$$p_t + \sum_{m=1}^{t-1} p_m = \frac{1}{N-t+1}\bar{K}z_{N-t+1}^{(N)} + \frac{N-t}{N-t+1}\sum_{m=1}^{t-1} p_m.$$

Since the lemma is true for round $t-1$, we have

$$p_t + \sum_{m=1}^{t-1} p_m = (N-t)\bar{K}\left[\begin{array}{c} \frac{1}{(N-t)(N-t+1)}z_{N-t+1}^{(N)} + \frac{1}{(N-t+1)(N-t+2)}z_{N-t+2}^{(N)} \\ +\frac{1}{(N-t+2)(N-t+3)}z_{N-t+3}^{(N)} + \cdots + \frac{1}{(N-1)N}z_N^{(N)} \end{array}\right],$$

which establishes the Lemma. □

Proof of Proposition 8(i): We show that when all bidders in the chore auction (with $\bar{K} = N - K$ chores) follow their maxmin perfect bidding strategy, then each bidder obtains his strategic Shapley value allocation.

For $i > N-K$, the bidder with the i-th lowest value $z_i^{(N)}$ drops out in round $N-i+1$ and pledges compensation

$$\gamma_{N-i+1}^{i}\left(z_i^{(N)}, \mathbf{p}_{t-1}\right) = \frac{1}{i}\left(\bar{K}z_i^{(N)} - \sum_{m=1}^{N-i} p_m\right).$$

His payoff is

$$z_i^{(N)} - \frac{1}{i}\left(\bar{K} z_i^{(N)} - \sum_{m=1}^{N-i} p_m\right) = \frac{i-\bar{K}}{i} z_i^{(N)} + \frac{1}{i}\sum_{m=1}^{N-i} p_m.$$

Using Lemma 1, we can write his payoff as

$$\frac{i-\bar{K}}{i} z_i^{(N)} + \bar{K}\left[\frac{1}{i\,(i+1)} z_{i+1}^{(N)} + \frac{1}{(i+1)\,(i+2)} z_{i+2}^{(N)} + \cdots + \frac{1}{(N-1)\,N} z_N^{(N)}\right].$$

Using that $\bar{K} = N - K$ this becomes

$$\frac{1}{i}\left(i - (N-K)\right) z_i^{(N)} + \sum_{m=1}^{N-i} \frac{N-K}{(i+m-1)\,(i+m)} z_{i+m}^{(N)}.$$

For $i \leq N-K$, the bidder with the i-th lowest value does not drop out. He receives a $1/\bar{K} = 1/(N-K)$ share of the total compensation pledged in K rounds, i.e.,

$$\frac{1}{N-K}\sum_{i=1}^{K} p_i.$$

By Lemma 1, we have

$$\begin{aligned}\frac{1}{N-K}\sum_{i=1}^{K} p_i &= \bar{K}\left[\begin{array}{c}\frac{1}{(N-K)(N-K+1)} z_{N-K+1}^{(N)} + \frac{1}{(N-K+1)(N-K+2)} z_{N-K+2}^{(N)} \\ + \cdots + \frac{1}{(N-1)N} z_N^{(N)}\end{array}\right] \\ &= \sum_{m=N-K+1}^{N} \frac{N-K}{(m-1)\,m} z_m^{(N)},\end{aligned}$$

which establishes Proposition 8(i). The proof of Proposition 8(ii) is symmetric. □

References

[1] Boyce, J.: Allocation of goods by lottery. Economic Inquiry **32**, 457-476 (1994)

[2] Bose, S., Ozdenoren, E., Pape, A.: Optimal auctions with ambiguity. Theoretical Economics **4**, 411-438 (2006).

[3] Brams, S., Taylor, A.: *Fair Division. From Cake Cutting to Dispute Resolution.* Cambridge University Press (1996)

[4] Brooks, R., Landeo, C., Spier, K.: Trigger happy or gun shy? Dissolving common-value partnerships with Texas shootouts. RAND Journal of Economics **41**, 649-673 (2010)

[5] Chakravarty, S., Mitra, M., Sarkar, P.: *A Course on Cooperative Game Theory.* Cambridge University Press. (2015)

[6] Chen, Y., Katuscak, P., Ozdenoren, E.: Sealed bid auctions with ambiguity: theory and experiments. Journal of Economic Theory **136**, 513-535 (2007)

[7] Cramton, O., Gall, U., Sujarittanonta, P., Wilson, R.: Applicant auctions for internet top-level domains: resolving conflicts efficiently. Mimeo (2013)

[8] Cramton, P., Gibbons, R., Klemperer, P.: Dissolving a partnership efficiently. *Econometrica* **55**, 615-632 (1987)

[9] de Frutos, M.A., Kittsteiner, T.: Efficient partnership dissolution under buy-sell clauses. *RAND Journal of Economics* **39**, 184-198 (2008)

[10] de Frutos, M.A.: Asymmetric price-benefit auctions. Games and Economic Behavior **33**, 48-71 (2000)

[11] Dubins, E., Spanier, E.: How to cut a cake fairly. American Mathematical Monthly **68**, 1-17 (1961)

[12] Federal Communications Commission: The FCC Report to Congress on Spectrum Auctions. http://wireless.fcc.gov/auctions/data/papersAndStudies/fc970353.pdf (1997)

[13] Gilboa, I., Schmeidler, D.: Maxmin expected utility with non-unique proirs. Journal of Mathematical Economics **18**, 141-153 (1989)

[14] Green, J., Laffont, J.: Characterization of satisfactory mechanisms for the revelation of preferences for public goods. Econometrica **45**, 427–438 (1977)

[15] Holt, C., Sherman, R.: Waiting-line auctions. Journal of Political Economy **90**, 280-294 (1982)

[16] Hu, A., Zou, L.: Sequential auctions, price trends, and risk preferences. Journal of Economic Theory **158**, 319-335 (2015)

[17] Leo, G.: Taking turns. *Games and Economic Behavior* **102**, 525-547 (2017)

[18] Levin, D., Ozdenoren, E.: Auctions with uncertain numbers of bidders. Journal of Economic Theory **118**, 229–251 (2004)

[19] Li, S.: Obviously strategy-proof mechanisms. *American Economic Review* **108**, 3257-3287 (2017)

[20] Long, Y.: Optimal strategy-proof and budget balanced mechanisms to assign multiple objects, http://papers.ssrn.com/sol3/papers.cfm?abstract_id=2827387 (2016)

[21] Long, Y., Mishra, D., Sharma, T.: Balanced ranking mechanisms. Games and Economic Behavior **105**, 9-39 (2017)

[22] McAfee, R.P.: Amicable divorce: dissolving a partnership with simple mechanisms. Journal of Economic Theory **56**, 266-293 (1992)

[23] Mezzetti, C.: Sequential auctions with informational externalities and aversion to price risk: decreasing and increasing price sequences. Economic Journal **121**, 990-1016 (2011)

[24] Morgan, J.: Dissolving a partnership (un)fairly. Economic Theory **24**, 909-923 (2004)

[25] Moulin, H.: An application of the Shapley value to fair division with money. Econometrica **60**, 1131-1349 (1992)

[26] Moulin, H.: *Cooperative Microeconomics - A Game Theoretic Introduction.* Princeton University Press (1995)

[27] Myerson, R.: Graphs and cooperation in games. Mathematics of Operations Research **2**, 225-229 (1977)

[28] Pourbabaee, F.: Robust experimentation in the continuous time bandit problem. Econ Theory 73, 151–181 (2022).

[29] Roth, A., ed.: The Shapley Value: Essays in honor of Lloyd S. Shapley. Cambridge University Press (1988)

[30] Salo, A., Weber, M.: Ambiguity aversion in first-price sealed-bid auctions. Journal of Risk and Uncertainty **11**, 123-137 (1995)

[31] Serrano, R.: Nash Program. In The New Palgrave Dictionary of Economics, 2nd edition, S. Durlauf and L. Blume (eds.), McMillan, London (2008)

[32] Serrano, R.: Sixty-seven years of the Nash program: time for retirement? SERIEs Journal of the Spanish Economic Association **12**, 35-48 (2021)

[33] Shapley, L. S.. Notes on the n-person game – II: the value of an n-person game. RAND Corporation (1951)

[34] Shapley, L. S.. Core of convex games. International Journal of Game Theory **11**, 101-106 (1971)

[35] Steinhaus, H.: The problem of fair division. *Econometrica* **16**, 101-104 (1948)

[36] Stong, S.: Ambiguity aversion in the all-pay auction and war of attrition. Journal of Public Economic Theory **20**, 822-839 (2018)

[37] Sung, J.: Optimal contracting under mean-volatility joint ambiguity uncertainties. Econ Theory **74**, 593–642 (2022)

[38] Su, F. E.: Rental harmony: Sperner's lemma in fair division. American Mathematics Monthly **106**, 930-942 (1999)

[39] Van Essen, M., Wooders, J.: Dissolving partnerships dynamically. Journal of Economic Theory **166**, 212-241 (2016)

[40] Van Essen, M., Wooders, J.: Allocating positions fairly: auctions and Shapley value. *Journal of Economic Theory* **196**, 1-47 (2021)

[41] Walker, M.: On the nonexistence of a dominant strategy mechanism for making optimal public decisions. Econometrica **48**, 1521–1540 (1980)

[42] Wasser, C.: Bilateral $k + 1$-price auctions with asymmetric shares and values. Games and Economic Behavior **82**, 350-368 (2013)