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Equilibrium in a market with intermediation is Walrasian

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Abstract. We show that a profit maximizing monopolistic intermediary may behave approximately like a Walrasian auctioneer by setting bid and ask prices nearly equal to Walrasian equilibrium prices. In our model agents choose to trade either through the intermediary or privately. Buyers (sellers) trading through the intermediary potentially trade immediately at the ask (bid) price, but sacrifice the spread as gains. A buyer or seller who trades privately shares all the gains to trade with this trading partner, but risks costly delay in finding a partner. We show that as the cost of delay vanishes, the equilibrium bid and ask prices converge to the Walrasian equilibrium prices.

JEL classification: C72, C78, L12

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1 Introduction

The fiction of a benevolent auctioneer is sometimes used to explain the following paradoxical aspect of competitive equilibrium. In a competitive equilibrium each agent takes prices as given, but when all agents behave in this manner, how prices come to be equilibrium prices is left unexplained. The role of the auctioneer is to adjust prices until markets clear, thus resolving the paradox. Our objective is to show that a self-interested monopolist intermediary may effectively play the role of a benevolent Walrasian auctioneer by setting nearly Walrasian bid and ask prices.

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In our model buyers and sellers have the option of trading through a monopolistic intermediary or trading privately. Given the intermediary's bid and ask price, at each date buyers and sellers remaining in the market choose whether to enter the mediated market or the private trading market. Buyers (sellers) entering the mediated market potentially trade with the intermediary immediately at his ask (bid) price, but sacrifice the difference between the bid and the ask (i.e., the spread). Agents entering the private trading market risk costly delays in finding a trading partner, but share all the gains to trade with a partner once one is found. In equilibrium each agent follows an optimal policy of entering the mediated and private trading market given (a) the bid and ask price, (b) the price negotiated in the private trading market, and (c) the entry policy of all other agents. In addition, the intermediary's bid and ask prices maximize his profits.

Our main result is that when buyers and sellers are patient, the intermediary sets bid and ask prices nearly equal to Walrasian equilibrium prices. Thus, the intermediary sets prices which are (nearly) market clearing. In contrast, for the economy we consider it is known from Rubinstein and Wolinsky (1985) that the market outcome is not Walrasian when all trade is decentralized. Therefore, we find that some degree of centralized trade, like that provided by an intermediary, is necessary in order for the market outcome to be Walrasian.

To obtain our results, we compare three trading procedures that differ with respect to the degree of trade centralization. At one extreme is a Walrasian procedure in which trade is centralized (i.e., there is no search). At the other extreme is a market with only private trade, which makes for complete decentralization. Our model of an intermediated market lies between the two, and allows both centralized trade (though the intermediary) and decentralized trade (through a private trading market). For all three models, the underlying exchange economy is the same and thus the source of differences in market outcomes is the difference in the degree of centralized trade allowed.

The exchange economy we consider is a time-differentiated commodities market with two goods, money and an indivisible good, at each of an infinite number of dates. A buyer who enters the market at date t and who subsequently at date s exchanges p units of date-s money for a unit of date-s indivisible good obtains utility $\delta^{s-t}(1-p)$, where δ is the discount factor. A seller who enters at date t' prior to s and who takes the other side of such an exchange obtains utility $\delta^{s-t'}p$. A greater measure of sellers than buyers enters the market at date zero, and equal measures of sellers and buyers enter the market at every date thereafter.

A Walrasian trading procedure provides a benchmark against which the other trading procedures can be compared. Under the Walrasian procedure at each date there is a spot market for each good and buyers and sellers can trade at the prevailing spot market prices without search. Thus the procedure is interpreted as one in which trade is centralized. A Walrasian equilibrium is a sequence of spot market prices and a trading date for each agent such that the market for the indivisible good at each date clears and the date at which each agent trades maximizes his utility given the sequence of spot market prices. From Wooders

(1994) it is known that the Walrasian equilibrium price of the indivisible good (in units of money) is zero at each date for the time-differentiated commodities market under consideration.

Our model of the intermediated market is the focus of the present paper. We show that in equilibrium the intermediary's bid price is zero and his ask price is positive, but less than the reservation price of buyers. Each buyer trades in the mediated market at the date he enters. Thus, the intermediary and buyers capture all gains. We also show that as the discount factor approaches one (i.e., the cost of delay vanishes), in the limit the intermediary's bid and ask are both zero, and thus he buys and sells the indivisible good at each date at its Walrasian price.

Last we discuss the situation where there is only private trading, in which case trade is completely decentralized. Rubinstein (1989) shows for this case that the price of the indivisible good is positive, and therefore different from its Walrasian price, even in the limit as the discount factor approaches one.[1] Thus, when only decentralized trade is possible, the outcome is not Walrasian.[2] Comparing equilibrium payoffs of buyers and sellers in an intermediated market with payoffs when there is only private trade reveals that the intermediary captures all gains arising from elimination of delay.

Intermediated markets are of interest as they represent market structures intermediate between markets where trade is entirely decentralized and Walrasian markets. The real-estate market, the stock market, and the used car market all have the property that agents may trade either through a dealer or privately. Our model of an intermediated market is a stylized representation of such markets. One can also view our model of an intermediated market as a market-like mechanism whose equilibria are nearly Walrasian.

The paper is organized as follows. Section 2 describes the time-differentiated commodities market and its Walrasian equilibrium. Section 3 describes the model of the intermediated market and Section 4 characterizes its equilibria. Section 5 discusses the equilibrium when there is only private trade. We conclude by discussing related models of intermediated markets.

2 The economy and its Walrasian equilibrium

We consider an economy where each agent has an interest in carrying out only one transaction. Each buyer (seller) is concerned with the date and the price at which he obtains (supplies) a unit of indivisible good. It is convenient to represent such an economy as an overlapping-generations economy with infinitely-lived agents.

The set of agents is denoted by E, where $E \subset \mathbb{R}$. The Lebesgue measure on the line is denoted by μ. Let $\{E_B, E_S\}$ be a Lebesgue measurable partition of E, where E_i is the set of agents of type i. The indices B and S refer to "buyers" and "sellers," respectively. Time is indexed by $t \in \{0, 1, \ldots\}$. Let $\{E^0, E^1, \ldots\}$ be another Lebesgue measurable partition of E, where E^t is the set of agents entering (or born) at date t. The set of generation t agents of type i is $E_i^t = E_i \cap E^t$. The demographics of the economy are such that there is a greater measure of generation zero sellers than generation zero buyers, and an equal measure of sellers and buyers at every generation thereafter. In particular, there exist numbers $\nu > 0$ and $\Delta > 0$ such that $\mu(E_S^0) = \nu + \Delta$, $\mu(E_B^0) = \nu$, and $\mu(E_S^t) = \mu(E_B^t) = \nu$ for each $t > 0$. Therefore, by date t a measure $\nu t + \Delta$ of sellers has been born while only a measure νt of buyers has been born.

At each date there are two goods. Good 0 is divisible and plays the role of money, while good 1 is indivisible. Each buyer is endowed with zero units of the indivisible good and one unit of money at the date he is born. Each seller is endowed with a unit of the indivisible good and no money at the date he is born. Both goods are costlessly storable.

Let p_s denote the price of a unit of the indivisible good at date s in terms of units of money at date s. Let $\mathbf{p} = (p_0, p_1, \ldots)$. Agents are impatient, discounting future gains using discount factor δ, where $\delta < 1$. A buyer who enters at date t and trades at date $\tau \in \{t, t+1, \ldots\}$, given prices $\mathbf{p}$, has utility $u_t(\tau, \mathbf{p}; B) = \delta^{\tau - t}(1 - p_\tau)$. A seller who enters at date t and trades at date $\tau \in \{t, t+1, \ldots\}$, given prices $\mathbf{p}$, has utility $u_t(\tau, \mathbf{p}; S) = \delta^{\tau - t} p_\tau$. Thus, each seller has a reservation price of zero for his unit of the indivisible good, and each buyer demands a unit of the indivisible good with a limit price of one. The (undiscounted) utility of a buyer and seller who trade at date t are $1 - p_t$ and p_t, respectively. The utility of never trading is zero.

Since an agent can trade a unit of the indivisible good only following his entry into the market, the choice set for an agent entering at date t is $X^t = \{t, t+1, \ldots\} \cup \{\infty\}$, where the element ∞ denotes never trading. An *assignment* is a function $x : E \to X^0$ such that $x(E^t) \subseteq X^t$ for each t. We assume that x is measurable.

Following Wooders (1994), a Walrasian equilibrium is a price sequence and assignment such that (i) the assignment is market-clearing at each date, and (ii) the trading date assigned to each agent maximizes the agent's utility given the price sequence.[3]

Definition: *A* **Walrasian equilibrium** *is a pair* $(\mathbf{p}, x)$ *satisfying for each* $t \geq 0$:

(i) $\mu(\{j \in E_B^0 \cup \ldots \cup E_B^t : x(j) = t\}) = \mu(\{j \in E_S^0 \cup \ldots \cup E_S^t : x(j) = t\})$;

(ii) *For each* $i \in \{S, B\} : u_t(x(j), \mathbf{p}; i) = \max_{\tau \in X^t} u_t(\tau, \mathbf{p}; i)$, $\forall j \in E_i^t$.

We say $\mathbf{p}$ is *Walrasian* if there exists an assignment x such that $(\mathbf{p}, x)$ is a Walrasian equilibrium. The following Theorem is a special case of the Theorem 1 in Wooders (1994), and is proven there.

Theorem 1: *If* $\mathbf{p}$ *is Walrasian then* $\mathbf{p} = (0, 0, \ldots)$.

The intuition underlying this result is straightforward. If $\mathbf{p}$ were Walrasian and were $p_t > 0$ for some t, then each seller entering by date t would eventually supply a unit of indivisible good since he would obtain a strictly positive utility by supplying a unit of indivisible good at date t, but would obtain only a utility of zero by never trading. Since the measure of sellers entering by date t is greater than the measure of buyers entering by t, market-clearing would then imply that there would be a date $m > t$ such that a positive measure of sellers entering by date t would supply a unit of indivisible good at date m. Moreover, utility maximization would imply $\delta^{m-t} p_m \geq p_t$, since otherwise each such seller would obtain a higher utility by supplying a unit at date t than he would by supplying a unit at date m. An induction argument establishes that if $\mathbf{p}$ were Walrasian and were $p_t > 0$ for some t, then for every $n \geq t$ there would be an $m > n$ such $\delta^{m-t} p_m > p_t$. Choosing n sufficiently large yields $p_m > 1$, but this would imply market-clearing cannot be satisfied since each generation m seller would supply a unit at a price no lower than p_m, while no buyer would demand a unit at a price greater than one.

We have taken the Walrasian trading procedure to be one where (*i*) at each date there is only a spot market for each good, and (*ii*) each agent concentrates his purchases and sales on spot markets at one date. This setup parallels the one in our model of the intermediated market and the model of the market with only private trade. In both models, (*i*) exchanges at date t involve only date t money and date t indivisible good, and (*ii*) each buyer and seller participates in only one exchange. Defining the Walrasian trading procedure in this fashion isolates the differing degrees of centralized trade under different trading procedures (rather than, say, the presence or absence of futures markets) as the source of the differences in the market outcomes.

3 A model of an intermediated market

In this section, for the economy just described, we study the equilibrium of a trading procedure where at each date, each buyer and seller born, but not having yet traded, chooses whether to trade with a monopolistic intermediary or whether to trade privately. Buyers (sellers) entering the mediated market potentially trade immediately with the intermediary at the ask (bid) price, but sacrifice the spread as potential gains to trade. The private trading market is modeled as a random matching market. An agent entering the matching market shares all the gains to trade with his partner once matched, but may experience costly delay in being matched.

In our model of an intermediated market, the bid P_b and the ask P_a are chosen by the intermediary to maximize his profits. The intermediary is not endowed with an inventory of the traded good, nor can he accumulate one. The intermediary may only cross trades and so, if unequal measures of buyers and sellers enter the mediated market, the intermediary must ration the type of agent entering in greater measure. The intermediary rations only the type of agent entering the mediated market in greater measure and rations agents of the same type with the same probability. Let m_i denote the measure of agents of type $i \in \{S, B\}$ born but not having yet traded, and let λ_i denote the proportion of those agents entering the mediated market. Using the notation "$\neg i$" to refer to agents not of type i, the probability that a type i agent trades when entering the mediated market, denoted by ρ_i, is

$$\rho_i = \begin{cases} \frac{\lambda_{\neg i} m_{\neg i}}{\lambda_i m_i} & \text{if } \lambda_{\neg i} m_{\neg i} \leq \lambda_i m_i \\ 1 & \text{if } \lambda_{\neg i} m_{\neg i} > \lambda_i m_i . \end{cases} \quad (1)$$

Those agents who are rationed remain in the market, and at the next date again choose whether to enter the mediated or matching market.

Attention is restricted to situations where P_a, P_b, m_i, and λ_i are stationary for each $i \in \{S, B\}$. In a steady state, at each date a measure $\lambda_i \rho_i m_i$ of type i agents trade in the mediated market and then exit. (It is unambiguous to refer to the volume of trade in the intermediated market as $\lambda_i \rho_i m_i$ since by (1) we have $\lambda_S \rho_S m_S = \lambda_B \rho_B m_B$.) Although the intermediary carries no inventory, since the measure of each type of agent entering the market is deterministic, the intermediary would not benefit by doing so. Note that at each date a measure ν or greater of each type of agent is born, and therefore $m_i \geq \nu$ for each $i \in \{S, B\}$.

When the intermediary crosses a trade, transferring a unit of the indivisible good from a seller to a buyer, the seller receives a price of P_b while the buyer pays a price of P_a. The buyer's (undiscounted) utility is $1 - P_a$ and the seller's utility is P_b. Agents are von Neumann-Morgenstern expected utility maximizers and have rational conjectures about their probability of trading when entering the mediated or the matching market. Hence defining $R_B \equiv 1 - P_a$ and $R_S \equiv P_b$, the expected reward to a type i agent to entering the mediated market is $\rho_i R_i$. The difference $1 - R_B - R_S = P_a - P_b$ is the "spread" and represents the profit to the intermediary from crossing a single trade. The intermediary's steady state profit is $(1 - R_B - R_S)\lambda_i \rho_i m_i$. Since profit is proportional to volume, our assumption that the intermediary does not unnecessarily ration agents (i.e., he rations only the type of agent entering the mediated market in greater measure) is natural.

Those agents not entering the mediated market enter the matching market. The probability that a type i agent finds a partner when entering the matching market depends upon the measure of each type of agent entering the matching market. In particular, the probability an agent of type i is matched is

$$\alpha_i = k \frac{(1 - \lambda_{\neg i}) m_{\neg i}}{(1 - \lambda_i) m_i + (1 - \lambda_{\neg i}) m_{\neg i}}, \quad (2)$$

where $(1 - \lambda_i) m_i$ is the measure of agents of type i entering the matching market, and $k \in (0, 1]$ is an exogenous parameter indexing the efficiency of the random matching process. (The same "matching technology," for $k = 1$, is used in Gale (1987).) When each match ends with trade, at each date a measure $\alpha_i (1 - \lambda_i) m_i$ of type i agents trade in the matching market and then exit. Those agents who are not matched again choose whether to enter the mediated or the matching market at the next date.

Since there is an (undiscounted) unit gain to trade in any match, the net surplus of a match is one minus the sum of the buyer's and seller's disagreement payoff. Let N_i denote the surplus negotiated in the matching market by a type i agent when matched. We assume that when the net surplus is non-negative, each match ends with trade. In this case, $N_S + N_B = 1$ and a matched buyer and seller exchange a unit of the indivisible good at a price of N_S. If the net surplus is negative, then bargaining ends with disagreement and N_i is defined to be zero for each $i \in \{S, B\}$. The expected reward to an agent of type i to entering the matching market is $\alpha_i N_i$.

When ρ_i, α_i, R_i, and N_i are stationary, the problem of choosing an optimal policy for entering the mediated and the matching market is a stationary discounted dynamic programming problem. Let V_i denote the expected utility of an agent of type i under the optimal policy. It is well known (see Theorem 2.1 of Ross (1983), for example) that V_i satisfies the optimality equation

$$V_i = \max\{\rho_i R_i + (1-\rho_i)\delta V_i, \alpha_i N_i + (1-\alpha_i)\delta V_i\}. \tag{3}$$

The disagreement payoff of a matched agent of type i is δV_i, so the net surplus of a match is $1 - \delta V_S - \delta V_B$. Since each agent is able to obtain a utility of zero by never trading, attention is restricted to situations where $V_S \geq 0$ and $V_B \geq 0$.

A matched buyer and seller are assumed to negotiate a price for the indivisible good that evenly splits the net surplus of their match when the net surplus is nonnegative. Therefore

$$N_i = \begin{cases} \frac{1-\delta V_S - \delta V_B}{2} + \delta V_i & \text{if } 1 - \delta V_S - \delta V_B \geq 0 \\ 0 & \text{otherwise.} \end{cases} \tag{4}$$

Since $V_i \geq 0$, we have $N_i \geq 0$. Using Nash rather than strategic bargaining simplifies the analysis, without qualitatively changing our results.

The value to an agent of type i of entering the mediated market and following the optimal policy thereafter, is $\rho_i R_i + (1-\rho_i)\delta V_i$. The value of entering the matching market and following the optimal policy thereafter is $\alpha_i N_i + (1-\alpha_i)\delta V_i$. It is well known that there is a stationary policy which is optimal. (See Theorem 2.2 of Ross (1983).) In particular, if $\rho_i R_i + (1-\rho_i)\delta V_i$ is greater than $\alpha_i N_i + (1-\alpha_i)\delta V_i$, then the policy of entering the mediated market at each date is optimal. Therefore, the proportion of type i agents entering each market is related to the value of entering each market as follows

$$\rho_i R_i + (1-\rho_i)\delta V_i \begin{pmatrix} > \\ < \end{pmatrix} \alpha_i N_i + (1-\alpha_i)\delta V_i \Rightarrow \lambda_i = \begin{pmatrix} 1 \\ 0 \end{pmatrix}. \tag{5}$$

Either policy is optimal when the value of entering each market is equal.

In a steady state, at each date exits from the mediated market plus exits from the matching market are exactly balanced by entry. Thus, for each $i \in \{S, B\}$, m_i is related to λ_i, ρ_i, and α_i by the equation

$$\lambda_i \rho_i m_i + \alpha_i (1-\lambda_i) m_i = \nu. \tag{6}$$

Equation (6) presumes that each match ends with trade, which is indeed the case when entering the matching market is optimal for both types.[4] Since buyers and sellers exit in pairs from each market, we require that

$$m_S - m_B = \Delta. \tag{7}$$

In other words, the steady-state difference between the measure of sellers born, but not having yet traded, and the measure of buyers born, but not having yet traded, must be the same as the difference in the measures at date zero.

A vector $\{R_i, V_i, m_i, \lambda_i, \alpha_i, \rho_i\}_{i=S,B}$ satisfying (1)-(7) represents a steady state of the intermediated market when each agent follows an optimal policy of entering the mediated and matching market given the bid and ask, the price negotiated in the private trading market, and the entry policy of every other agent. We refer to such a vector as a *post bid–ask equilibrium*.

Definition. *The vector* $\{R_i, V_i, m_i, \lambda_i, \alpha_i, \rho_i\}_{i=S,B}$ *is a* **post bid-ask equilibrium (PBAE)** *if it satisfies (1)-(7).*

Given a *PBAE* $\{R_i, V_i, m_i, \lambda_i, \alpha_i, \rho_i\}_{i=S,B}$, the (steady state) profit to the intermediary is $(1 - R_B - R_S)\lambda_i \rho_i m_i$. A *PBAE* need not be a "full" equilibrium as the bid (i.e., $1 - R_B$) and ask (i.e., R_S) may not maximize the intermediary's profit. We define an equilibrium with intermediation as follows.

Definition. *A PBAE* $\{R_i^*, V_i^*, m_i^*, \lambda_i^*, \alpha_i^*, \rho_i^*\}_{i=S,B}$ *is an* **equilibrium with intermediation** *if for every other PBAE* $\{R_i, V_i, m_i, \lambda_i, \alpha_i, \rho_i\}_{i=S,B}$ *we have* $(1 - R_B^* - R_S^*)\lambda_i^* \rho_i^* m_i^* \geq (1 - R_B - R_S)\lambda_i \rho_i m_i$.

We are interested in intermediation when there is an active private trading market (i.e., $\lambda_i < 1$ for some $i \in \{S, B\}$) and we have implicitly restricted attention to this case since, were the private trading market inactive, then α_i would not given by (2). If $\lambda_S = \lambda_B = 1$, then an agent entering the private trading market finds a partner with probability zero and therefore entering the mediated market is optimal even if $P_a = 1$ and $P_b = 0$. Situations like this one in which the intermediary maintains a large spread are not, however, robust to even one type of agent making arbitrarily small "trembles" when choosing which market to enter. As we now show, if agents tremble when choosing a market, and therefore $\lambda_i < 1$ for some $i \in \{S, B\}$, then $V_S + V_B$ is bounded below by $\frac{k}{2-2\delta+\delta k}$.

Proposition: *Let* $\lambda_i < 1$ *for some* $i \in \{S, B\}$. *Then equations (2)-(4) imply* $V_S + V_B \geq \frac{k}{2-2\delta+\delta k}$.

Proof. Suppose contrary to the proposition that $V_S + V_B < \frac{k}{2-2\delta+\delta k}$. Then $1 - \delta V_S - \delta V_B > 0$ and equations (3) and (4) imply that

$$V_S + V_B \geq (\alpha_S + \alpha_B)\left(\frac{1 - \delta V_S - \delta V_B}{2}\right) + \delta(V_S + V_B).$$

Equation (2) implies $\alpha_S + \alpha_B = k$. Rearranging the inequality yields a contradiction. □

An implication of this result is that for the volume of trade in the mediated market to be positive when agents tremble, then the spread can be no greater than $1 - \frac{k}{2-2\delta+\delta k}$, since otherwise at least one type of agent obtains a higher payoff entering the private trading market. Clearly the constant returns to scale nature of the matching technology plays an important role in this argument. For other matching technologies than the one considered here, the sum $\alpha_S + \alpha_B$ might approach zero as trembles vanish, the measure of agents in the matching market therefore approaching zero. In that case, the sum $V_S + V_B$ would be bounded below only by zero and the intermediary could maintain a large spread for small trembles.

The definition of *PBAE* also rules out situations where the mediated market is inactive. (If $\lambda_S = \lambda_B = 0$ then ρ_i is not given by (1).) If the mediated market is inactive, then an agent entering the mediated market trades with probability zero since the intermediary only crosses trades. This kind of situation, in which the intermediary cannot attract entry regardless of its bid and ask, is easily upset by trembles (as above) or by allowing the intermediary to make purchases without making sales. Explicitly modelling trembles yields the same results as those obtained here where we have accounted for trembles implicitly in our definition of equilibrium.

4 Equilibrium with intermediation is Walrasian

The main result of this section is that in every equilibrium with intermediation the bid is zero and the ask, which depends on the discount factor, is positive but less than one. As the cost of delay vanishes, the ask goes to zero, and the intermediary buys and sells the indivisible good at each date at its Walrasian equilibrium price of zero. Our first result is that an equilibrium with intermediation exists.

Theorem 2: *An equilibrium with intermediation exists*.

Theorem 3 characterizes the set of equilibria with intermediation.

Theorem 3: *In every equilibrium with intermediation the intermediary's bid is* 0*, his ask is* $\frac{(1-\delta)(2-k)}{2-2\delta+\delta k}$ (*which is positive but less than the reservation price of buyers), and all trade is in the mediated market. Specifically, every equilibrium with intermediation* $\{R_i, V_i, m_i, \lambda_i, \alpha_i, \rho_i\}_{i=S,B}$ *satisfies:*

(i) $R_B = V_B = \frac{k}{2-2\delta+\delta k}, R_S = V_S = 0;$

(ii) $m_B = \nu, m_S = \nu + \Delta;$

(iii) $\alpha_B = k, \alpha_S = 0;$

(iv) $\lambda_B = 1, \frac{\nu}{\nu+\Delta} \leq \lambda_S < 1;$

(v) $\rho_B = 1, \rho_S = \frac{\nu}{\lambda_S(\nu+\Delta)}.$

An equilibrium with intermediation has the following characteristics at each date t: Each generation t buyer enters the mediated market (since $\lambda_B = 1$) and trades immediately (since $\rho_B = 1$). A measure $\lambda_S(\nu + \Delta)$ of sellers enters the mediated market, where $\nu \leq \lambda_S(\nu + \Delta) < \nu + \Delta$, and each seller trades with probability $\frac{\nu}{\lambda_S(\nu+\Delta)}$. Although the date at which any given generation t seller trades is not determined, each generation t seller either supplies a unit of the indivisible good at a price of zero or never supplies a unit of the indivisible good. The volume of trade in the mediated market is ν, the spread is $1 - R_B - R_S = \frac{(1-\delta)(2-k)}{2-2\delta+\delta k}$, and the intermediary's profit is $\frac{(1-\delta)(2-k)}{2-2\delta+\delta k}\nu$. The intermediary and buyers capture all gains to trade.

Of primary interest is the equilibrium behavior of the intermediary when the cost of delay vanishes. As the discount factor approaches one, the ask obtained in the limit is zero since $\lim_{\delta\to 1} \frac{(1-\delta)(2-k)}{2-2\delta+\delta k} = 0$. The spread also goes to zero as the cost of delay vanishes, since the bid is zero regardless of the cost of delay. In the limit, the intermediary buys and sells the indivisible good at each date at its Walrasian equilibrium price of zero. Thus, we have the following result.

Theorem 4: *Every equilibrium with intermediation is Walrasian in the limit as the discount factor approaches one, i.e.,* $\lim_{\delta\to 1} P_a = 0$, $P_b = 0\, \forall \delta \in (0, 1)$, $\lim_{\delta\to 1} V_B = 1$, *and* $\lim_{\delta\to 1} V_S = 0$.

To understand the result that the intermediary's bid and ask are nearly Walrasian when the cost of delay is small, it is useful to note that there are *PBAE* in which the intermediary and sellers capture all gains to trade. Consider, for example, the following one: $V_S = \frac{k}{2-2\delta+\delta k}, V_B = 0, m_S = 2\nu + \Delta, m_B = 2\nu, \lambda_S = 1, \lambda_B = \frac{1}{2}, \alpha_S = k, \alpha_B = 0, \rho_S = \frac{\nu}{2\nu+\Delta}, \rho_B = 0, R_S = V_S \frac{1-\delta(1-\rho_S)}{\rho_S}$, and $R_B = 0$. In this *PBAE* the bid is $\frac{k}{2-2\delta+\delta k} \frac{1-\delta(1-\rho_S)}{\rho_S}$, the ask is 1, and at each date all sellers enter the mediated market. The price negotiated in the matching market, were there a match, is $N_S = \frac{1-\delta V_S-\delta V_B}{2} + \delta V_S = \frac{1-\delta+\delta k}{2-2\delta+\delta k}$. Thus were a seller to enter the matching market at every date, his expected utility would be $k \frac{N_S}{1-\delta+\delta k} = \frac{k}{2-2\delta+\delta k}$. The intermediary, therefore, would need only bid $\frac{k}{2-2\delta+\delta k}$ to induce sellers to enter the mediated market were $\rho_S = 1$. Since $\rho_S < 1$ in this *PBAE* and since delay is costly when a seller obtains a positive price, the intermediary's bid must be greater than $\frac{k}{2-2\delta+\delta k}$.

This *PBAE* is not an equilibrium with intermediation since the intermediary obtains the same volume at a smaller spread by setting a bid of zero and an ask of $\frac{(1-\delta)(2-k)}{2-2\delta+\delta k}$. Although sellers may be rationed in a *PBAE* with this bid and ask, the intermediary need not compensate sellers for being rationed since sellers are indifferent between entering the matching market (and not trading since $\alpha_S = 0$) and entering the mediated market (and trading at a price of zero, possibly after some delay).

In the next section we show that when agents only have the opportunity to trade privately, then the price of the indivisible good at each date is positive even as the cost of delay vanishes. We conclude that some degree of centralized trade, like that provided by an intermediary, is necessary for the market outcome to be Walrasian.

5 Equilibrium with only private trading

By removing the possibility of trading through the intermediary from the model of intermediation, one obtains a model where trade is completely decentralized. In this case Rubinstein (1989) shows, when α_i is the steady state matching probability of agents of type i, that the expected utility of a type i agent, denote by $\hat{V}_i$, is[5]

$$\hat{V}_i = \frac{\alpha_i}{2(1-\delta)+\delta(\alpha_S+\alpha_B)}.$$

For the matching technology of the present paper we have by (2) that $\alpha_S + \alpha_B = k$ and, therefore, that $\hat{V}_i = \frac{\alpha_i}{2(1-\delta)+\delta k}$ and $\hat{V}_S + \hat{V}_B = \frac{k}{2(1-\delta)+\delta k}$. As the cost of delay vanishes the price of the indivisible good at each date is given by $\lim_{\delta\to 1} \frac{1-\delta\hat{V}_S-\delta\hat{V}_B}{2} + \delta\hat{V}_S = \frac{\alpha_S}{k}$, which is greater than its Walrasian equilibrium price of zero. Thus, as the costs of delay vanish, neither buyers nor sellers obtain their Walrasian equilibrium payoff. This non-Walrasian result is well known from Rubinstein and Wolinsky (1985), which embeds the alternating offer game of Rubinstein (1982) into a model where there is only private trade.

It is natural to ask how the entry of an intermediary into a market where heretofore all trade was private affects the welfare of buyers and sellers. We measure flow-welfare of traders by the sum of the expected utilities of buyers and sellers and, since a buyer and seller have a unit gain to trade, we measure the flow cost of delay by one minus the flow welfare. Denoting by V_B^* and V_S^*, respectively, the expected utilities of buyers and sellers in an equilibrium with intermediation, we have by Theorem 3 that $V_B^* = \frac{k}{2-\delta(2-k)}$ and $V_S^* = 0$. Therefore $\hat{V}_S + \hat{V}_B = V_S^* + V_B^*$, which yields the following corollary to Theorem 3.

Corollary 1: *In an equilibrium with intermediation, (i) the flow welfare is the same as in the equilibrium with only private trading, and (ii) the intermediary captures all the gains to trade arising from the elimination of delay.*

Nonetheless entry of an intermediary, and therefore the introduction of the possibility of centralized trade, has a significant effect on the distribution of gains to trade to buyers and sellers, shifting the gains toward the type present in the market in smaller measure. We conclude by noting that $V_S^* + V_B^*$ is increasing in the efficiency of the matching process (i.e., is increasing in k), and therefore the gains to trade captured by the intermediary decreases in k.

6 Concluding remarks

The intermediary in the present paper is a monopolist and is "large" in the sense that his choice of bid and ask prices influences the composition of the private trading market. Rubinstein and Wolinsky (1987) consider a model of intermediation with many "small" intermediaries (or middlemen), where the activity of any one intermediary has no influence on market aggregates. They show that the distribution of the gains to trade is biased in favor of buyers when intermediaries take ownership of the good as opposed to when they trade on consignment. It is an open question whether the introduction of small intermediaries makes the market outcome more competitive.

Other authors have studied models of "large" intermediaries that differ from our model in significant respects. In Yavas (1994) and in Gehrig (1993) the search market operates for only one period. Moresi (1990) characterizes the steady state that prevails in the search market for given bid and ask prices, but does not determine the intermediary's profit maximizing bid and ask. In the present paper the search market operates perpetually and the intermediary sets bid and ask prices to maximize his profits.

7 Appendix

Before proving Theorems 2 and 3 it is useful to prove the following lemma.

Lemma 1: *If* $\{R_i, V_i, m_i, \lambda_i, \alpha_i, \rho_i\}_{i=S,B}$ *is a PBAE then (i)* $\lambda_i > 0$ *for each* $i \in \{S, B\}$, *(ii)* $V_S + V_B \geq \frac{k}{2-2\delta+\delta k}$, *and (iii)* $R_i = V_i \frac{1-\delta(1-\rho_i)}{\rho_i}$ *for each* $i \in \{S, B\}$.

Proof. Let $q = \{R_i, V_i, m_i, \lambda_i, \alpha_i, \rho_i\}_{i=S,B}$ be a *PBAE*.

Proof of Part (i). Suppose contrary to the Lemma that $\lambda_i = 0$ for some $i \in \{S, B\}$. If $\lambda_S = \lambda_B = 0$, then ρ_i is not given by (1), contradicting q is a *PBAE*. If $\lambda_i = 0$ and $\lambda_{\neg i} > 0$ we have $\rho_{\neg i} = 0$ by (1) and $\alpha_{\neg i} > 0$ by (2). Since $\lambda_{\neg i} \neq 0$ then

$$V_{\neg i} = \rho_{\neg i} R_{\neg i} + (1 - \rho_{\neg i})\delta V_{\neg i} \geq \alpha_{\neg i} N_{\neg i} + (1 - \alpha_{\neg i})\delta V_{\neg i}, \tag{8}$$

where the inequality follows from (5) and the equality follows from (3). The equality in (8), $\rho_{\neg i} = 0$, and $\delta < 1$, imply that $V_{\neg i} = 0$. The inequality in (8), $V_{\neg i} = 0$, and $\alpha_{\neg i} > 0$ imply that $N_{\neg i} = 0$. By supposition $\lambda_i = 0$. Since $\lambda_i \neq 1$, then (3) and (5) yield

$$V_i = \alpha_i N_i + (1 - \alpha_i)\delta V_i. \tag{9}$$

Case I: Suppose that $1 - \delta V_S - \delta V_B \geq 0$. In this case we have $N_{\neg i} = \frac{1-\delta V_S - \delta V_B}{2} + \delta V_{\neg i}$ by (4). Since $N_{\neg i} = 0$ and $V_{\neg i} = 0$ we have that $1 - \delta V_S - \delta V_B = 0$. Therefore, we also have that $V_i = \frac{1}{\delta}$. Using $V_i = \frac{1}{\delta}$, $1 - \delta V_S - \delta V_B = 0$, and $N_i = \frac{1-\delta V_S - \delta V_B}{2} + \delta V_i$, then (9) yields $\frac{1}{\delta} = 1$, which is a contradiction.

Case II: Suppose that $1 - \delta V_S - \delta V_B < 0$. In this case we have $N_S = N_B = 0$ by (4). Equation (9) then implies that $V_i = 0$. But $V_i = 0$ and $V_{\neg i} = 0$ (from above) contradicts that $1 - \delta V_S - \delta V_B < 0$.

Proof of Part (ii). If $\lambda_S = \lambda_B = 1$, then α_i is not given by (2), contradicting that q is a *PBAE*. The result follows from the Proposition of Sect. 3.

Proof of Part (iii). By Part (i) we have $\lambda_i > 0$ for each $i \in \{S, B\}$. Since $\lambda_i \neq 0$ for each $i \in \{S, B\}$, then (3) and (5) imply that $V_i = \rho_i R_i + (1 - \rho_i)\delta V_i$ for each $i \in \{S, B\}$. □

Proof of Theorem 2. We show that the vector $q^* = \{R_i^*, V_i^*, m_i^*, \lambda_i^*, \alpha_i^*, \rho_i^*\}_{i=S,B}$ given by

$$R_B^* = \tfrac{k}{2-2\delta+\delta k},\ V_B^* = \tfrac{k}{2-2\delta+\delta k},\ m_B^* = \nu,\quad \lambda_B^* = 1,\quad \alpha_B^* = k,\ \rho_B^* = 1,$$

and

$$R_S^* = 0,\quad V_S^* = 0,\quad m_S^* = \nu + \Delta,\ \lambda_S^* = \tfrac{\nu}{\nu+\Delta},\ \alpha_S^* = 0,\ \rho_S^* = 1,$$

is an equilibrium with intermediation. It is easy to verify that q^* is a *PBAE*. We then need to show that for every other *PBAE* $\{R_i, V_i, m_i, \lambda_i, \alpha_i, \rho_i\}_{i=S,B}$ we have:

$$(1 - R_B - R_S)\lambda_i \rho_i m_i \leq (1 - R_B^* - R_S^*)\lambda_i^* \rho_i^* m_i^*.$$

Lemma 1(i) and equation (1) yield $\rho_i > 0$ for each $i \in \{S, B\}$, and therefore $\frac{1-\delta(1-\rho_i)}{\rho_i}$ is well defined. It is easy to see that $\frac{1-\delta(1-\rho_i)}{\rho_i} \geq 1$. Lemma 1$(iii)$, $\frac{1-\delta(1-\rho_i)}{\rho_i} \geq 1$, and $V_i \geq 0$ imply $R_i \geq V_i$ for each $i \in \{S, B\}$. Lemma 1(ii) and $R_i \geq V_i$ for each $i \in \{S, B\}$ imply that $1 - R_B - R_S \leq 1 - \frac{k}{2-2\delta+\delta k}$. The inequality is obtained by noting that $\lambda_i \rho_i m_i \leq \nu$ by (6). □

Proof of Theorem 3. Suppose that $q = \{R_i, V_i, m_i, \lambda_i, \alpha_i, \rho_i\}_{i=S,B}$ is an equilibrium with intermediation. We first show that $\lambda_i \rho_i m_i = \nu$ and $R_B + R_S = \frac{k}{2-2\delta+\delta k}$. By (6) we have that $\lambda_i \rho_i m_i \leq \nu$ and by Part (ii) and (iii) of Lemma 1 we have that $R_B + R_S \geq \frac{k}{2-2\delta+\delta k}$. If either $\lambda_i \rho_i m_i < \nu$ or $R_B + R_S > \frac{k}{2-2\delta+\delta k}$, then

$$(1 - R_B - R_S)\lambda_i \rho_i m_i < \left(1 - \frac{k}{2 - 2\delta + \delta k}\right)\nu,$$

where the right hand side of the inequality is the profit to the intermediary from the PBAE given in Theorem 2. This contradicts that q is an equilibrium with intermediation.

We next show that $R_B + R_S = \frac{k}{2-2\delta+\delta k}$ and Lemma 1 imply that (a) $V_S + V_B = \frac{k}{2-2\delta+\delta k}$, and (b) For each $i \in \{S, B\} : V_i > 0$ implies $\rho_i = 1$. Since $R_B + R_S = \frac{k}{2-2\delta+\delta k}$ and since $R_i \geq V_i$ for each $i \in \{S, B\}$ by Part (iii) of Lemma 1, then (a) holds by Part (ii) of Lemma 1. Suppose contrary to (b) that $V_i > 0$ and $\rho_i < 1$. Then $R_i > V_i$ by Part (iii) of Lemma 1. Since $R_i > V_i$ and $R_{\neg i} \geq V_{\neg i}$, then $R_S + R_B > V_S + V_B = \frac{k}{2-2\delta+\delta k}$ which is a contradiction.

We now show that $\lambda_B = 1$. Since $\lambda_i \rho_i m_i = \nu$ we have by (6) that $\alpha_i(1 - \lambda_i)m_i = 0$. Therefore either $\lambda_i = 1$ or $\alpha_i = 0$. But α_i is zero only if $\lambda_{\neg i} = 1$. Therefore, either $\lambda_i = 1$ or $\lambda_{\neg i} = 1$. Suppose that $\lambda_B < 1$. Then $\lambda_S = 1$. Moreover,

$m_S = m_B + \Delta > \nu$ and $\lambda_S \rho_S m_S = \nu$ imply that $\rho_S < 1$. Observation (a) yields $1 - \delta V_S - \delta V_B > 0$, and therefore by (4) we have that $N_S = \frac{1-\delta V_S - \delta V_B}{2} + \delta V_S > 0$. Since $\lambda_B < 1$, it follows from (2) that $\alpha_S > 0$. By (3), we have

$$V_S \geq \alpha_S N_S + (1 - \alpha_S)\delta V_S .$$

Thus $N_S > 0$ and $\alpha_S > 0$, and therefore $V_S > 0$, which contradicts (b) since $\rho_S < 1$.

We now show that $V_S = 0$ and $V_B = \frac{k}{2-2\delta+\delta k}$. In a *PBAE* either $\lambda_S < 1$ or $\lambda_B < 1$. (If both $\lambda_S = \lambda_B = 1$, then the matching market is inactive and α_i is not given by (2).) Therefore, $\lambda_B = 1$ implies that $\lambda_S < 1$. Moreover, $\lambda_B = 1$ implies by (2) that $\alpha_S = 0$. Since $\lambda_S \neq 1$, then (3) and (5) imply that $V_S = \alpha_S N_S + (1 - \alpha_S)\delta V_S$. As $\alpha_S = 0$ and $\delta < 1$, we have $V_S = 0$. Part (iii) of Lemma 1 implies that $R_S = 0$. From (a) it follows that $V_B = \frac{k}{2-2\delta+\delta k}$. From $V_B > 0$ and (b), we have $\rho_B = 1$. Part (iii) of Lemma 1 and $V_B = \frac{k}{2-2\delta+\delta k}$ then imply that $R_B = \frac{k}{2-2\delta+\delta k}$.

It is only left to be shown that $m_S = \nu + \Delta$, $m_B = \nu$, $\lambda_S \geq \frac{\nu}{\nu+\Delta}$, and $\rho_S = \frac{\nu}{\lambda_S(\nu+\Delta)}$. Since $\rho_B = 1$, by (1) we have $\lambda_S m_S \geq \lambda_B m_B$. Moreover, $\lambda_B \rho_B m_B = \nu$ and $\lambda_B = \rho_B = 1$ imply that $m_B = \nu$, and therefore $m_S = \nu + \Delta$ by (7). Then $\lambda_S m_S \geq \lambda_B m_B$ implies that $\lambda_S \geq \frac{\nu}{\nu+\Delta}$, and $\lambda_S \rho_S m_S = \nu$ implies that $\rho_S = \frac{\nu}{\lambda_S(\nu+\Delta)}$. □

Endnotes

[1] We use results from Model A of Rubinstein (1989), rather than from Rubinstein and Wolinsky (1985) which first reports a non-Walrasian result, since the bargaining game considered there more closely fits the bargaining game in our model of intermediation.

[2] It has been debated whether the market equilibrium found in Rubinstein and Wolinsky (1985) is indeed non-Walrasian. As Gale (1987) writes of their model (p. 26) “Since a positive constant flow of agent enters the market at each date, the set of all agents has infinite measure. The corresponding exchange economy is not well defined.” Interpreting Rubinstein and Wolinsky (1985) as a model of a time-differentiated commodities market as we do has the advantage that Walrasian equilibrium is well-defined. See Wooders (1994) or Sect. 7.5 of Osborne and Rubinstein (1990) for further discussion.

[3] Our definition of Walrasian equilibrium is similar to the one in Schmidt and Aliprantis (1993). Both definitions provide for only a spot market price for each good at each date.

[4] If $\lambda_i < 1$ for each $i \in \{S, B\}$, then (5) and (3) imply that $V_i = \alpha_i N_i + (1 - \alpha_i)\delta V_i$ for each $i \in \{S, B\}$. Moreover, we have that $N_i = \frac{1-\delta V_S - \delta V_B}{2} + \delta V_i$ if $1 - \delta V_S - \delta V_B \geq 0$, and $N_i = 0$ otherwise. This system of equations has a unique solution where $V_i = \frac{\alpha_i}{2-\delta(2-k)}$ for each $i \in \{S, B\}$. This implies $\delta V_S + \delta V_B = \frac{\delta k}{2-\delta(2-k)} < 1$, and therefore each match ends with trade.

[5] Although in Rubinstein (1986) bargaining is strategic, the value equations obtained there are the same as those obtained under Nash bargaining, and thus it is appropriate to use his results here.

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