1 Supplemental Appendix

This appendix contains the proof of Proposition 3 and omitted proofs for the goods auction.

Proof of Proposition 3: By the Revelation Principle, for any mechanism and BNE of the mechanism, there is an equivalent direct mechanism with truthtelling as a BNE. Hence it is without loss of generality to take ξ to be a direct mechanism.

Let $m_\xi(x)$ be the expected payment of a bidder who reports value x when all the other bidders report their values truthfully. Since the bidders with the K highest reports win an item, the expected payoff of a bidder with value x who reports $\hat{x}$ is

$$\pi(\hat{x}, x) = H_{N-K}^{(N-1)}(\hat{x})x - m_\xi(\hat{x}),$$

where $H_{N-K}^{(N-1)}(\hat{x}) = \Pr(Z_{N-K}^{(N-1)} < \hat{x})$ is the probability that the $N-K$-th lowest of $N-1$ values is less than $\hat{x}$.

Differentiating $\pi(\hat{x}, x)$ with respect to $\hat{x}$ yields

$$\frac{d\pi(\hat{x}, x)}{d\hat{x}} = h_{N-K}^{(N-1)}(\hat{x})x - m'_\xi(\hat{x}).$$

Since truthtelling is a BNE then

$$m'_\xi(x) = h_{N-K}^{(N-1)}(x)x \text{ for all } x \in [0, \bar{x}].$$

By the Fundamental Theorem of Calculus we have

$$m_\xi(x) = m_\xi(0) + \int_0^x t h_{N-K}^{(N-1)}(t)dt.$$

Since the mechanism is budget balanced, the ex-ante expected payment of a bidder is zero, i.e.,

$$\int_0^{\bar{x}} m_\xi(x) f(x)dx = m_\xi(0) + \int_0^{\bar{x}} \left[\int_0^x t h_{N-K}^{(N-1)}(t)dt\right] f(x)dx = 0.$$

The double integral above is

$$\frac{1}{N}\int_0^{\bar{x}}\left[\int_0^x t\frac{N!}{(N-K-1)!\,(K-1)!}F(t)^{N-K-1}[1-F(t)]^{K-1}f(t)dt\right]f(x)dx.$$

Reversing the order of integration and factoring out K yields

$$\begin{aligned}
& \frac{K}{N}\int_0^{\bar{x}}\int_t^{\bar{x}} t\frac{N!}{(N-K-1)!K!}F(t)^{N-K-1}[1-F(t)]^{K-1}f(x)f(t)dxdt \\
= & \frac{K}{N}\int_0^{\bar{x}} t\frac{N!}{(N-K-1)!K!}F(t)^{N-K-1}[1-F(t)]^{K}f(t)dt \\
= & \frac{K}{N}E[Z_{N-k}^{(N)}].
\end{aligned}$$

Hence

$$m_\xi(0) = -\frac{K}{N}E[Z_{N-k}^{(N)}].$$

Thus the expected payoff to a bidder with value x is

$$H_{N-K}^{(N-1)}(x)x - m_\xi(x) = H_{N-K}^{(N-1)}(x)x - \left(-\frac{K}{N}E[Z_{N-k}^{(N)}] + \int_0^x t h_{N-K}^{(N-1)}(t)dt\right).$$

Integrating the RHS by parts, we have

$$H_{N-K}^{(N-1)}(x)x - m_\xi(x) = \frac{K}{N}E[Z_{N-k}^{(N)}] + \int_0^x H_{N-K}^{(N-1)}(t)dt,$$

which establishes the result. □

Proof of Proposition 1′: Let $\beta = (\beta_1, \ldots, \beta_{N-K})$ be a symmetric equilibrium in increasing and differentiable strategies. For each $t \leq N-K$, let $\pi_t(\hat{x}, x|\mathbf{z}_{t-1})$ be the expected payoff to a bidder with value x who in round t deviates from equilibrium and bids as though his value is $\hat{x}$ (i.e., he bids $\beta_t(\hat{x}|\mathbf{z}_{t-1})$), when $\mathbf{z}_{t-1}$ is the profile of values of the $t-1$ bidders to drop so far. In this case we will sometimes say the bidder "bids $\hat{x}$". Let

$$\Pi_t(x|\mathbf{z}_{t-1}) = \pi_t(x, x|\mathbf{z}_{t-1})$$

be the bidder's equilibrium payoff in round t.

(a) For each $\mathbf{z}_{t-1}$:

(a.i) β_t satisfies the differential equation given in Proposition 1'(i).

(a.ii) if $x \geq z_{t-1}$ then $x \in \arg\max_{\hat{x}} \pi_t(\hat{x}, x|\mathbf{z}_{t-1})$, i.e., it is optimal for each bidder to follow β_t in round t; if $x < z_{t-1}$ then $z_{t-1} \in \arg\max_{\hat{x}} \pi_t(\hat{x}, x|\mathbf{z}_{t-1})$.

(b) For each $\mathbf{z}_{t-1}$:

$$\frac{d\Pi_t(x|\mathbf{z}_{t-1})}{dx} \geq 0\,.$$

We prove by induction that the claim is true for each $t \in \{1, \ldots, N-K\}$, thereby establishing Proposition 1. Note that since equilibrium is in increasing strategies, at any round t the sequence of dropout prices $(p_1, \ldots, p_{t-1})$ reveals the $t-1$ lowest values $\mathbf{z}_{t-1} = (z_1, \ldots, z_{t-1})$.

Let $\mathbf{z}_{N-K-1}$ be arbitrary and consider an active bidder whose value is x but who bids as though it is $\hat{x} \geq z_{N-K-1}$. There are two cases to consider: (i) $x \geq z_{N-K-1}$ and (ii) $x < z_{N-K-1}$.

Case (i): $x \geq z_{N-K-1}$. With a bid of $\hat{x} \geq z_{N-K-1}$, if $\hat{x} > z_{N-K}$ then a rival bidder drops out first at the price $\beta_{N-K}(z_{N-K}|\mathbf{z}_{N-K-1})$, the bidder wins an item, and he receives compensation of

$$x - \frac{1}{K}\left(\beta_{N-K}(z_{N-K}|\mathbf{z}_{N-K-1}) + \sum_{i=1}^{N-K-1} p_i\right).$$

If $\hat{x} < z_{N-K}$ then the bidder drops before any rival and he obtains compensation $\beta_{N-K}(\hat{x}|\mathbf{z}_{N-K-1})$. Hence $\pi_{N-K}(\hat{x}, x|\mathbf{z}_{N-K-1}) =$

$$\int\limits_{z_{N-K-1}}^{\hat{x}} u(x - \tfrac{1}{K}\left(\beta_{N-K}(z_{N-K}|\mathbf{z}_{N-K-1}) + \textstyle\sum_{i=1}^{N-K-1} p_i\right))h_{N-K}^{(N-1)}(z_{N-K}|z_{N-K-1})dz_{N-K}$$
$$+ \int\limits_{\hat{x}}^{\bar{x}} u(\beta_{N-K}(\hat{x}|\mathbf{z}_{N-K-1}))h_{N-K}^{(N-1)}(z_{N-K}|z_{N-K-1})dz_{N-K}.$$

Differentiating with respect to $\hat{x}$ yields $\partial \pi_{N-K}(\hat{x}, x|\mathbf{z}_{N-K-1})/\partial \hat{x} =$

$$\begin{aligned}
&u'(\beta_{N-K}(\hat{x}|\mathbf{z}_{N-K-1}))\beta'_{N-K}(\hat{x}|\mathbf{z}_{N-K-1})(1 - H_{N-K}^{(N-1)}(\hat{x}|z_{N-K-1})) \\
&-u(\beta_{N-K}(\hat{x}|\mathbf{z}_{N-K-1}))h_{N-K}^{(N-1)}(\hat{x}|z_{N-K-1}) \\
&+u\left(x - \frac{1}{K}(\beta_{N-K}(\hat{x}|\mathbf{z}_{N-K-1}) + \sum_{i=1}^{N-K-1} p_i)\right) h_{N-K}^{(N-1)}(\hat{x}|z_{N-K-1})
\end{aligned} \quad (1)$$

A necessary condition for β to be an equilibrium is that $\partial \pi_{N-K}(\hat{x}, x|\mathbf{z}_{N-K-1})/\partial \hat{x}|_{\hat{x}=x} = 0$, i.e.,

$$\begin{aligned}
&u'(\beta_{N-K}(x|\mathbf{z}_{N-K-1}))\beta'_{N-K}(x|\mathbf{z}_{N-K-1}) \\
&= \left[u(\beta_{N-K}(x|\mathbf{z}_{N-K-1})) - u\left(x - (\tfrac{1}{K}\beta_{N-K}(x|\mathbf{z}_{N-K-1}) + \textstyle\sum_{i=1}^{N-K-1} p_i)\right)\right]]\lambda_{N-K}^{N}(x).
\end{aligned} \quad (2)$$

where

$$\frac{h_{N-K}^{(N-1)}(x|z_{N-K-1})}{1 - H_{N-K}^{(N-1)}(x|z_{N-K-1})} = \frac{Kf(x)}{1 - F(x)} = \lambda_{N-K}^{N}(x)$$

Alternatively, since types can be inferred from dropout prices, we can write the necessary condition as

$$\begin{aligned}
&u'(\beta_{N-K}(x; \mathbf{p}_{N-K-1}))\beta'_{N-K}(x; \mathbf{p}_{N-K-1}) \\
&= \left[u(\beta_{N-K}(x|\mathbf{p}_{N-K-1})) - ux - \tfrac{1}{K}(\beta_{N-K}(x|\mathbf{p}_{N-K-1}) + \textstyle\sum_{i=1}^{N-K-1} p_i)\right]]\lambda_{N-K}^{N}(x)
\end{aligned}$$

which establishes (a.i) for $t = N - K$.

The necessary condition holds for all x and, in particular, it holds for $x = \hat{x}$, i.e.,

$$\begin{aligned}
&u'(\beta_{N-K}(\hat{x}|\mathbf{z}_{N-K-1}))\beta'_{N-K}(\hat{x}|\mathbf{z}_{N-K-1}) \\
&= \left[u(\beta_{N-K}(\hat{x}|\mathbf{z}_{N-K-1})) - u\left(\hat{x} - \tfrac{1}{K}(\beta_{N-K}(\hat{x}|\mathbf{z}_{N-K-1}) + \textstyle\sum_{i=1}^{N-K-1} p_i)\right)\right]]\lambda_{N-K}^{N}(\hat{x})
\end{aligned} \quad (3)$$

Substituting (3) into (1) and simplifying yields

$$\frac{\partial \pi_{N-K}(\hat{x}, x|\mathbf{z}_{N-K-1})}{\partial \hat{x}} = \left[\begin{array}{c} u\left(x - \frac{1}{K}(\beta_{N-K}(\hat{x}|\mathbf{z}_{N-K-1}) + \sum_{i=1}^{N-K-1} p_i)\right) \\ -u\left(\hat{x} - \frac{1}{K}(\beta_{N-K}(\hat{x}|\mathbf{z}_{N-K-1}) + \sum_{i=1}^{N-K-1} p_i)\right) \end{array}\right] h_{N-K}^{(N-1)}(\hat{x}|z_{N-K-1}).$$

Clearly, $\partial\pi_{N-K}(\hat{x}, x|\mathbf{z}_{N-K-1})/\partial\hat{x}|_{\hat{x}=x} = 0$. Moreover, for $\hat{x} \geq z_{N-K-1}$ we have

$$\frac{\partial^2\pi_{N-K}(\hat{x}, x|\mathbf{z}_{N-K-1})}{\partial\hat{x}\partial x} = u'\left(x - \frac{1}{K}(\beta_{N-K}(\hat{x}|\mathbf{z}_{N-K-1}) + \sum_{i=1}^{N-K-1} p_i)\right) h_{N-K}^{(N-1)}(\hat{x}|z_{N-K-1}) \geq 0,$$

where the inequality holds since $u' > 0$ and $h_{N-K}^{(N-1)}(\hat{x}|z_{N-K-1}) \geq 0$. Hence, if $x \geq z_{N-K-1}$ then $x \in \arg\max_{\hat{x}} \pi_{N-K}(\hat{x}, x|\mathbf{z}_{N-K-1})$ by Lemma 0 of McAfee (1992).

Case (ii): $x < z_{N-K-1}$. It is clearly never optimal for a bidder to bid as though his type is less than z_{N-K-1}, i.e., bid less than $\beta_{N-K}(z_{N-K-1}|\mathbf{z}_{N-K-1})$, since he receives more compensation with a bid of $\beta_{N-K}(z_{N-K-1}|\mathbf{z}_{N-K-1})$. (For either bid he drops out for sure since the other bidders have values above $\mathbf{z}_{N-K-1}$.)

For $\hat{x} \geq z_{N-K-1}$ we have

$$\frac{\partial\pi_{N-K}(\hat{x}, x|\mathbf{z}_{N-K-1})}{\partial\hat{x}} = \begin{bmatrix} u\left(x - \frac{1}{K}(\beta_{N-K}(\hat{x}|\mathbf{z}_{N-K-1}) + \sum_{i=1}^{N-K-1} p_i)\right) \\ -u\left(\hat{x} - \frac{1}{K}(\beta_{N-K}(\hat{x}|\mathbf{z}_{N-K-1}) + \sum_{i=1}^{N-K-1} p_i)\right) \end{bmatrix} h_{N-K}^{(N-1)}(\hat{x}|z_{N-K-1}) < 0$$

and thus $z_{N-K-1} \in \arg\max_{\hat{x}} \pi_{N-K}(\hat{x}, x|\mathbf{z}_{N-K-1})$. Hence (a.ii) is true for $t = N-K$.

To prove (b), note that $d\Pi_{N-K}(x|\mathbf{z}_{N-K-1})/dx$ is

$$\begin{aligned} & \left.\frac{\partial\pi_{N-K}(\hat{x}, x|\mathbf{z}_{N-K-1})}{\partial\hat{x}}\right|_{\hat{x}=x} + \left.\frac{\partial\pi_{N-K}(\hat{x}, x|\mathbf{z}_{N-K-1})}{\partial x}\right|_{\hat{x}=x} \\ = & \int_{z_{N-K-1}}^{x} u'\left(x - \frac{1}{K}(\beta_{N-K}(z_{N-K}|\mathbf{z}_{N-K-1}) + \sum_{i=1}^{N-K-1} p_i)\right) h_{N-K}^{(N-1)}(z_{N-K}|z_{N-K-1})dz_{N-K} \\ \geq & \; 0, \end{aligned}$$

where the second equality holds since $\partial\pi_{N-K}(\hat{x}, x|\mathbf{z}_{N-K-1})/\partial\hat{x}|_{\hat{x}=x} = 0$. Hence (b) holds for $t = N-K$.

Assume the claim is true for rounds $t+1$ through $N-1$. We show it is true for round t. Let $\mathbf{z}_{t-1}$ be arbitrary. If $x < z_{t-1}$ then, by the same argument as before, $z_{t-1} \in \arg\max_{\hat{x}} \pi_t(\hat{x}, x|\mathbf{z}_{t-1})$.

Suppose $x \geq z_{t-1}$. Consider an active bidder in the t-th round whose value is x and who bids as though his value is $\hat{x} \geq z_{t-1}$. We need to distinguish between two cases: (i) $\hat{x} \in [z_{t-1}, x]$ and (ii) $\hat{x} > x$, since his payoff function differs in each case. In what follows, we denote the payoff to a bid of $\hat{x}$ as $\pi_t^L(\hat{x}, x|\mathbf{z}_{t-1})$ if $\hat{x} \in [z_{t-1}, x]$ and as $\pi_t^H(\hat{x}, x|\mathbf{z}_{t-1})$ if $\hat{x} \geq x$.

Case (i): Suppose $\hat{x} \in [z_{t-1}, x]$. If $z_t \in [z_{t-1}, \hat{x}]$ the bidder continues to round $t+1$ where, by the induction hypothesis, he optimally bids x and he has an expected payoff of $\Pi_{t+1}(x|\mathbf{z}_{t-1}, z_t)$. If $z_t \geq \hat{x}$ he receives compensation of $\beta_t(\hat{x}|\mathbf{z}_{t-1})$. Hence his payoff is

$$\begin{aligned}\pi_t^L(\hat{x}, x|\mathbf{z}_{t-1}) = \quad & \textstyle\int_{z_{t-1}}^{\hat{x}} \Pi_{t+1}(x|\mathbf{z}_{t-1}, z_t) h_t^{(N-1)}(z_t|z_{t-1}) dz_t \\ & + \textstyle\int_{\hat{x}}^{\bar{x}} u\left(\beta_t(\hat{x}|\mathbf{z}_{t-1})\right) h_t^{(N-1)}(z_t|z_{t-1}) dz_t.\end{aligned}$$

Differentiating with respect to $\hat{x}$ yields $\partial \pi_t^L(\hat{x}, x|\mathbf{z}_{t-1})/\partial \hat{x} =$

$$\begin{aligned}&\Pi_{t+1}(x|\mathbf{z}_{t-1}, \hat{x}) h_t^{(N-1)}(\hat{x}|z_{t-1}) - u\left(\beta_t(\hat{x}|\mathbf{z}_{t-1})\right) h_t^{(N-1)}(\hat{x}|z_{t-1}) \\ &+ u'\left(\beta_t(\hat{x}|\mathbf{z}_{t-1})\right) \beta_t'(\hat{x}|\mathbf{z}_{t-1})(1 - H_t^{(N-1)}(\hat{x}|z_{t-1})).\end{aligned}$$

Since

$$\Pi_{t+1}(x|\mathbf{z}_{t-1}, x) = u(\beta_{t+1}(x|\mathbf{z}_{t-1}, x)),$$

and

$$\frac{h_t^{(N-1)}(x|z_{t-1})}{1 - H_t^{(N-1)}(x|z_{t-1})} = (N-t)\frac{f(x)}{1 - F(x)} = \lambda_t^N(x),$$

the necessary condition for equilibrium that $\partial \pi_t^L(\hat{x}, x|\mathbf{z}_{t-1})/\partial \hat{x}|_{\hat{x}=x} \geq 0$ can be written as

$$\begin{aligned}& u'\left(\beta_t(x|\mathbf{z}_{t-1})\right) \beta_t'(x|\mathbf{z}_{t-1}) \\ \geq \quad & [u\left(\beta_t(x|\mathbf{z}_{t-1})\right) - u(\beta_{t+1}(x|\mathbf{z}_{t-1}, x))]\lambda_t^N(x).\end{aligned} \tag{4}$$

Also, for $\hat{x} \in [z_{t-1}, x]$ we have

$$\frac{\partial^2 \pi_t^L(\hat{x}, x|\mathbf{z}_{t-1})}{\partial \hat{x} \partial x} = \frac{d}{dx}\Pi_{t+1}(x|\mathbf{z}_{t-1}, \hat{x}) h_t^{(N-1)}(\hat{x}|z_{t-1}) \geq 0,$$

where the inequality follows since (b) is true for round $t+1$ by the induction hypothesis.

Case (ii): Suppose $\hat{x} \geq x$. If $z_t \in [z_{t-1}, x]$, then the bidder continues to round $t+1$ and, by the induction hypothesis, he bids x and obtains $\Pi_{t+1}(x|\mathbf{z}_{t-1}, z_t)$. If $z_t \in [x, \hat{x}]$, then he continues to round $t+1$ and, by the induction hypothesis, he bids z_t and receives compensation of $\beta_{t+1}(z_t|\mathbf{z}_{t-1}, z_t)$. If $z_t > \hat{x}$ then in round t he receives compensation of $\beta_t(\hat{x}|\mathbf{z}_{t-1})$. His payoff at round t is therefore

$$\begin{aligned}\pi_t^H(\hat{x}, x|\mathbf{z}_{t-1}) = \quad & \textstyle\int_{z_{t-1}}^{x} \Pi_{t+1}(x|\mathbf{z}_{t-1}, z_t) h_t^{(N-1)}(z_t|z_{t-1}) dz_t \\ & + \textstyle\int_{x}^{\hat{x}} u(\beta_{t+1}(z_t|\mathbf{z}_{t-1}, z_t)) h_t^{(N-1)}(z_t|z_{t-1}) dz_t, \\ & + \textstyle\int_{\hat{x}}^{\bar{x}} u\left(\beta_t(\hat{x}|\mathbf{z}_{t-1})\right) h_t^{(N-1)}(z_t|z_{t-1}) dz_t.\end{aligned}$$

Differentiating with respect to $\hat{x}$ yields

$$\begin{aligned}\frac{\partial \pi_t^H(\hat{x}, x|\mathbf{z}_{t-1})}{\partial \hat{x}} \quad = \quad & u(\beta_{t+1}(\hat{x}|\mathbf{z}_{t-1}, \hat{x})) h_t^{(N-1)}(\hat{x}|z_{t-1}) - u\left(\beta_t(\hat{x}|\mathbf{z}_{t-1})\right) h_t^{(N-1)}(\hat{x}|z_{t-1}) \\ & + u'\left(\beta_t(\hat{x}|\mathbf{z}_{t-1})\right) \beta_t'(\hat{x}|\mathbf{z}_{t-1})(1 - H_t^{(N-1)}(\hat{x}|z_{t-1})).\end{aligned}$$

A necessary condition for equilibrium is that $\partial \pi_t^H(\hat{x}, x|\mathbf{z}_{t-1})/\partial \hat{x}|_{\hat{x}=x} \leq 0$, i.e.,

$$\begin{aligned} & u'\left(\beta_t(x|\mathbf{z}_{t-1})\right) \beta_t'(x|\mathbf{z}_{t-1}) \\ \leq \quad & [u\left(\beta_t(x|\mathbf{z}_{t-1})\right) - u(\beta_{t+1}(x|\mathbf{z}_{t-1}, x))] \lambda_t^N(x).\end{aligned} \tag{5}$$

Equations (4) and (5) imply that

$$\begin{aligned} & u'\left(\beta_t(x|\mathbf{z}_{t-1})\right) \beta_t'(x|\mathbf{z}_{t-1}) \\ = \quad & [u\left(\beta_t(x|\mathbf{z}_{t-1})\right) - u(\beta_{t+1}(x|\mathbf{z}_{t-1}, x))] \lambda_t^N(x).\end{aligned} \tag{6}$$

Since the bid functions are increasing, we can replace $\mathbf{z}_{t-1}$ with $\mathbf{p}_{t-1}$ and replace $\beta_{t+1}(x|\mathbf{z}_{t-1}, x)$ with $\beta_{t+1}(x; \mathbf{p}_{t-1}, \beta_t(x; \mathbf{p}_{t-1}))$, writing the first order condition as

$$\begin{aligned} & u'\left(\beta_t(x; \mathbf{p}_{t-1})\right) \beta_t'(x; \mathbf{p}_{t-1}) \\ = \quad & [u\left(\beta_t(x; \mathbf{p}_{t-1})\right) - u(\beta_{t+1}(x; \mathbf{p}_{t-1}, \beta_t(x; \mathbf{p}_{t-1})))] \lambda_t^N(x),\end{aligned}$$

which establishes (a.i) for round t.

Equation (6) holds for all x and, in particular, it holds for $x = \hat{x}$, i.e.,

$$\begin{aligned} & u'\left(\beta_t(\hat{x}|\mathbf{z}_{t-1})\right)\beta_t'(\hat{x}|\mathbf{z}_{t-1}) \\ = \quad & [u\left(\beta_t(\hat{x}|\mathbf{z}_{t-1})\right) - u(\beta_{t+1}(\hat{x}|\mathbf{z}_{t-1}, \hat{x}))]\lambda_t^N(\hat{x}). \end{aligned}$$

Substituting this expression into the expression for $\partial\pi_t^H(\hat{x}, x|\mathbf{z}_{t-1})/\partial\hat{x}$ yields

$$\frac{\partial\pi_t^H(\hat{x}, x|\mathbf{z}_{t-1})}{\partial\hat{x}} = 0 \text{ for } \hat{x} \geq x.$$

Furthermore,

$$\frac{\partial^2\pi_t^H(\hat{x}, x|\mathbf{z}_{t-1})}{\partial\hat{x}\partial x} = 0 \text{ for } \hat{x} \geq x.$$

We have shown that

$$\left.\frac{\partial\pi_t^L(\hat{x}, x|\mathbf{z}_{t-1})}{\partial\hat{x}}\right|_{\hat{x}=x} = \left.\frac{\partial\pi_t^H(\hat{x}, x|\mathbf{z}_{t-1})}{\partial\hat{x}}\right|_{\hat{x}=x} = 0$$

and

$$\frac{\partial^2\pi_t^L(\hat{x}, x|\mathbf{z}_{t-1})}{\partial\hat{x}\partial x} \geq 0 \text{ for } \hat{x} \in [z_{t-1}, x] \text{ and } \frac{\partial^2\pi_t^H(\hat{x}, x|\mathbf{z}_{t-1})}{\partial\hat{x}\partial x} \geq 0 \text{ for } \hat{x} \geq x.$$

Hence (a.ii) is true for round t by Van Essen and Wooders' (2016) extension of McAfee's (1992) Lemma 0.

To establish (b) is true for round t, observe that

$$\begin{aligned} \Pi_t(x|\mathbf{z}_{t-1}) \quad = \quad & \int_{z_{t-1}}^{x} \Pi_{t+1}(x|\mathbf{z}_{t-1}, z_t)h_t^{(N-1)}(z_t|z_{t-1})dz_t \\ & + \int_{x}^{\bar{x}} u\left(\beta_t(x|\mathbf{z}_{t-1})\right) h_t^{(N-1)}(z_t|z_{t-1})dz_t. \end{aligned}$$

Differentiating and simplifying yields

$$\frac{d\Pi_t(x|\mathbf{z}_{t-1})}{dx} = \int\limits_{z_{t-1}}^{x} \frac{d}{dx}\Pi_{t+1}(x|\mathbf{z}_{t-1}, z_t)h_t^{(N-1)}(z_t|z_{t-1})dz_t \geq 0,$$

where the equality follows from $\Pi_{t+1}(x|\mathbf{z}_{t-1}, x) = u(\beta_{t+1}(x|\mathbf{z}_{t-1}, x))$ and (6), and the inequality follows since $d\Pi_{t+1}(x|\mathbf{z}_{t-1}, z_t)/dx \geq 0$ by the induction hypothesis. □

Proof of Proposition 2.2: The proof is by induction. By Proposition 1′(i), at round $N-K$ the differential equation for the equilibrium bid function is

$$\beta'_{N-K}(x|\mathbf{p}_{N-K-1}) = \left[\frac{K+1}{K}\beta_{N-K}(x|\mathbf{p}_{N-K-1}) + \frac{1}{K}\sum_{i=1}^{N-K-1} p_i - x\right]\lambda^N_{N-K}(x).$$

Multiplying both sides by $1-F(x)$ we obtain

$$\beta'_{N-K}(x|\mathbf{p}_{N-K-1})\,(1-F(x)) - (K+1)\beta_{N-K}(x|\mathbf{p}_{N-K-1})f(x) = \left[\frac{1}{K}\sum_{i=1}^{N-K-1} p_i - x\right]Kf(x)$$

i.e.,

$$\frac{d}{dx}\left(\beta_{N-K}(x|\mathbf{p}_{N-K-1})\,(1-F(x))^{K+1}\right) = \left[\frac{1}{K}\sum_{i=1}^{N-K-1} p_i - x\right]Kf(x)\,(1-F(x))^K.$$

By the Fundamental Theorem of Calculus we have

$$\beta_{N-K}(x|\mathbf{p}_{N-K-1})\,(1-F(x))^{K+1} = -\int_0^x [s - \frac{1}{K}\sum_{i=1}^{N-K-1} p_i]Kf(s)\,(1-F(s))^K\,ds + C.$$

The LHS of this equation is zero when $x = \bar{x}$ (since $F(\bar{x}) = 1$), which implies

$$C = \int_0^{\bar{x}} [s - \frac{1}{K}\sum_{i=1}^{N-K-1} p_i]Kf(s)\,(1-F(s))^K\,ds.$$

Since

$$\int_x^{\bar{x}} s(K+1)\frac{(1-F(s))^K}{(1-F(x))^{K+1}}f(s)ds = E\left[Z^{(N)}_{N-K}|Z^{(N)}_{N-K} > x > Z^{(N)}_{N-K-1}\right],$$

then

$$\beta_{N-K}(x|\mathbf{p}_{N-K-1}) = \frac{K}{K+1}E\left[Z^{(N)}_{N-K}|Z^{(N)}_{N-K} > x > Z^{(N)}_{N-K-1}\right] - \frac{1}{K+1}\sum_{i=1}^{N-K-1} p_i$$

which establishes the result for round $N-K$.

Assume in round t that

$$\beta_t(x;\mathbf{p}_{t-1}) = \frac{K}{N-t+1}E\left[Z_{N-K}^{(N)}|Z_t^{(N)} > x > Z_{t-1}^{(N)}\right] - \frac{1}{N-t+1}\sum_{i=1}^{t-1} p_i.$$

We need to show that $\beta_{t-1}(x;\mathbf{p}_{t-2})$ is as given in Proposition 2. The differential equation for round $t-1$ is

$$\beta_{t-1}'(x;\mathbf{p}_{t-2}) = \left[\beta_{t-1}(x|\mathbf{p}_{t-2}) - \beta_t(x|\mathbf{p}_{t-2},\beta_{t-1}(x|\mathbf{p}_{t-2}))\right]\lambda_{t-1}^N(x).$$

By the induction hypothesis

$$\begin{aligned}\beta_t(x;\mathbf{p}_{t-2},\beta_{t-1}(x|\mathbf{p}_{t-2})) &= \frac{K}{N-t+1}E\left[Z_{N-K}^{(N)}|Z_t^{(N)} > x > Z_{t-1}^{(N)}\right] \\ &\quad -\frac{1}{N-t+1}\left(\sum_{i=1}^{t-2} p_i + \beta_{t-1}(x|\mathbf{p}_{t-2})\right).\end{aligned}$$

Hence,

$$\beta_{t-1}'(x;\mathbf{p}_{t-2}) = \left[\begin{array}{c}\frac{N-t+2}{N-t+1}\beta_{t-1}(x|\mathbf{p}_{t-2}) - \frac{K}{N-t+1}E\left[Z_{N-K}^{(N)}|Z_t^{(N)} > x > Z_{t-1}^{(N)}\right] \\ +\frac{1}{N-t+1}\sum_{i=1}^{t-2} p_i\end{array}\right]\lambda_{t-1}^N(x).$$

Multiplying both sides by $(1-F(x))^{N-t+2}$ yields $\frac{d}{dx}\left(\beta_{t-1}(x|\mathbf{p}_{t-2})(1-F(x))^{N-t+2}\right) =$

$$\left[\frac{\sum_{i=1}^{t-2} p_i}{N-t+1} - KE\left[Z_{N-K}^{(N)}|Z_t^{(N)} > x > Z_{t-1}^{(N)}\right]\right] f(x)(1-F(x))^{N-t+1}.$$

By the Fundamental Theorem of Calculus and since $F(\bar{x}) = 1$ then

$$\beta_{t-1}(x|\mathbf{p}_{t-2}) = \int_x^{\bar{x}}\left[\frac{\sum_{i=1}^{t-2} p_i}{N-t+1} - KE\left[Z_{N-K}^{(N)}|Z_t^{(N)} > s > Z_{t-1}^{(N)}\right]\right] f(s)\frac{(1-F(s))^{N-t+1}}{(1-F(x))^{N-t+2}}ds.$$

Since (to be established momentarily)

$$\begin{aligned}&\int_x^{\bar{x}} E\left[Z_{N-K}^{(N)}|Z_t^{(N)} > s > Z_{t-1}^{(N)}\right] f(s)\frac{(N-t+2)(1-F(s))^{N-t+1}}{(1-F(x))^{N-t+2}}ds \\ =\ & E\left[Z_{N-K}^{(N)}|Z_{t-1}^{(N)} > x > Z_{t-2}^{(N)}\right]\end{aligned}$$

then

$$\beta'_{t-1}(x;\mathbf{p}_{t-2}) = \frac{K}{N-t+2}\left(E\left[Z^{(N)}_{N-K}|Z^{(N)}_{t-1} > x > Z^{(N)}_{t-2}\right] - \frac{1}{K}\sum_{i=1}^{t-2} p_i\right)$$

which completes the proof.

Finally we establish the equality we just used. We have

$$\int_x^{\bar{x}} E\left[Z^{(N)}_{N-K}|Z^{(N)}_{t} > s > Z^{(N)}_{t-1}\right] f(s)\frac{(N-t+2)\left(1-F(s)\right)^{N-t+1}}{\left(1-F(x)\right)^{N-t+2}}ds$$

$$= \int_x^{\bar{x}}\left(\int_s^{\bar{x}} \frac{r(N-t+1)!\left[F(r)-F(s)\right]^{N-K-t}\left[1-F(r)\right]^K f(r)dr}{(N-K-t)!K!\left[1-F(s)\right]^{N-t+1}}\right)$$

$$\times\frac{(N-t+2)f(s)\left(1-F(s)\right)^{N-t+1}}{\left(1-F(x)\right)^{N-t+2}}ds$$

$$= \int_x^{\bar{x}}\left(\int_x^{r} r\frac{(N-t+1)!}{(N-K-t)!K!}\frac{\left[F(r)-F(s)\right]^{N-K-t}\left[1-F(r)\right]^K}{\left(1-F(x)\right)^{N-t+2}}\right)(N-t+2)f(r)f(s)dsdr$$

$$= \int_x^{\bar{x}} r\frac{(N-t+2)!}{(N-K-t+1)!K!}\frac{\left[F(r)-F(s)\right]^{N-K-t+1}\left[1-F(r)\right]^K}{\left(1-F(x)\right)^{N-t+2}}f(r)dr$$

$$E\left[Z^{(N)}_{N-K}|Z^{(N)}_{t-1} > x > Z^{(N)}_{t-2}\right].\square$$

Proof of Proposition 4.2: To save space we write β_t rather than $\beta_t(x;\mathbf{p}_{t-1})$. At round $t = N-K$, the differential equation that characterizes equilibrium behavior is

$$\frac{d}{dx}\left(e^{-\alpha(\frac{K+1}{K}\beta_{N-K})}\left(1-F(x)\right)^{K+1}\right) = -\left(e^{\frac{\alpha}{K}\sum_{i=1}^{N-K-1}p_i}\right)e^{-\alpha x}\left(K+1\right)f(x)\left(1-F(x)\right)^K.$$

From the Fundamental Theorem of Calculus, $e^{-\alpha(\frac{K+1}{K}\beta_{N-K})}\left(1-F(x)\right)^{K+1} =$

$$-\left(e^{\frac{\alpha}{K}\sum_{i=1}^{N-K-1}p_i}\right)\int_0^x e^{-\alpha z}\left(K+1\right)f(z)\left(1-F(z)\right)^K dz + C$$

At $x = \bar{x}$, the LHS of the above equation is equal to zero and hence

$$C = \left(e^{\frac{\alpha}{K}\sum_{i=1}^{N-K-1}p_i}\right)\int_0^{\bar{x}} e^{-\alpha z}\left(K+1\right)f(z)\left(1-F(z)\right)^K dz.$$

So

$$e^{-\alpha\left(\frac{K+1}{K}\beta_{N-K}\right)} = \left(e^{\frac{\alpha}{K}\sum_{i=1}^{N-K-1} p_i}\right) \frac{\int_x^{\bar{x}} e^{-\alpha z} (K+1) f(s) (1-F(s))^K ds}{(1-F(x))^{K+1}}.$$

Taking logs of both sides we have

$$-\alpha \frac{K+1}{K}\beta_{N-K} = \ln\left(\frac{\int_x^{\bar{x}} e^{-\alpha z} (K+1) f(s) (1-F(s))^K ds}{(1-F(x))^{K+1}}\right) + \alpha \frac{1}{K}\sum_{i=1}^{N-K-1} p_i,$$

and hence

$$\begin{aligned}
\beta_{N-K}^{\alpha}(x; \mathbf{p}_{N-K-1}) &= -\frac{K}{\alpha(K+1)} \ln\left(\frac{\int_x^{\bar{x}} e^{-\alpha z} (K+1) f(z) (1-F(z))^K dz}{(1-F(x))^{K+1}}\right) \\
&\quad -\frac{1}{K+1}\sum_{i=1}^{N-K-1} p_i. \\
&= -\frac{K}{\alpha(K+1)} \ln\left(E\left[e^{-\alpha Z_{N-K}^{(N)}} | Z_{N-K}^{(N)} > x > Z_{N-K-1}^{(N)}\right]\right) \\
&\quad -\frac{1}{K+1}\sum_{i=1}^{N-K-1} p_i \\
&= -\frac{K}{\alpha(K+1)} \ln\left(D_{N-K}^{\alpha}(x)\right) - \frac{1}{K+1}\sum_{i=1}^{N-K-1} p_i.
\end{aligned}$$

Next, we solve for the round $t-1$ bid function. Assume that in round $t \leq N-K$, bidders follow the bid function

$$\beta_t^{\alpha}(x; \mathbf{p}_{t-1}) = -\frac{N-t}{(N-t+1)\,\alpha} \ln\left(D_t^{\alpha}(x)\right) - \frac{1}{N-t+1}\sum_{i=1}^{t-1} p_i.$$

Note that this implies that $\beta_t^{\alpha}(x; \mathbf{p}_{t-2}, \beta_{t-1}^{\alpha}(x; \mathbf{p}_{t-2})) =$

$$-\frac{N-t}{(N-t+1)\,\alpha} \ln\left(D_t^{\alpha}(x)\right) - \frac{1}{N-t+1}\sum_{i=1}^{t-2} p_i - \frac{1}{N-t+1}\beta_{t-1}^{\alpha}(x; \mathbf{p}_{t-2}).$$

After some manipulation, the differential equation for round $t-1$ from Proposition $1'$ can be written as

$$\begin{aligned}&\frac{d}{dx}\left(e^{-\alpha\frac{N-t+2}{N-t+1}\beta_{t-1}}\left(1-F(x)\right)^{N-(t-1)+1}\right)\\ =\ &-e^{\alpha\left(\frac{1}{N-t+1}\sum_{i=1}^{t-2}p_i\right)}D_t^{\alpha}(x)^{\frac{N-t}{N-t+1}}(N-t+2)\left(1-F(x)\right)^{N-t+1}f(x).\end{aligned}$$

From the Fundamental Theorem of Calculus we have $e^{-\alpha\frac{N-t+2}{N-t+1}\beta_{t-1}}\left(1-F(x)\right)^{N-(t-1)+1}=$

$$-\int_0^x e^{\alpha\left(\frac{1}{N-t+1}\sum_{i=1}^{t-2}p_i\right)}D_t^{\alpha}(s)^{\frac{N-t}{N-t+1}}(N-t+2)\left(1-F(s)\right)^{N-t+1}f(s)ds+C.$$

At $x=\bar{x}$, the LHS of the above equation is equal to zero and hence

$$C=\int_0^{\bar{x}} e^{\alpha\left(\frac{1}{N-t+1}\sum_{i=1}^{t-2}p_i\right)}D_t^{\alpha}(s)^{\frac{N-t}{N-t+1}}(N-t+2)\left(1-F(s)\right)^{N-t+1}f(s)ds.$$

Rearranging yields $\beta_{t-1}(x;\mathbf{p}_{t-2})=$

$$\begin{aligned}&-\frac{1}{N-t+2}\sum_{i=1}^{t-2}p_i-\frac{N-t+1}{(N-t+2)\,\alpha}\ln\left[\frac{\int_x^{\bar{x}}D_t^{\alpha}(z)^{\frac{N-t}{N-t+1}}(N-t+2)\left(1-F(z)\right)^{N-t+1}f(z)dz}{\left(1-F(x)\right)^{N-t+2}}\right]\\ =\ &-\frac{1}{N-t+2}\sum_{i=1}^{t-2}p_i-\frac{N-t+1}{(N-t+2)\,\alpha}\ln(D_{t-1}^{\alpha}(x))\end{aligned}$$

where the second equality holds since

$$D_{t-1}^{\alpha}(x)=E\left[\left(D_t^{\alpha}(Z_{t-1}^{(N)})\right)^{\frac{N-t}{N-t+1}}|Z_{t-1}^{(N)}>x>Z_{t-2}^{(N)}\right].\square$$

Proof of Proposition 5: Here we establish the inequalities for the goods auction.

We show that for $t=1,\ldots,N-K$ and $\mathbf{p}_{t-1}$ that $\beta_t^0(x;\mathbf{p}_{t-1})>\beta_t^{\alpha}(x;\mathbf{p}_{t-1})$ for $x<\bar{x}$. The proof is by induction. For $t=N-K$, since e^x is a convex function, then by Jensen's Inequality, for $x<\bar{x}$ we have

$$e^{E[-\alpha Z_{N-K}^{(N)}|Z_{N-K}^{(N)}>x>Z_{N-K-1}^{(N)}]}<E[e^{-\alpha Z_{N-K}^{(N)}}|Z_{N-K}^{(N)}>x>Z_{N-K-1}^{(N)}].$$

Noting that the RHS is $D^\alpha_{N-K}(x)$, taking the log of both sides, and then multiplying through by $-K/((K+1)\,\alpha)$ yields

$$\frac{K}{K+1}E[Z^{(N)}_{N-K}|Z^{(N)}_{N-K} > x > Z^{(N)}_{N-K-1}] > -\frac{K}{(K+1)\alpha}\ln(D^\alpha_{N-K}(x))$$

Adding $-\frac{1}{K+1}\sum_{i=1}^{N-K-1} p_i$ to both sides yields $\beta^0_{N-K}(x;\mathbf{p}_{N-K-1}) > \beta^\alpha_{N-1}(x;\mathbf{p}_{N-K-1})$ for $x < \bar{x}$.

For $t \leq N-K$, define

$$\Sigma^0_t(x) = \frac{K}{N-t+1}E[Z^{(N)}_{N-K}|Z^{(N)}_t > x > Z^{(N)}_{t-1}],$$

and

$$\Sigma^\alpha_t(x) = -\frac{1}{\alpha}\ln\left(D^\alpha_t(x)^{\frac{N-t}{N-t+1}}\right),$$

where $D^\alpha_t(x)$ is defined in Proposition P4.2. We have that

$$e^{-\alpha\Sigma^\alpha_t(x)} = D^\alpha_t(x)^{\frac{N-t}{N-t+1}}.$$

We established above that $\Sigma^0_{N-K}(x) > \Sigma^\alpha_{N-K}(x)$.

Assume for $t \leq N-K-1$ that $\Sigma^0_{t+1}(x) > \Sigma^\alpha_{t+1}(x)$ for $x < \bar{x}$. We show that $\Sigma^0_t(x) > \Sigma^\alpha_t(x)$ for $x < \bar{x}$. Since $-\alpha\Sigma^0_{t+1}(x) < -\alpha\Sigma^\alpha_{t+1}(x)$ and e^x is increasing, then

$$e^{-\alpha\Sigma^0_{t+1}(x)} < e^{-\alpha\Sigma^\alpha_{t+1}(x)} \text{ for } x < \bar{x},$$

or

$$e^{-\alpha\Sigma^0_{t+1}(x)} < D^\alpha_{t+1}(x)^{\frac{N-t-1}{N-t}} \text{ for } x < \bar{x},$$

Thus

$$E[e^{-\alpha\Sigma^0_{t+1}(Z^{(N)}_t)}|Z^{(N)}_t > x > Z^{(N)}_{t-1}] < E[D^\alpha_{t+1}(Z^{(N)}_t)^{\frac{N-t-1}{N-t}}|Z^{(N)}_t > x > Z^{(N)}_{t-1}] = D^\alpha_t(x).$$

Since e^x is convex, then

$$e^{E[-\alpha\Sigma^0_{t+1}(Z^{(N)}_t)|Z^{(N)}_t > x > Z^{(N)}_{t-1}]} < E[e^{-\alpha\Sigma^0_{t+1}(Z^{(N)}_t)}|Z^{(N)}_t > x > Z^{(N)}_{t-1}].$$

and hence

$$e^{E[-\alpha\Sigma^0_{t+1}(Z_t^{(N)})|Z_t^{(N)}>x>Z_{t-1}^{(N)}]} < D_t^\alpha(x).$$

Taking logs of both sides of this inequality yields

$$E[-\alpha\Sigma^0_{t+1}(Z_t^{(N)})|Z_t^{(N)} > x > Z_{t-1}^{(N)}] < \ln\left(D_t^\alpha(x)\right).$$

Multiplying both sides by $-\frac{N-t}{(N-t+1)\alpha}$ yields

$$\begin{aligned}\int_x^{\bar{x}} \Sigma^0_{t+1}(z)\frac{(N-t)[1-F(z)]^{N-t}f(z)}{(1-F(x))^{N-t+1}}dz &= \frac{K}{N-t+1}E\left[Z_{N-K}^{(N)}|Z_t^{(N)} > x > Z_{t-1}^{(N)}\right].\\ &> -\frac{N-t}{(N-t+1)\alpha}\ln\left(D_t^\alpha(x)\right).\end{aligned}$$

Adding $-\frac{1}{N-t+1}\sum_{i=1}^{t-1}p_i$ to both sides yields $\beta_t^0(x;\mathbf{p}_{t-1}) > \beta_t^\alpha(x;\mathbf{p}_{t-1})$ for $x < \bar{x}$.

We now show that for each $t = 1, \ldots, N-K$ and $\mathbf{p}_{t-1}$ that $\beta_t^\alpha(x;\mathbf{p}_{t-1}) > \gamma_t(x;\mathbf{p}_{t-1})$ for $x < \bar{x}$. The proof is by induction. We first show $\beta_{N-K}^\alpha(x;\mathbf{p}_{N-K-1}) > \gamma_{N-K}(x;\mathbf{p}_{N-K-1})$.

Since $e^{-\alpha s} < e^{-\alpha x}$ for $x < s < \bar{x}$ then

$$D_{N-K}^\alpha(x) = E[e^{-\alpha Z_{N-K}^{(N)}}|Z_{N-K}^{(N)} > x > Z_{N-K-1}^{(N)}] < e^{-\alpha x}.$$

Taking logs of both sides and rearranging yields

$$-\frac{K}{(K+1)\alpha}\ln(D_{N-K}^\alpha(x)) > \frac{K}{K+1}x,$$

Adding $-\frac{1}{K+1}\sum_{i=1}^{N-K-1}p_i$ to both sides yields $\beta_{N-K}^\alpha(x;\mathbf{p}_{t-1}) > \gamma_{N-K}(x;\mathbf{p}_{N-K-1})$ for $x < \bar{x}$.

Assume for $t \leq N-K-1$ that $\Sigma^\alpha_{t+1}(x) > Kx/(N-t)$ for $x < \bar{x}$. Since $\Sigma^\alpha_{t+1}(x)$ is increasing, then for $s > x$ we have $\Sigma^\alpha_{t+1}(s) > \Sigma^\alpha_{t+1}(x) > Kx/(N-t)$ or $-\alpha\Sigma^\alpha_{t+1}(s) < -\alpha\Sigma^\alpha_{t+1}(x) < -\alpha Kx/(N-t)$ and thus

$$e^{-\alpha\Sigma^\alpha_{t+1}(s)} = D_{t+1}^\alpha(s)^{\frac{N-t-1}{N-t}} < e^{-\alpha\Sigma^\alpha_{t+1}(x)} < e^{-\alpha K\frac{x}{N-t}}.$$

Hence

$$E[D_{t+1}^{\alpha}(Z_t^{(N)})^{\frac{N-t-1}{N-t}}|Z_t^{(N)} > x > Z_{t-1}^{(N)}] = D_t^{\alpha}(x) < e^{-\alpha K \frac{x}{N-t}}.$$

Taking logs of both sides yields

$$\ln(D_t^{\alpha}(x)) < -\alpha K \frac{x}{N-t},$$

and so

$$-\frac{N-t}{(N-t+1)\alpha} \ln(D_t^{\alpha}(x)) > \frac{Kx}{N-t+1}.$$

Hence $\Sigma_t^{\alpha}(x) > Kx/(N-t+1)$ for $x < \bar{x}$. Adding $-\sum_{i=1}^{t-1} p_i$ to each side gives us

$$\beta_t^{\alpha}(x; \mathbf{p}_{t-1}) > \gamma_t(x; \mathbf{p}_{t-1}) \text{ for } x < \bar{x}. \ \Box$$

Proof of Proposition 6: We first show that $\beta_t^{\alpha}(x; \mathbf{p}_{t-1})$ is decreasing in α. The proof is by induction. Suppose $\tilde{\alpha} > \alpha$. Since the transformation $y = x^{\frac{\alpha}{\tilde{\alpha}}}$ is concave, then by Jensen's inequality we have that

$$\begin{aligned} \left(D_{N-K}^{\tilde{\alpha}}(x)\right)^{\frac{\alpha}{\tilde{\alpha}}} &= \left(E[e^{-\tilde{\alpha} Z_{N-K}^{(N)}}|Z_{N-K}^{(N)} > x > Z_{N-K-1}^{(N)}]\right)^{\frac{\alpha}{\tilde{\alpha}}} \\ &> E[\left(e^{-\tilde{\alpha} Z_{N-K}^{(N)}}\right)^{\frac{\alpha}{\tilde{\alpha}}}|Z_{N-K}^{(N)} > x > Z_{N-K-1}^{(N)}] \\ &= D_{N-K}^{\alpha}(x) \end{aligned}$$

for $x < \bar{x}$. Taking logs and rearranging yields

$$-\frac{1}{K+1} \sum_{i=1}^{N-K-1} p_i - \frac{K}{(K+1)\alpha} \ln D_{N-K}^{\alpha}(x) > -\frac{1}{K+1} \sum_{i=1}^{N-K-1} p_i - \frac{K}{(K+1)\tilde{\alpha}} \ln D_{N-K}^{\tilde{\alpha}}(x).$$

Hence, $\beta_{N-K}^{\alpha}(x; \mathbf{p}_{N-K-1}) > \beta_{N-K}^{\tilde{\alpha}}(x; \mathbf{p}_{N-K-1})$.

Let

$$\Sigma_{t+1}^{\alpha}(x) = -\frac{1}{\alpha} \ln \left(D_t^{\alpha}(x)^{\frac{N-t-1}{N-t}}\right).$$

Suppose $\beta^{\alpha}_{t+1}(x;\mathbf{p}_t) > \beta^{\tilde{\alpha}}_{t+1}(x;\mathbf{p}_t)$ and hence $\Sigma^{\alpha}_{t+1}(x) > \Sigma^{\tilde{\alpha}}_{t+1}(x)$. We show that $\beta^{\alpha}_{t}(x;\mathbf{p}_{t-1}) > \beta^{\tilde{\alpha}}_{t}(x;\mathbf{p}_{t-1})$. Jensen's inequality and $\Sigma^{\alpha}_{t+1}(x) > \Sigma^{\tilde{\alpha}}_{t+1}(x)$ imply

$$\begin{aligned}\left(E[e^{-\tilde{\alpha}\Sigma^{\tilde{\alpha}}_{t+1}(Z^{(N)}_t)}|Z^{(N)}_t > x > Z^{(N)}_{t-1}]\right)^{\frac{\alpha}{\tilde{\alpha}}} &> E[e^{-\alpha\Sigma^{\tilde{\alpha}}_{t+1}(Z^{(N)}_t)}|Z^{(N)}_t > x > Z^{(N)}_{t-1}] \\ &> E[e^{-\alpha\Sigma^{\alpha}_{t+1}(Z^{(N)}_t)}|Z^{(N)}_t > x > Z^{(N)}_{t-1}].\end{aligned}$$

Simple algebra yields

$$\begin{aligned}\Sigma^{\alpha}_t(x) &= -\frac{N-t}{(N-t+1)\alpha}\ln E[e^{-\alpha\Sigma^{\alpha}_{t+1}(Z^{(N)}_t)}|Z^{(N)}_t > x > Z^{(N)}_{t-1}] \\ &> -\frac{N-t}{(N-t+1)\tilde{\alpha}}\ln E[e^{-\tilde{\alpha}\Sigma^{\tilde{\alpha}}_{t+1}(Z^{(N)}_t)}|Z^{(N)}_t > x > Z^{(N)}_{t-1}] \\ &= \Sigma^{\tilde{\alpha}}_t(x)\end{aligned}$$

and therefore that $\beta^{\alpha}_{t}(x;\mathbf{p}_{t-1}) > \beta^{\tilde{\alpha}}_{t}(x;\mathbf{p}_{t-1})$.

Next we prove that the $\lim_{\alpha\to\infty}\beta^{\alpha}_t(x;\mathbf{p}_{t-1}) = \gamma_t(x;\mathbf{p}_{t-1})$. The bid function $\beta^{\alpha}_t(x;\mathbf{p}_{t-1})$ can be written as

$$\beta^{\alpha}_t(x;\mathbf{p}_{t-1}) = -\frac{1}{\alpha}\ln\left(D^{\alpha}_t(x)^{\frac{N-t}{N-t+1}}\right) - \sum_{i=1}^{t-1}\frac{1}{N-t+1}p_i.$$

By the definition of $D^{\alpha}_t(x)$ and iteratively applying Jensen's Inequality we obtain

Likewise, since $y^{\frac{N-t-1}{N-t+1}}$ is concave, repeating the same argument yields

$$D^{\alpha}_t(x)^{\frac{N-t}{N-t+1}} \geq E[e^{-\frac{\alpha K}{N-t+1}Z^{(N)}_{N-K}}|Z^{(N)}_t > x > Z^{(N)}_{t-1}]. \tag{7}$$

Thus we have

$$\frac{1}{\alpha}\ln(D^{\alpha}_t(x)^{\frac{N-t}{N-t+1}}) \geq \frac{1}{\alpha}\ln\left(E[e^{-\frac{\alpha K}{N-t+1}Z^{(N)}_{N-K}}|Z^{(N)}_t > x > Z^{(N)}_{t-1}]\right).$$

The round t equilibrium bid function therefore is bounded above by

$$\beta^{\alpha}_t(x;\mathbf{p}_{t-1}) \leq -\frac{1}{\alpha}\ln\left(E[e^{-\frac{\alpha K}{N-t+1}Z^{(N)}_{N-K}}|Z^{(N)}_t > x > Z^{(N)}_{t-1}]\right) - \sum_{i=1}^{t-1}\frac{1}{N-t+1}p_i.$$

By Proposition 5 we have that

$$\gamma_t(x;\mathbf{p}_{t-1}) \le \beta_t^{\alpha}(x;\mathbf{p}_{t-1}).$$

We complete the proof by establishing that $\lim_{\alpha\to\infty} -\frac{1}{\alpha}\ln\left(E[e^{-\frac{\alpha K}{N-t+1}Z_{N-K}^{(N)}}|Z_t^{(N)} > x > Z_{t-1}^{(N)}]\right) = \frac{xK}{N-t+1}$, i.e.,

$$\lim_{\alpha\to\infty} -\frac{1}{\alpha}\ln\left(\int_x^{\bar{x}} e^{-\frac{\alpha K z}{N-t+1}}h(s)ds\right) = \frac{Kx}{N-t+1},$$

where

$$h(s) = \frac{(N-t+1)!}{(N-K-t)!K!}\frac{[F(s)-F(x)]^{N-K-t}(1-F(s))^K}{[1-F(x)]^{N-t+1}}f(s).$$

The result then follows from the squeeze theorem.

We now establish the above limit. Applying l'Hopital's rule, this limit equals

$$\lim_{\alpha\to\infty}\frac{K}{N-t+1}\frac{\int_x^{\bar{x}} ze^{-\frac{\alpha K z}{N-t+1}}h(z)dz}{\int_x^{\bar{x}} e^{-\frac{\alpha K z}{N-t+1}}h(z)dz}.$$

Setting $\tilde{\alpha} = \frac{\alpha K}{N-t+1}$ the desired result is equivalent to showing that

$$\lim_{\tilde{\alpha}\to\infty}\frac{\int_x^{\bar{x}} ze^{-\tilde{\alpha}z}h(z)dz}{\int_x^{\bar{x}} e^{-\tilde{\alpha}t}h(z)dz} = x$$

This was demonstrated in the proof of Proposition 6 of Van Essen and Wooders (2016). Hence, we have that $\lim_{\alpha\to\infty}\beta_t^{\alpha}(x;\mathbf{p}_{t-1}) = -\sum_{i=1}^{t-1}\frac{1}{N-t+1}p_i + \frac{xK}{N-t+1}$. $\square$