UNIVERSIDAD CARLOS III DE MADRID working papers

Working Paper 12-26
Economics Series
September 2012

Departamento de Economía
Universidad Carlos III de Madrid
Calle Madrid, 126
28903 Getafe (Spain)
Fax (34) 91 624 98 75

# DYNAMIC MARKETS FOR LEMONS: PERFORMANCE, LIQUIDITY, AND POLICY INTERVENTION*

Diego Moreno[1] and John Wooders[2]

**Abstract**

The inefficiency of competitive markets for lemons raises fundamental questions about market performance and the role of policy intervention. We study the performance of dynamic markets, and show that when the time horizon is finite decentralized markets perform better and high quality is more liquid than centralized ones. When frictions are small, decentralized markets become completely illiquid at all but the first and the last date. When the time horizon is infinite, decentralized markets yield the static competitive surplus, whereas centralized markets have separating equilibria that yield a greater surplus. Subsidizing low quality or taxing high quality tends to increase surplus in both decentralized and centralized markets.

**Keywords:** Decentralized Dynamic Market for Lemons, Adverse Selection, Efficiency, Liquidity, Policy Intervention.

*We gratefully acknowledge financial support from Spanish Ministry of Science and Innovation, grants SEJ2007-67436 and ECO2011-29762. This paper builds on Moreno and Wooders (2001), http://econ.arizona.edu/docs/Working_Papers/Misc%20Years/quality_y2.pdf.

[1]Departamento de Economía, Universidad Carlos III de Madrid, diego.moreno@uc3m.es

[2]Department of Economics, University of Technology Sydney, jwooders@gmail.com.

## Notation

A Market for Lemons

| | |
|---|---|
| $\tau$: | the good's quality, $\tau \in \{H, L\}$. |
| $u^\tau$: | value to buyers of a unit of $\tau$-quality. |
| $c^\tau$ | cost to sellers of $\tau$-quality. |
| $q^\tau$: | fraction of sellers of $\tau$-quality. |
| $t$: | a date at which the market is open, $t \in \{1,\ldots,T\}$. |
| $\delta$: | traders' discount factor. |
| $u(q)$ | $= qu^H + (1-q)u^L$. |
| $\bar{q}$: | $= \dfrac{c^H - u^L}{u^H - u^L}$, i.e., $u(\bar{q}) = c^H$. |
| $\bar{S}$: | $= m^L(u^L - c^L)$. |
| $\hat{q}$: | $= \dfrac{c^H - c^L}{u^H - c^L}$, i.e., $u(\hat{q}) - c^H = (1-\hat{q})(u^L - c^L)$. |
| $\tilde{S}$: | $= \left[m^L + m^H(1-\hat{q})\right](u^L - c^L)$. |

A Decentralized Market for Lemons

| | |
|---|---|
| $r_t^\tau$: | reservation price at date $t$ of sellers of $\tau$-quality. |
| $\lambda_t^\tau$: | probability that a seller of $\tau$-quality who is matched at date $t$ trades. |
| $m_t^\tau$: | stock of $\tau$-quality sellers in the market at date $t$. |
| $q_t^\tau$: | fraction of $\tau$-quality sellers in the market at date $t$. |
| $V_t^\tau$: | expected utility of a seller of $\tau$-quality at date $t$. |
| $V_t^B$: | expected utility of a buyer at date $t$. |
| $S^{DE}$: | surplus in a decentralized market equilibrium – see equation (2). |
| $\rho_t^\tau$: | probability of a price offer of $r_t^\tau$ at date $t$. |

A Dynamic Competitive Market for Lemons

| | |
|---|---|
| $s_t^\tau$: | supply of $\tau$-quality good at date $t$, |
| $u_t$: | expected value to buyers of a unit supplied at date $t$. |
| $d_t$: | demand at date $t$. |
| $S^{CE}$: | surplus in a dynamic competitive equilibrium – see equation (3). |
| $\overline{T}$: | smallest integer $t$ such that $\delta^{t-1}(c^H - c^L) \leq u^L - c^L$. |
| $\widetilde{T}$: | smallest integer $t$ such that $\delta^{t-1}(u^H - c^L) \leq u^L - c^L$. |

# 1 Introduction

Akerlof's finding that the competitive equilibrium of a market for lemons may be inefficient is a cornerstone of the theory of markets with adverse selection. Since adverse selection pervades real good markets (e.g., cars, housing, labor) as well as markets for financial assets (e.g., insurance, stocks), this result has significant welfare implications, and calls for research on fundamental questions that remain open: How do dynamic markets for lemons perform? What determines the liquidity of different qualities of the good? What is the role of frictions in alleviating or aggravating adverse selection? Which market structures perform better, centralized ones, in which trade is multilateral and agents are price-takers, or decentralized ones, in which trade is bilateral and prices are negotiated? Is there a role for government intervention? Our analysis provides answers to these questions.

We consider a simple market in which there is an equal measure of buyers and sellers initially present, and there is no further entry over time. Sellers differ in the quality of the unit of the good they hold, which may be high or low. A seller knows the quality of his good, but quality is unknown to buyers prior to purchase. Buyers are homogeneous and value each quality more highly than sellers. We assume that the expected value to buyers of a random unit is below the cost of a high quality unit, since in this case only low quality units trade in Akerlof's (static) competitive equilibrium, i.e., the lemons problem arises.

We study the performance of decentralized markets for lemons in which trade is bilateral and time consuming, and buyers and sellers bargain over prices. These features are common in markets for real goods and financial assets. We characterize the unique decentralized market equilibrium, and we identify the dynamics of transaction prices, trading patterns, the liquidities of the different qualities, and the market composition (i.e., the fractions of units of the different qualities in the market). Also, we study the asymptotic properties of equilibrium as frictions vanish. Using our characterization of market equilibrium, we identify policy interventions that are welfare improving. Finally, we compare the performance of decentralized and centralized dynamic markets.

In the decentralized market we study, at each date a fraction of the buyers and sellers remaining in the market are randomly paired. In every pair, the buyer makes a take-it-or-leave-it price offer. If the seller accepts, then the agents trade at that price and exit the market. If the seller rejects the offer, then the agents split and both remain in the market at the next date. In this market there are trading "frictions" since meeting a partner is time-consuming and traders discount future gains.

In this market, equilibrium dynamics are non-stationary and involve a delicate balance: At each date, the price offers of the buyers must be optimal given the sellers' reservation prices, the market composition, and the buyers' payoff to remaining in the market. While the market composition is determined by past price offers, the sellers' reservation prices are determined by future price offers. Thus, even if the horizon is finite a market equilibrium cannot be computed recursively.

We begin by studying the equilibria of decentralized markets that open over a finite horizon. Perishable goods such as fresh fruit or event tickets, as well as financial assets such as (put or call) options or thirty-year bonds, are noteworthy examples. We show that equilibrium is unique when frictions are small, and we identify the key features of equilibrium dynamics: at the first date, both a *low* price (accepted only by low quality sellers) and *negligible* prices (rejected by both types of sellers) are offered; at the last date, both a *high* price (accepted by both types of sellers) and a low price are offered; and at all the intervening dates, all three types of prices – high, low and negligible – are offered. Since some offers are rejected, trade involves delay. In contrast to the static competitive equilibrium, some high quality units trade while not all low quality units trade.

Remarkably, the surplus realized in the decentralized market equilibrium exceeds the surplus realized in the static competitive equilibrium: the gain realized from trading high quality units more than offsets the loss resulting from trading low quality units with delay. Moreover, the surplus realized increases as frictions decrease, and thus decentralized markets yield more than the competitive surplus even in the limit as frictions vanish. Surprisingly, in the limit there is trade only at the first and last dates, and the market is completely illiquid at all intervening dates (i.e., buyers offer negligible prices).

A decentralized market that operates over an infinite horizon has multiple equilibria. Our analysis focuses on the equilibrium that is obtained as the limit of the sequence of equilibria of increasingly long finite-horizon markets. In this equilibrium, at the first date buyers make low and negligible price offers, and at every date thereafter buyers make only high and negligible price offers in proportions that do not change over time. We show that all units trade eventually, although the expected delay becomes infinite as frictions vanish. In contrast to prior results in the literature, each trader obtains his static competitive payoff even when frictions are significant. Thus, the cost of delay exactly equals the surplus realized from trading high quality units.

Our characterization of dynamic market equilibrium yields insights into the determinants of market liquidity and the effectiveness of alternative policies designed to increase market efficiency. We take the liquidity of a quality to be the ease with which a unit of that quality is sold, i.e., the probability it trades. In markets that open over a finite horizon, we show that the liquidity of high quality decreases as traders become more patient and, counter-intuitively, as the probability of meeting a partner increases. Indeed, as noted earlier, as frictions vanish trade freezes at all but the first and the last date. In markets that open over an infinite horizon, the liquidity of each quality decreases as traders become more patient, and is unaffected by the probability of meeting a partner.

Policy intervention may alleviate or aggravate the adverse selection problem. A subsidy on buyers of low quality increases the liquidity of high quality units and raises net surplus (i.e., surplus net of the present value cost of the subsidy). As frictions vanish, the subsidy raises net surplus when the horizon is finite, but it has no effect on net surplus (i.e., it amounts to a pure transfer to low quality sellers) when the horizon is infinite.

Not every subsidy is welfare enhancing. In markets that open over a finite horizon, a subsidy on high quality may reduce the net surplus. Moreover, it always does so as frictions vanish. In markets that open over an infinite horizon, a subsidy on high quality is completely wasteful as it reduces the net surplus by the present value cost of the subsidy.

### Dynamic Competitive Equilibrium

In order to evaluate the impact of market micro-structure on performance, we also study dynamic markets in which trade is centralized and prices clear the market at each date. In markets that open over a finite horizon, we show that if traders are patient, then in every dynamic competitive equilibrium all low quality units trade at the first date and no high quality units ever trade. Hence the surplus realized is the same as in the static competitive equilibrium. When the horizon is finite, decentralized markets (which yield more than the static competitive surplus) perform better than centralized markets.

In a centralized market in which traders are sufficiently impatient, or in which the horizon is infinite, there are dynamic competitive equilibria in which all low quality units trade immediately at a low price and all high quality units trade with delay at a high price. These *separating* equilibria, in which different qualities trade at different dates, yield a surplus greater than the static competitive surplus. Consequently, when the horizon is infinite, centralized markets perform better than decentralized markets (which, as noted earlier, yield the static competitive surplus).

We show that, as frictions vanish, the surplus at the most efficient separating equilibrium of a centralized market equals the surplus in the equilibrium of a decentralized market that opens over a finite horizon. Intuitively this result holds since the same incentive constraints operate in both markets. In this separating dynamic competitive equilibrium high quality trades with sufficiently long delay that low quality sellers are willing to trade immediately at a low price rather than waiting to trade at a high price. Likewise, in a decentralized market equilibrium high price offers are made with a sufficiently small probability that low quality sellers are willing to immediately accept a low price, rather than waiting for a high price.

### Related Literature

Our work relates to a strand of literature that examines the mini-micro foundations of competitive equilibrium. This literature has established that decentralized trade of homogeneous goods tends to yield competitive outcomes when trading frictions are small – see, e.g., Gale (1987, 1996) or Binmore and Herrero (1988) when bargaining is under complete information, and Moreno and Wooders (2002) and Serrano (2002) when bargaining is under incomplete information.

More recent work has studied decentralized markets with adverse selection. Moreno and Wooders (2010) studies markets with stationary entry, and finds that payoffs are competitive as frictions vanish. For the one time entry case, on which the present paper focuses, Blouin (2003) studies a market that opens over an infinite horizon in which only one of three exogenously given prices may emerge from bargaining. He finds that equilibrium payoffs are not competitive. In contrast, in markets open over an infinite horizon, we find that payoffs are competitive even when frictions are not negligible. An essential feature of our model, which explains the differing result, is that prices are fully endogenous. Camargo and Lester (2011) study a market where traders have stochastic discount rates that are bounded away from one and vary randomly from one period to the next. (In contrast, in our setting traders have a fixed discount, and we study the market equilibria for discount rates near one, as well as for discount rates approaching one.) They assume that buyers may offer only one of two exogenously given prices, and they find that all units trade in finite time.

Several other recent papers study the role of public information in overcoming the adverse selection problem. Kim (2011) studies a continuous time version of the model of Moreno and Wooders (2010), and shows that if buyers observe the length of time a seller has been in the market, or the number of times a seller has been matched in the past, then market equilibria are more efficient than when this information is absent. Bilancini and Boncinelli (2011) study a market for lemons with finitely many buyers and sellers, and show that if the number of sellers in the market is public information, then in equilibrium all units trade in finite time.

Lauermann and Wolinsky (2011) explore the role of trading rules in a search model with adverse selection, and show that information is aggregated more effectively in auctions than under sequential search by an informed buyer. Morris and Shin (2012) show that when traders coordinate their participation decisions, even a small amount of adverse selection may have a corrosive effect on market confidence and lead to outcomes with bad welfare properties.

Our paper relates to a second strand of literature that studies competitive equilibrium in dynamic markets. Wooders (1998) studies markets with homogeneous goods. Janssen and Roy (2002), for markets with adverse selection and a continuum of qualities, show that dynamic competitive equilibria exist and involve all qualities trading in finite time, with the lowest qualities trading first and the highest qualities trading last. Even though in our setting a dynamic competitive equilibrium as they define it does not exist, the separating dynamic competitive equilibria that we identify has analogous features.

The paper is organized as follows. Section 2 describes our market for lemons, Section 3 studies decentralized markets, and Section 4 studies centralized markets. Proofs are presented in the Appendix.

## 2 A Market for Lemons

Consider a market for an indivisible commodity whose quality can be either high ($H$) or low ($L$). There is a positive measure of buyers and sellers. The measure of sellers with a unit of quality $\tau \in \{H, L\}$ is $m^\tau > 0$. For simplicity, we assume that the measure of buyers ($m^B$) is equal to the measure of sellers, i.e., $m^B = m^H + m^L$. Each buyer wants to purchase a single unit of the good. Each sellers owns a single unit of the good. A seller knows the quality of his good, but quality is unobservable to buyers.

Preferences are characterized by values and costs: the value to a buyer of a unit of high (low) quality is $u^H$ ($u^L$); the cost to a seller of a unit of high (low) quality is $c^H$ ($c^L$). Thus, if a buyer and a seller trade at the price $p$, the buyer obtains a utility of $u - p$ and the seller obtains a utility of $p - c$, where $u = u^H$ and $c = c^H$ if the unit traded is of high quality, and $u = u^L$ and $c = c^L$ if it is of low quality. A buyer or seller who does not trade obtains a utility of zero.

We assume that both buyers and sellers value high quality more than low quality (i.e., $u^H > u^L$ and $c^H > c^L$), and that buyers value each quality more highly than sellers (i.e., $u^H > c^H$ and $u^L > c^L$). Also we restrict attention to markets in which the lemons problem arises; that is, we assume that the fraction of sellers of $\tau$-quality in the market, denoted by

$$q^\tau := \frac{m^\tau}{m^H + m^L},$$

is such that the expected value to a buyer of a randomly selected unit of the good, given by

$$u(q^H) := q^H u^H + q^L u^L,$$

is below the cost of high quality, $c^H$. Equivalently, we may state this assumption as

$$\bar{q} := \frac{c^H - u^L}{u^H - u^L} > q^H.$$

Note that $\bar{q} > q^H$ implies $c^H > u^L$.

Therefore, we assume throughout that $u^H > c^H > u^L > c^L$ and $\bar{q} > q^H$. Under these parameter restrictions only low quality trades in the unique competitive equilibrium, even though there are gains to trade for both qualities – see Figure 1. For future reference, we describe this equilibrium in Remark 1 below.

Figure 1 goes here.

**Remark 1.** *The market has a unique static competitive equilibrium. In equilibrium all low quality units trade at the price $u^L$, and no high quality unit trades. Thus, the surplus,*

$$\bar{S} = m^L(u^L - c^L), \tag{1}$$

*is captured by low quality sellers.*

# 3 A Decentralized Market for Lemons

In this section we study the market described in Section 2 when trade is *decentralized.* We assume that the market is open for $T$ consecutive dates. All traders are present at the market open, and there is no further entry. Traders discount utility at a common rate $\delta \in (0,1)$, i.e., if a unit of quality $\tau$ trades at date $t$ and price $p$, then the buyer obtains a utility of $\delta^{t-1}(u^\tau - p)$ and the seller obtains a utility of $\delta^{t-1}(p - c^\tau)$. At each date every buyer (seller) in the market meets a randomly selected seller (buyer) with probability $\alpha \in (0,1)$. In each pair, the buyer offers a price at which to trade. If the offer is accepted by the seller, then the agents trade and both leave the market. If the offer is rejected by the seller, then the agents split and both remain in the market at the next date. A trader who is unmatched in the current date also remains in the market at the next date. An agent observes only the outcomes of his own matches.

In this market, a *pure strategy for a buyer* is a sequence of price offers $(p_1, \ldots, p_T) \in \mathbb{R}_+^T$. A *pure strategy for a seller* is a sequence of reservation prices $r = (r_1, \ldots, r_T) \in \mathbb{R}_+^T$, where $r_t$ is the smallest price that the seller accepts at date $t \in \{1, \ldots, T\}$.[1]

The strategies of buyers may be described by a sequence $\lambda = (\lambda_1, \ldots, \lambda_T)$, where $\lambda_t$ is a *c.d.f.* with support on $\mathbb{R}_+$ specifying the probability distribution over price offers at date $t \in \{1, \ldots, T\}$. Given $\lambda$, the maximum expected utility of a seller of quality $\tau \in \{H, L\}$ at date $t \leq T$ is defined recursively as

$$V_t^\tau = \max_{x \in \mathbb{R}_+} \left\{ \alpha \int_x^\infty (p_t - c^\tau)\, d\lambda_t(p_t) + \left(1 - \alpha \int_x^\infty d\lambda_t(p_t)\right) \delta V_{t+1}^\tau \right\},$$

where $V_{T+1}^\tau = 0$. In this expression, the payoff to a seller of quality $\tau$ who receives a price offer $p_t$ is $p_t - c^\tau$ if $p_t$ is at least his reservation price $x$, and it is $\delta V_{t+1}^\tau$, his continuation utility, otherwise. Since all sellers of quality $\tau$ have the same maximum expected utility, then their equilibrium reservation prices are identical. Therefore we restrict attention to strategy distributions in which all sellers of quality $\tau \in \{H, L\}$ use the same sequence of reservation prices $r^\tau \in \mathbb{R}_+^T$.

Let $(\lambda, r^H, r^L)$ be a *strategy distribution.* For $t \in \{1, \ldots, T\}$, the probability that a matched seller of quality $\tau \in \{H, L\}$ trades, denoted by $\lambda_t^\tau$, is

$$\lambda_t^\tau = \int_{r_t^\tau}^\infty d\lambda_t.$$

[1] Ignoring, as we do, that a trader may condition his actions on the history of his prior matches is inconsequential – see Osborne and Rubinstein (1990), pages 154-162.

The stock of sellers of quality $\tau$ in the market at date $t+1$, denote by $m_{t+1}^{\tau}$, is

$$m_{t+1}^{\tau} = (1 - \alpha\lambda_t^{\tau})\, m_t^{\tau},$$

where $m_1^{\tau} = m^{\tau}$. The fraction of sellers of quality $\tau$ in the market at date $t$, denoted by $q_t^{\tau}$, is

$$q_t^{\tau} = \frac{m_t^{\tau}}{m_t^H + m_t^L}.$$

The maximum expected utility of a buyer at date $t \leq T$ is defined recursively as

$$V_t^B = \max_{x \in \mathbb{R}_+} \left\{ \alpha \sum_{\tau \in \{H,L\}} q_t^{\tau} I(x, r_t^{\tau})(u^{\tau} - x) + \left(1 - \alpha \sum_{\tau \in \{H,L\}} q_t^{\tau} I(x, r_t^{\tau})\right) \delta V_{t+1}^B \right\},$$

where $V_{T+1}^B = 0$. Here $I(x, y)$ is the indicator function whose value is 1 if $x \geq y$, and 0 otherwise. In this expression, the payoff to a buyer who offers the price $x$ is $u^{\tau} - x$ when matched to a $\tau$-quality seller who accepts the offer (i.e., $I(x, r_t^{\tau}) = 1$), and it is $\delta V_{t+1}^B$, her continuation utility, otherwise.

A strategy distribution $(\lambda, r^H, r^L)$ is a *decentralized market equilibrium (DE)* if for each $t \in \{1, \ldots, T\}$:

$(DE.\tau)$ $r_t^{\tau} - c^{\tau} = \delta V_{t+1}^{\tau}$ for $\tau \in \{H, L\}$, and

$(DE.B)$ for every $p_t$ in the support of $\lambda_t$ we have

$$\alpha \sum_{\tau \in \{H,L\}} q_t^{\tau} I(p_t, r_t^{\tau})(u^{\tau} - p_t) + \left(1 - \alpha \sum_{\tau \in \{H,L\}} q_t^{\tau} I(p_t, r_t^{\tau})\right) \delta V_{t+1}^B = V_t^B.$$

Condition $DE.\tau$ ensures that each type $\tau$ seller is indifferent between accepting or rejecting an offer of his reservation price. Condition $DE.B$ ensures that price offers that are made with positive probability are optimal.

The *surplus* realized in a decentralized market equilibrium can be calculated as

$$S^{DE} = m^B V_1^B + m^H V_1^H + m^L V_1^L. \tag{2}$$

Proposition 1 establishes basic properties of decentralized market equilibria.

**Proposition 1.** *Assume that $T < \infty$, and let $(\lambda, r^H, r^L)$ be a DE. Then for all $t$:*

*(1.1) $r_t^H = c^H > r_t^L$, $V_t^H = 0$, and $q_{t+1}^H \geq q_t^H$.*

*(1.2) Only the high price $p_t = c^H$, or the low price $p_t = r_t^L$, or negligible prices $p_t < r_t^L$ may be offered with positive probability.*

The intuition for these results is straightforward: Since buyers make price offers, they keep sellers at their reservation prices.[2] Since agents who do not trade obtain a zero payoff, then sellers' reservation prices at date $T$ are equal to their costs, i.e., $r_T^\tau = c^\tau$. Thus, buyers never offer a price above $c^H$ at $T$, and therefore the expected utility of high quality sellers at $T$ is zero, i.e., $V_T^H = 0$. Hence $r_{T-1}^H = c^H$. Also, since delay is costly (i.e., $\alpha\delta < 1$), low quality sellers accept price offers below $c^H$, i.e., $r_{T-1}^L < c^H$. A simple induction argument shows that $r_t^H = c^H > r_t^L$ for all $t$. Obviously, any price above $r_t^H$ (accepted by both types of sellers) or in the interval $(r_t^L, r_t^H)$ (accepted only by low quality sellers) is suboptimal and is therefore made with probability zero. Moreover, since $r_t^H > r_t^L$ then the proportion of high quality sellers in the market (weakly) increases over time (i.e., $q_{t+1}^H \geq q_t^H$) as low quality sellers (who accept offers of both $r_t^H$ and $r_t^L$) exit the market at either the same or a faster rate than high quality sellers (who only accept offers of $r_t^H$).

In a decentralized market equilibrium a buyer may offer: (i) a *high price*, $p = r_t^H = c^H$, which is accepted by both types of sellers, thus getting a unit of high quality with probability $q_t^H$ and of low quality with probability $q_t^L = 1 - q_t^H$; or (ii) a *low price* $p = r_t^L$, which is accepted by low quality sellers and rejected by high quality sellers, thus trading only if the seller in the match has a unit of low quality; or (iii) a *negligible price, $p < r_t^L$*, which is rejected by both types of sellers. In order to complete the description of a decentralized market equilibrium we need to determine the probabilities with which each of these three price offers are made.

Let $(\lambda, r^H, r^L)$ be a market equilibrium. Recall that $\lambda_t^\tau$ is the probability that a matched $\tau$-quality seller trades at date $t$ (i.e., gets an offer greater than or equal to $r_t^\tau$). For $\tau \in \{H, L\}$ denote by $\rho_t^\tau$ the probability of a price offer equal to $r_t^\tau$. Since

[2] This is a version of the Diamond Paradox in our context.

prices greater than $c^H$ are offered with probability zero by Proposition 1, then the probability of a high price offer (i.e., an offer of $r_t^H = c^H$) is $\rho_t^H = \lambda_t^H$. And since prices in the interval $(r_t^L, r_t^H)$ are offered with probability zero, then the probability of a low price offer (i.e., an offer of $r_t^L$) is $\rho_t^L = \lambda_t^L - \lambda_t^H$. Hence the probability of a negligible price offer is $1 - (\rho_t^H + \rho_t^L) = 1 - \lambda_t^L$.

Henceforth we ignore the distribution of negligible price offers, which is inconsequential, and describe a DE by a collection $(\rho^H, \rho^L, r^H, r^L)$. Proposition 2 establishes some simple properties of equilibrium price offers.

**Proposition 2.** *Assume that $T < \infty$, and let $(\rho^H, \rho^L, r^H, r^L)$ be a DE. Then:*

*(2.1) At every date $t \in \{1, \ldots, T\}$ either high or low prices are offered with positive probability, i.e., $\rho_t^H + \rho_t^L > 0$.*

*(2.2) At date 1 high prices are offered with probability zero, i.e., $\rho_1^H = 0$.*

*(2.3) At date $T$ negligible prices are offered with probability zero, i.e., $1 - \rho_T^H - \rho_T^L = 0$.*

The intuition for (2.2) is clear: Since the expected utility of a random unit in the market at date 1 is less than $c^H$ by assumption, then high price offers are suboptimal; i.e., $\rho_1^H = 0$. Likewise, the intuition for (2.3) is simple: At date $T$ the sellers' reservation prices are equal to their costs. Hence a buyer obtains a positive payoff by offering the low price. Since a buyer who does not trade obtains zero, then negligible price offers are suboptimal, i.e., $\rho_T^H + \rho_T^L = 1$.

The intuition for (2.1) is a bit more involved: Suppose, for example, that all buyers make negligible offers at date $t$, i.e., $\rho_t^H = \rho_t^L = 0$. Let $t'$ be the first date following $t$ where a buyer makes a non-negligible price offer. Since there is no trade between $t$ and $t'$, then the distribution of qualities is the same at $t$ and $t'$, i.e., $q_t^H = q_{t'}^H$. Thus, an impatient buyer is better off by offering at date $t$ the price she offers at $t'$, which implies that negligible prices are suboptimal at $t$; i.e., $\rho_t^H + \rho_t^L = 1$. Hence $\rho_t^H > 0$ and/or $\rho_t^L > 0$.

In a decentralized market that opens for a single date, i.e., $T = 1$, the sellers' reservation prices are their costs. Thus, (2.2) and (2.3) imply that only low price offers are made (i.e., $\rho_1^H = 0$ and $\rho_1^L = 1$) and only low quality trades. We state this result in Remark 2 below. When $T = 1$, in contrast to the static competitive equilibrium (see Remark 1), in the DE buyers capture the surplus.

**Remark 2.** *If $T = 1$, then the unique DE is $(\rho^H, \rho^L, r^H, r^L) = (0, 1, c^H, c^L)$. Hence all matched low quality sellers trade at the price $c^L$, and none of the high quality sellers trade. Traders' expected utilities are $V_1^B = \alpha(u^L - c^L)$ and $V_1^H = V_1^L = 0$, and the surplus, $S^{DE} = m^L \alpha(u^L - c^L)$, is captured by buyers.*

The intuition for this result is clear: a buyer's payoff is $u(q^H) - c^H < 0$ (because $q^H < \bar{q}$ by assumption) if he offers $c^H$ and is $(1 - q^H)(u^L - c^L) > 0$ if he offers $c^L$. Therefore only a low price offer is optimal.

Write

$$\hat{q} := \frac{c^H - c^L}{u^H - c^L},$$

i.e., in a market that opens for a single date $\hat{q}$ is the fraction of high quality sellers that makes a buyer indifferent between an offer of $c^H$ and an offer of $c^L$. It is easy to see that $\bar{q} < \hat{q}$. Since $q^H < \bar{q}$ by assumption, then $q^H < \hat{q}$.

Proposition 3 below establishes that a market that opens for two or more periods has a decentralized market equilibrium which is identified by some basic properties of price offers. (Precise expressions for the reservation prices and the probabilities of high and low price offers in this equilibrium are provided in the Appendix.) Since $c^H > u^L > c^L$ and $\hat{q} < 1$, the assumptions of Proposition 3 hold when frictions are small, i.e., when $\alpha$ and $\delta$ are close to one.

**Proposition 3.** *If $1 < T < \infty$ and frictions are small (i.e., $\alpha\delta(c^H - u^L) > (1 - \delta)(u^L - c^L)$, $\alpha\delta(c^H - c^L) > u^L - c^L$ and $\alpha(1 - q^H)\hat{q} > \hat{q} - q^H$), then the following properties uniquely determine a DE:*

*(3.1) Low and negligible prices are offered at date 1, i.e., $\rho_1^L > 0$ and $1 - \rho_1^H - \rho_1^L > 0$.*

*(3.2) High, low and negligible prices are offered at every intermediate date, i.e., $\rho_t^H > 0$, $\rho_t^L > 0$, and $1 - \rho_t^H - \rho_t^L > 0$ for $t \in \{2, \ldots, T - 1\}$.*

*(3.3) High and low prices are offered at date $T$, i.e., $\rho_T^H > 0$ and $\rho_T^L > 0$.*

*In this equilibrium the payoff to low quality sellers is*

$$V_1^L = \left(1 - \alpha\delta^{T-1}(1-\hat{q})\right)(u^L - c^L),$$

*the payoff to buyers is*

$$V_1^B = \alpha\delta^{T-1}(1-\hat{q})(u^L - c^L),$$

*and the surplus is*

$$S^{DE} = \left[m^L + m^H\alpha\delta^{T-1}(1-\hat{q})\right](u^L - c^L).$$

*Thus, the payoff to buyers (low quality sellers) is above (below) their competitive payoff, and decreases (increases) with $T$ and increases (decreases) with $\alpha$ and $\delta$. Also, the surplus is above the competitive surplus $\bar{S}$, and decreases with $T$ and increases with $\alpha$ and $\delta$.*

Proposition 4 establishes that under stronger requirements on $\alpha$ and $\delta$, properties (3.1)-(3.3) of Proposition 3 are satisfied in every DE, and therefore the market has a unique DE. These requirements are strengthenings of the conditions in Proposition 3. The requirement $\alpha\delta^{T-1}(c^H - c^L) > u^L - c^L$ holds for $\alpha$ and $\delta$ sufficiently close to one since $c^H > u^L > c^L$. The requirement $\delta\left[\alpha\left(1-q^H\right)\hat{q} - \hat{q} + q^H\right](c^H - c^L) > q^H(1-\hat{q})(u^L - c^L)$ reduces to $\delta(c^H - c^L) > u^L - c^L$ for $\alpha = 1$, which holds for $\delta$ close to one.

**Proposition 4.** *If $T < \infty$ and frictions are sufficiently small (i.e., $\alpha\delta(c^H - u^L) > (1-\delta)(u^L - c^L)$, $\alpha\delta^{T-1}(c^H - c^L) > u^L - c^L$ and $\delta\left[\alpha(1-q^H)\hat{q} - \hat{q} + q^H\right](c^H - c^L) > q^H(1-\hat{q})(u^L - c^L)$), then there is a unique DE.*

We now discuss the properties of the DE. It is easy to describe the trading patterns: at the first date some matched low quality sellers trade, but no high quality sellers trade. At intermediate dates, some matched sellers of both types trade. At the last date all matched low quality sellers and some matched high quality sellers trade. At every date but the first there is trade at more than one price, since both $c^H$ and $r_t^L < c^H$ are offered.

Price dispersion is key feature of equilibrium: Suppose instead that all buyers offer the high price $r_t^H = c^H$ at some date $t$, i.e., $\rho_t^H = 1$. Then for $\alpha$ and $\delta$ near one the reservation price of low quality sellers will be near $c^H$, and hence above $u^L$, prior to $t$. Thus, a low price offer (which if accepted buys a unit of low quality, whose value is only $u^L$) is suboptimal prior to $t$. Hence only high and negligible offers are made prior to $t$, and thus $q_t^H = q^H$; but then a high price offer is suboptimal at $t$ since $q_t^H < \bar{q}$. Hence $\rho_t^H < 1$. Likewise, suppose that all buyers offer the low price $r_t^L$ at some date $t < T$, i.e., $\rho_t^L = 1$. Then all matched low quality sellers trade, and hence $\alpha$ near one implies $q_{t+1}^H > \hat{q}$, and therefore $q_T^H > \hat{q}$. But $q_T^H > \hat{q}$ implies that $r_T^H = c^H$ is the only optimal price offer at date $T$, which contradicts that $\rho_T^H < 1$. Hence $\rho_t^L < 1$.

A more involved argument establishes that all three types of price offers – high, low, and negligible – are made at every date except the first and last (i.e., $\rho_t^H > 0$, $\rho_t^L > 0$, and $1 - \rho_t^H - \rho_t^L > 0$ for $t \in \{2, \ldots, T-1\}$). Identifying the probabilities of the different price offers is delicate: Their past values determine the current market composition, and their future values determine the sellers' reservation prices. In equilibrium, at each intermediate date the market composition and the sellers' reservation prices make buyers indifferent between a high, a low or a negligible price offer. Closed form expressions for these probabilities are derived in the proof of Proposition 3.

The comparative statics of buyer payoffs are intuitive: In equilibrium negligible price offers are optimal at every date except the last. Hence only at the last date does a buyer capture any gains to trade. Consequently, buyer payoffs are increasing in $\alpha$. Also, decreasing $T$ or increasing $\delta$ reduces delay costs and therefore increases buyer payoffs. Low quality sellers capture surplus whenever high price offers are made, i.e., at every date except the first. The probability of a high price offer decreases with both $\alpha$ and $\delta$ (see equation 10 in the proof of Proposition 3), and thus the payoff to low quality sellers also decreases.

In the DE, units of both qualities trade, although with delay. Somewhat surprisingly, the surplus generated in the DE is greater than the (static) competitive equilibrium surplus, $\bar{S}$. Thus the gains realized from trading high quality units more than offsets the loss from trading low quality units with delay.

Counter-intuitively, shortening the horizon over which the market opens is advantageous since the surplus decreases with $T$ (for $T \geq 2$); i.e., it is maximal when $T = 2$. In the proof of Proposition 3, we show that at the last date the fraction of high quality sellers and the probability of a high price are both independent of $T$. Moreover, when frictions are small trade concentrates in the first and last periods. Thus, shortening the horizon decreases delay costs without significantly reducing the measures of high and low quality units that trade.

Proposition 5 below identifies the probabilities of high, low, and negligible price offers as frictions vanish. A remarkable feature of equilibrium is that at every intermediate date all price offers are negligible; that is, all trade concentrates at the first and last date. Thus, the market freezes, i.e., *both* qualities become completely illiquid. And since the market is active for only two dates (the first and the last), not surprisingly the equilibrium is independent of $T$ so long as $1 < T < \infty$.

**Proposition 5.** *If $1 < T < \infty$, then as $\alpha$ and $\delta$ approach one the DE described in Proposition 3 is eventually the unique equilibrium, and the sequences of probabilities of high and low price offers approach $(\tilde{\rho}^H, \tilde{\rho}^L)$ given by*

*(5.1)* $\tilde{\rho}_1^H = 0$ *and* $0 < \tilde{\rho}_1^L = \dfrac{\hat{q} - q^H}{\hat{q}\,(1 - q^H)} < 1.$

*(5.2)* $\tilde{\rho}_t^L = \tilde{\rho}_t^H = 0$ *for* $1 < t < T$.

*(5.3)* $0 < \tilde{\rho}_T^H = \dfrac{u^L - c^L}{u^H - c^L} < 1$ *and* $\tilde{\rho}_T^L = 1 - \tilde{\rho}_T^H$.

*Thus, trade concentrates at the first and the last dates. Moreover, the payoff to buyers remains above their competitive payoff and approaches $\tilde{V}_1^B = (1 - \hat{q})\,(u^L - c^L)$, the payoff to low quality sellers remains below their competitive payoff and approaches $\tilde{V}_1^L = \hat{q}(u^L - c^L)$, and the surplus remains above the competitive surplus and approaches*

$$\tilde{S} = \lim_{(\alpha,\delta)\to(1,1)} S^{DE} = \left[m^L + m^H\,(1 - \hat{q})\right](u^L - c^L),$$

*independently of $T$.*

Decentralized Market Equilibria when $T = \infty$

We now consider decentralized markets that open for infinitely many dates. In these markets, given a strategy distribution $(\lambda, r^H, r^L)$ one calculates the maximum expected utility of each type of trader at each date by solving a dynamic optimization problem. The definition of decentralized market equilibrium, however, remains the same.

Proposition 6 identifies a DE when frictions are small. This equilibrium is the limit, as $T$ approaches infinity, of the equilibrium described in Proposition 3. More precisely, for every integer $T$, denote by $(\rho^H(T), \rho^L(T), r^H(T), r^L(T))$ the DE identified in Proposition 3 for a market that opens for $T$ dates. For each $t$ and $\tau$, let $\hat{\rho}_t^\tau$ and $\hat{r}_t^\tau$ be the limits $\lim_{T\to\infty} \rho_t^\tau(T)$ and $\lim_{T\to\infty} r_t^\tau(T)$. These limits are well defined. Proposition 6 shows that $(\hat{\rho}^H, \hat{\rho}^L, \hat{r}^H, \hat{r}^L)$ is a DE of a market that opens over an infinite horizon. Although there are multiple equilibria when $T = \infty$, this limiting equilibrium is a natural selection since for *every* finite $T$ the DE identified in Proposition 3 is the unique equilibrium for $\alpha$ and $\delta$ close to one. The assumption in Proposition 6 that frictions are small holds for $\alpha$ and $\delta$ close to one since $c^H - c^L > u^L - c^L$ and $\bar{q} < 1$.

**Proposition 6.** *If $T = \infty$ and frictions are small (i.e., $\alpha\delta(c^H - u^L) > (1-\delta)(u^L - c^L)$ and $\alpha(1-q^H)\bar{q} > \bar{q} - q^H$), then $(\hat{\rho}^H, \hat{\rho}^L, \hat{r}^H, \hat{r}^L)$, the limit of the sequence of the DE identified in Proposition 3, given by*

*(6.1)* $\hat{r}_t^H = c^H$, $\hat{r}_t^L = u^L$ *for all* $t$,

*(6.2)* $\hat{\rho}_1^H = 0$, $\hat{\rho}_1^L = \dfrac{\bar{q} - q^H}{\alpha\bar{q}(1-q^H)}$, *and*

*(6.3)* $\hat{\rho}_t^L = 0$, $\hat{\rho}_t^H = \dfrac{1-\delta}{\alpha\delta}\dfrac{u^L - c^L}{c^H - u^L}$ *for* $t > 1$,

*is a DE. In this equilibrium, the traders' payoffs are $V_1^B = V_1^H = 0$ and $V_1^L = u^L - c^L$, and the surplus is the competitive surplus, i.e., $S^{DE} = \bar{S}$, independently of the values of $\alpha$ and $\delta$.*

In the equilibrium of Proposition 6 all units trade eventually. At the first date, some low quality units trade but no high quality units trade. At subsequent dates, units of both qualities trade with the same constant probability. The traders' payoffs are competitive independently of $\alpha$ and $\delta$, and hence so is the surplus, even if frictions are non-negligible (provided they are small). This result is in contrast to most of the literature (e.g., Gale (1986), Moreno and Wooders (2010)), which shows that payoffs are competitive only as frictions vanish.[3]

An Example

Table 1 illustrates our results for a market in which $u^H = 1$, $c^H = .6$, $u^L = .4$, $c^L = .2$, $m^H = .2$, and $m^L = .8$. The left hand side of the table shows the equilibrium values of $\rho_t^H$ and $\rho_t^L$ for a market open for 10 dates. The bottom rows give the traders' payoffs and the surplus. For the values of $\alpha$ and $\delta$ given in the table, the DE is unique.

| | $T = 10$ | | | | $T = \infty$ | | | |
|---|---|---|---|---|---|---|---|---|
| | $\delta = \alpha = .95$ | | $\delta = \alpha = .99$ | | $\delta = \alpha = .95$ | | $\delta = \alpha = .99$ | |
| $t$ | $\rho_t^H$ | $\rho_t^L$ | $\rho_t^H$ | $\rho_t^L$ | $\rho_t^H$ | $\rho_t^L$ | $\rho_t^H$ | $\rho_t^L$ |
| 1 | .0000 | .7155 | .0000 | .7426 | .0000 | .5263 | .0000 | .5051 |
| 2 | .0421 | .0224 | .0070 | .0062 | .0554 | .0000 | .0102 | .0000 |
| 3 | .0416 | .0235 | .0070 | .0062 | .0554 | .0000 | .0102 | .0000 |
| 4 | .0411 | .0247 | .0070 | .0063 | ⋮ | ⋮ | ⋮ | ⋮ |
| 5 | .0405 | .0259 | .0069 | .0063 | ⋮ | ⋮ | ⋮ | ⋮ |
| 6 | .0399 | .0273 | .0069 | .0064 | ⋮ | ⋮ | ⋮ | ⋮ |
| 7 | .0394 | .0286 | .0069 | .0065 | ⋮ | ⋮ | ⋮ | ⋮ |
| 8 | .0388 | .0301 | .0069 | .0065 | ⋮ | ⋮ | ⋮ | ⋮ |
| 9 | .0382 | .0639 | .0068 | .0132 | ⋮ | ⋮ | ⋮ | ⋮ |
| 10 | .3040 | .6960 | .2602 | .7398 | ⋮ | ⋮ | ⋮ | ⋮ |
| $V^L$ | .1401 | | .1096 | | .2000 | | .2000 | |
| $V^B$ | .0599 | | .0904 | | .0000 | | .0000 | |
| $S^{DE}$ | .1720 | | .1781 | | .1600 | | .1600 | |

Table 1: Decentralized Market Equilibria

[3] Kim (2011) also finds that surplus is competitive even if frictions are non-negligible.

Comparing the results for $\alpha = \delta = .95$ to $\alpha = \delta = .99$, we see that the probabilities of high and low price offers decline when frictions are smaller: when $\alpha = a = .99$ more than 98% of all offers are negligible prices. As frictions vanish, at intermediate dates all price offers are negligible, i.e., the market freezes – see Proposition 5.2. The surplus realized exceeds the static competitive surplus (of $m^L(u^L - c^L) = .16$ ), and it does so even in the limit as frictions vanish.

Figure 2a and 2b show the dynamics of the stocks of sellers of high and low quality in the market, $m_t^H$ and $m_t^L$. Figure 2c shows that dynamics of the fraction of high quality sellers in the market, $q_t^H$. These figures illustrate several features of equilibrium as frictions become small: (i) high quality trades more slowly, (ii) low quality trades more quickly at the first date and at the last date, but trades more slowly at intermediate dates, (iii) the fraction of sellers of high quality in the market increases more quickly, but equals .5 at the market close regardless of the level of frictions. Even when frictions are very small (e.g., $\alpha = \delta = .99$) a substantial fraction of high quality sellers does not trade.

Figure 2 goes here.

The right hand side of Table 1 shows the equilibrium values of $\rho_t^H$ and $\rho_t^L$ in the DE described in Proposition 6 for a market that opens over an infinite horizon. In contrast to the example with $T = 10$, the DE yields exactly the competitive surplus even though frictions are non-negligible. After the first date, high and negligible price offers are made with constant probabilities. The probability that a unit of either quality trades is thus positive and constant over time, and hence all units trade eventually. As $\delta$ approaches one, the probability of a high price offer approaches zero, and the market becomes illiquid. The expected delay approaches infinity, although it does so at a speed that holds the surplus fixed at the competitive level.

### Market Liquidity

Assets that are easily bought or sold are known as *liquid* assets. We identify the liquidity of a unit of quality $\tau$ at date $t$ with the equilibrium probability that the unit trades at $t$, denoted by $l_t^{\tau}$. Thus, the liquidity of a high quality unit at date $t$ is $l_t^H = \alpha\rho_t^H$, and the liquidity of a low quality unit is $l_t^L = \alpha(\rho_t^H + \rho_t^L)$.[4] Proposition 7 shows how the liquidity of each quality changes in response to changes in the value of low quality units, the discount factor, and the matching probability. Its proof relies on the expressions for $\rho_t^H$ and $\rho_t^L$ given in the proofs of propositions 3 and 6.

**Proposition 7.** *In a decentralized market:*

*(7.1) If $1 < T < \infty$ and the assumptions of Proposition 3 hold, then for $t \in \{2, \ldots, T\}$:*

*(i) The liquidity of high quality decreases as the traders become more patient, i.e., $\partial l_1^H/\partial\delta = 0$ and $\partial l_t^H/\partial\delta < 0$.*

*(ii) The liquidity of high quality decreases as the matching probability increases, i.e., $\partial l_1^H/\partial\alpha = 0$ and $\partial l_t^H/\partial\alpha < 0$.*

*(iii) Increasing the value of low quality increases the liquidity of high quality at all dates but the first, i.e., $\partial l_1^H/\partial u^L = 0$ and $\partial l_t^H/\partial u^L > 0$.*

*(7.2) If $T = \infty$ and the assumptions of Proposition 6 hold, then for $t \in \{2, \ldots, T\}$:*

*(i) The liquidity of both qualities decreases as traders become more patient, i.e., $\partial l_1^H/\partial\delta = \partial l_1^L/\partial\delta = 0$ and $\partial l_t^H/\partial\delta = \partial l_t^L/\partial\delta < 0$.*

*(ii) The liquidity of both qualities is unchanged as the matching probability increases, i.e., $\partial l_1^H/\partial\alpha = \partial l_1^L/\partial\alpha = \partial l_t^H/\partial\alpha = \partial l_t^L/\partial\alpha = 0$.*

*(iii) Increasing the value of low quality decreases the liquidity of low quality in the first date, but increases the liquidity of both qualities at every subsequent date, i.e., $\partial l_1^H/\partial u^L = 0$ and $\partial l_1^L/\partial u^L < 0$, and $\partial l_t^H/\partial u^L = \partial l_t^L/\partial u^L > 0$.*

When the horizon is finite, high quality is less liquid as traders become more patient and, perhaps counter-intuitively, as the probability of meeting a partner increases. Indeed, both qualities become completely illiquid at intermediate dates as

[4] The difference between the bid and the ask price is a common measure of liquidity in centralized markets. In our model trade is decentralized and takes place at different prices. Therefore we focus on the ease with which an asset is sold.

both $\alpha$ and $\delta$ approach 1 – see Proposition 5. An increase in the value of low quality increases the liquidity of high quality, but it decreases the liquidity of low quality at date 1.[5]

For the market in Example 1, Table 2 illustrates the effect of an increase of the value of $u^L$ from .4 to .45. The liquidity of low quality decreases at the first date, but increases thereafter. The liquidity of high quality increases at all dates $t > 1$.

| | $T = 10$ | | | | $T = \infty$ | | | |
|---|---|---|---|---|---|---|---|---|
| | $u^L = .4$ | | $u^L = .45$ | | $u^L = .4$ | | $u^L = .45$ | |
| $t$ | $l_t^H$ | $l_t^L$ | $l_t^H$ | $l_t^L$ | $l_t^H$ | $l_t^L$ | $l_t^H$ | $l_t^L$ |
| 1 | .0000 | .6797 | .0000 | 0.64899 | .0000 | .5000 | .0000 | .3333 |
| 2 | .0400 | .0613 | .0575 | 0.08624 | .0526 | .0526 | .0877 | .0877 |
| 3 | .0395 | .0619 | .0565 | 0.08659 | .0526 | .0526 | .0877 | .0877 |
| 4 | .0390 | .0625 | .0555 | 0.08700 | ⋮ | ⋮ | ⋮ | ⋮ |
| 5 | .0385 | .0631 | .0544 | 0.08748 | ⋮ | ⋮ | ⋮ | ⋮ |
| 6 | .0379 | .0638 | .0533 | 0.08804 | ⋮ | ⋮ | ⋮ | ⋮ |
| 7 | .0374 | .0646 | .0523 | 0.08868 | ⋮ | ⋮ | ⋮ | ⋮ |
| 8 | .0368 | .0654 | .0512 | 0.08941 | ⋮ | ⋮ | ⋮ | ⋮ |
| 9 | .0363 | .0969 | .0501 | 0.13069 | ⋮ | ⋮ | ⋮ | ⋮ |
| 10 | .2888 | .9500 | .3610 | 0.95000 | ⋮ | ⋮ | ⋮ | ⋮ |

Table 2: The Effect on Liquidity of an Increase in $u^L$ ($\alpha = \delta = .95$)

When the horizon is infinite, the liquidity of both qualities decreases with $\delta$. Liquidity is independent of $\alpha$: an increase in $\alpha$ is exactly offset by a decrease in the probability of a high price offer (recall that only high and negligible price offers are made with positive probability – see Proposition 6). An increase in $u^L$ reduces the liquidity of low quality units at the first date but increases the liquidity of both qualities at every subsequent date.

[5] One can show that $\partial l_1^L / \partial u^L < 0$, but the sign of $\partial l_t^L / \partial u^L$ at intermediate dates is unclear.

An overall effect of a parameter change is its impact on the expected date at which a unit trades. For the example in Table 2, an increase in $u^L$ increases the probability that a unit of either quality trades at dates $t \geq 2$ from .0526 to .0877. The expected date at which a high quality unit trades decreases from $1 + 1/.0526 = 20.01$ to $1 + 1/.0877 = 12.40$. While the increase in $u^L$ nearly doubles the speed at which high quality trades, it has only a small effect on low quality since the probability that low quality trades at date 1 falls: the expected date at which low quality trades is reduced from 10.51 to 8.60.

### Policy Intervention

Our results characterizing the equilibrium of a decentralized market allow an assessment of the impact of programs aimed at improving market efficiency.[6] We study the impact of taxes and subsidies. Suppose, for example, that the government provides a per unit subsidy of $\mu > 0$ to buyers of low quality. (A tax is a negative subsidy.) Such a subsidy is feasible provided that quality is verifiable following purchase. Hence the instantaneous payoff to a buyer who purchases a unit of quality $\tau$ at price $p$ is $u^\tau + \mu - p$ rather than $u^\tau - p$. The impact of a subsidy may therefore be evaluated as an increase of the value of $u^L$. In this section we focus on the impact of subsidies on surplus, but their impact on liquidity can be assessed using our results in the prior section.

For a market that opens over a finite horizon, Proposition 3 provides analytic expressions for the traders' payoffs, as well as the surplus. Using these expressions we can evaluate the impact of subsidies and taxes.

A subsidy $\mu$ on low quality has no impact on the payoff of high quality sellers, increases the payoff of low quality sellers by $[1 - \alpha\delta^{T-1}(1 - \hat{q})]\mu$, increases the payoff of buyers by $\alpha\delta^{T-1}(1 - \hat{q})\mu$, and increases the surplus by $[m^L + m^H\alpha\delta^{T-1}(1 - \hat{q})]\mu$. The present value cost of the subsidy is bounded above by $\mu m^L$ since at most $m^L$ units receive the subsidy. Thus, the increase in surplus is larger than the cost of the subsidy. Hence the *net surplus*, i.e., the surplus minus the present value cost of the

[6] An example of such a program is the Public-Private Investment Program for Legacy Assets, by which the U.S. government provided financial assistance to investors who purchased legacy assets.

subsidy, increases. We summarize these results in Remark 3.

**Remark 3.** *In a decentralized market open over a finite horizon, a subsidy on low quality increases the payoff of low quality sellers and buyers, as well as the net surplus. A tax has the opposite effects.*

A subsidy $\mu^H$ on high quality affects payoffs and surplus through its impact on $\hat{q}$. It is easy to see that such a subsidy decreases the payoff of low quality sellers, increases the payoff of buyers, but its impact on the surplus is ambiguous. Nevertheless, the subsidy unambiguously lowers the net surplus as $\alpha$ and $\delta$ approach one: The (gross) surplus with the subsidy approaches $\tilde{S} = \left[m^L + m^H (1 - \hat{q})\right] (u^L - c^L)$, where $\hat{q} = (c^H - c^L)/(u^H + \mu^H - c^L)$. Since high quality trades only at the last date, the cost of the subsidy approaches $m^H \mu^H \tilde{\rho}_T^H$, where $\tilde{\rho}_T^H = (u^L - c^L)/(u^H + \mu^H - c^L)$. Thus the net surplus approaches

$$\left[m^L + m^H \left(1 - \frac{c^H - c^L}{u^H + \mu^H - c^L}\right)\right] (u^L - c^L) - m^H \mu^H \frac{u^L - c^L}{u^H + \mu^H - c^L},$$

which reduces to

$$m^L(u^L - c^L) + m^H \left(u^H - c^H\right) \frac{u^L - c^L}{u^H + \mu^H - c^L},$$

which is decreasing in $\mu^H$. Remark 4 summarizes the impact of a subsidy on high quality.

**Remark 4.** *In a decentralized market that opens over a finite horizon, a subsidy on high quality decreases the payoff of low quality sellers and increases the payoff of buyers, and may decrease the net surplus. Moreover, as $\alpha$ and $\delta$ approach one a subsidy on high quality unambiguously decreases the net surplus. A tax has the opposite effects.*

Table 3 illustrates the effect of several subsidies and taxes for the market of Example 1 when $\alpha = \delta = .95$ and $T = 10$.

| Subsidy/Tax | | Measures Trading | | Surplus | | | | PV Cost |
|---|---|---|---|---|---|---|---|---|
| $\mu^L$ | $\mu^H$ | High | Low | Buyers | Low | Total | Net | Subsidy/Tax |
| 0 | 0 | .0958 | .7927 | .0599 | .1121 | .1720 | .1720 | 0 |
| .05 | 0 | .1179 | .7936 | .0748 | .1401 | .2150 | .1790 | .0360 |
| .00 | .05 | .0932 | .7917 | .0634 | .1093 | .1727 | .1693 | .0034 |
| 0 | $-.05$ | .0988 | .7936 | .0559 | .1153 | .1712 | .1748 | $-.0036$ |
| .05 | 05 | .1148 | .7927 | .0792 | .1366 | .2159 | .1761 | .0398 |

Table 3: Effects of Subsidies and Taxes

The second row of Table 3 describes the effect of a subsidy $\mu^L = .05$ on low quality. Relative to the equilibrium without any subsidy or tax (described in the first row), the measure of high quality sellers that trades increases from .0958 to .1179 (i.e., from 47.92% to 58.96% of all high quality sellers). The measure of low quality sellers that trades increases modestly from .7927 to .7936. The surplus increases to .2150. The present value cost of the subsidy is .0360. Thus, the net surplus increases to $.2150 - .0360 = .1790$.

A subsidy $\mu^H = .05$ on high quality has a negative effect on the net surplus: The subsidy increases surplus to .1727. The present value cost of the subsidy is .0034, and therefore the net surplus *decreases* to .1693. This decrease is the result of a decrease in the measures of trade of both qualities. Naturally, a tax has the opposite effect, as shown the fourth row of Table 3.

If quality is not verifiable after purchase, a subsidy conditional on quality is not feasible. As shown in the last row of Table 3, an unconditional subsidy may increase surplus. to .2159. The subsidy $\mu^H = \mu^L = .05$ is more costly and has a smaller positive effect on the net surplus than a subsidy on low quality alone.

Proposition 6 provides analytic expressions for the traders' payoffs and the surplus in a market that opens over an infinite horizon. A subsidy of $\mu^L > 0$ on low quality increases the payoff of low quality sellers by $\mu^L$, and increases the surplus by $\mu^L m^L$. Since not all low quality trades at date 1, the present value cost of the subsidy is less than $\mu^L m^L$, and therefore the net surplus increases. We show that as $\delta$ approaches one, the present value cost of the subsidy approaches $\mu^L m^L$ (see the appendix), and thus the subsidy amounts to a transfer to low quality sellers. We summarize these results in Remark 5.

**Remark 5.** *In a decentralized market that opens over an infinite horizon, a subsidy on low quality increases the payoff of low quality sellers as well as the net surplus. As $\delta$ approaches one the subsidy amounts to a transfer to low quality sellers.*

A subsidy on high quality has no effect on either the traders' payoff or the surplus; its only effect is to decrease the probability of a low price offer at date 1, and thereby to increase the delay with which low quality trades. Interestingly, a tax on high quality raises revenue without affecting either the traders' payoffs or the surplus: A tax on high quality raises surplus by increasing the volume of trade of low quality units at date 1, while having no effect on trade in subsequent periods. (Specifically, the tax reduces $\bar{q}$ and increases $\hat{\rho}_1^L$, while leaving $\hat{\rho}_t^L$ and $\hat{\rho}_t^H$ unchanged – see Proposition 6.) While the traders' payoffs and the surplus remains unchanged, the net surplus (which includes tax revenue) increases. We summarize these results in Remark 6.

**Remark 6.** *In a decentralized market that opens over an infinite horizon, a subsidy on high quality has no effect on either the traders' payoffs or the surplus, and is therefore purely wasteful. Strikingly, a tax on high quality raises revenue without affecting payoffs or surplus.*

# 4 A Dynamic Competitive Market for Lemons

In this section we study the market described in Section 2 when trade is *centralized*. The market opens for $T$ consecutive dates, and traders' discount rate is $\delta \in (0, 1)$.

The supply and demand schedules are defined as follows. Let $p = (p_1, \ldots, p_T) \in \mathbb{R}_+^T$ be a sequence of prices. The utility to a seller of quality $\tau \in \{H, L\}$ who supplies at date $t$ is $\delta^{t-1}(p_t - c^\tau)$. Hence the maximum utility that a $\tau$-quality seller may attain is

$$v^\tau(p) = \max_{t \in \{1,\ldots,T\}} \{0, \delta^{t-1}(p_t - c^\tau)\}.$$

The *supply of* $\tau$*-quality good*, denoted $S^{\tau}(p)$, is the set of sequences $s^{\tau} = (s_1^{\tau}, \ldots, s_T^{\tau}) \in \mathbb{R}_+^T$ satisfying:

$(S.1)$ $\sum_{t=1}^{T} s_t^{\tau} \leq m^{\tau}$,

$(S.2)$ $s_t^{\tau} > 0$ implies $\delta^{t-1}(p_t - c^{\tau}) = v^{\tau}(p)$, and

$(S.3)$ $\left(\sum_{t=1}^{T} s_t^{\tau} - m^{\tau}\right) v^{\tau}(p) = 0.$

Condition $S.1$ requires that no more of good $\tau$ than is available, $m^{\tau}$, be supplied. Condition $S.2$ requires that supply be positive only at dates where it is optimal to supply. Condition $S.3$ requires that the total amount of good $\tau$ available be supplied when $\tau$-quality sellers may attain a positive utility (i.e., when $v^{\tau}(p) > 0$).

Denote by $u_t \in [u^L, u^H]$ the expected value to buyers of a unit supplied at date $t$. Then the utility to a buyer who demands a unit of the good at date $t$ is $\delta^{t-1}(u_t - p_t)$. If the sequence of buyers' expected values is $u = (u_1, \ldots, u_T)$, then the maximum utility a buyer may attain is

$$v^B(p, u) = \max_{t \in \{1,\ldots,T\}} \{0, \delta^{t-1}(u_t - p_t)\}.$$

The *market demand*, $D(p, u)$, is the set of sequences $d = (d_1, \ldots, d_T) \in \mathbb{R}_+^T$ satisfying:

$(D.1)$ $\sum_{t=1}^{T} d_t \leq m^B$,

$(D.2)$ $d_t > 0$ implies $\delta^{t-1}(u_t - p_t) = v^B(p, u)$, and

$(D.3)$ $\left(\sum_{t=1}^{T} d_t - m^B\right) v^B(p, u) = 0.$

Condition $D.1$ requires that the total demand not exceed the measure of buyers. Condition $D.2$ requires that the demand be positive only at dates where buying is optimal. Condition $D.3$ requires that demand be equal to the measure of buyers when buyers may attain a positive utility (i.e., when $v^B(p, u) > 0$).

We define dynamic competitive equilibrium along the lines in the literature – see e.g., Wooders (1998), and Janssen and Roy (2002).

A *dynamic competitive equilibrium (CE)* is a profile $(p, u, s^H, s^L, d)$ such that $s^H \in S^H(p)$, $s^L \in S^L(p)$, $d \in D(p, u)$, and for each $t$:

$(CE.1)$ $s_t^H + s_t^L = d_t$, and

$(CE.2)$ $s_t^H + s_t^L = d_t > 0$ implies $u_t = \dfrac{u^H s_t^H + u^L s_t^L}{s_t^H + s_t^L}$.

Condition $CE.1$ requires that the market clear at each date, and condition $CE.2$ requires that the expectations described by the vector $u$ be correct whenever there is trade.[7] For a market that opens for a single date (i.e., if $T = 1$), our definition reduces to Akerlof's. The surplus generated in a CE may be calculated as

$$S^{CE} = \sum_{\tau \in \{H,L\}} \sum_{t=1}^{T} s_t^\tau \delta^{t-1}(u^\tau - c^\tau). \tag{3}$$

As our next proposition shows, there is a CE where all low quality units trade at date 1 at the price $u^L$, and none of the high quality units ever trade. Moreover, if the market is open over a sufficiently short horizon, then every CE has these properties. Specifically, the horizon $T$ must be less than $\overline{T}$, which is defined by the inequality

$$\delta^{\overline{T}-2}(c^H - c^L) > u^L - c^L \geq \delta^{\overline{T}-1}(c^H - c^L).$$

Since $\overline{T}$ approaches infinity as $\delta$ approaches one, for a given $T$ the condition $T < \overline{T}$ holds when $\delta$ is near one, i.e., when traders are sufficiently patient.

**Proposition 8.** *There are CE in which all low quality units trade immediately at the price $u^L$ and none of the high quality units trade, e.g., $(p, u, s^H, s^L, d)$ given by $p_t = u_t = u^L$ for all $t$, $s_1^L = d_1 = m^L$, and $s_1^H = s_t^H = s_t^L = d_t = 0$ for $t > 1$ is a CE. In these CE the payoff to low quality sellers is $u^L - c^L$, the payoff to high quality sellers and buyers is zero, and the surplus is $\bar{S}$. Moreover, if $T < \overline{T}$, then every CE has these properties.*

The intuition for why high quality does not trade when $T < \overline{T}$ is clear: If high quality were to trade at $t \leq T$, then $p_t$ must be at least $c^H$. Hence the utility to low quality sellers is at least $\delta^{t-1}(c^H - c^L)$. Since

$$\delta^{t-1}(c^H - c^L) \geq \delta^{T-1}(c^H - c^L) \geq \delta^{\overline{T}-2}(c^H - c^L) > u^L - c^L > 0,$$

[7] Janssen and Roy (2002) focus on the CE on which the expected value to buyers of a random unit at dates when there is no trade is at least the value of the lowest quality for which there is a positive measure of unsold units. In our setting no CE with this property exists.

then all low quality sellers trade at prices greater than $u^L$. But at a price $p \in (u^L, c^H)$ only low quality sellers supply, and therefore the demand is zero. Hence all trade is at prices of at least $c^H$. Since $u(q^H) < c^H$ by assumption, and since in equilibrium all low quality is supplied, there must be a date at which there is trade and the expected value of a random unit supplied is below $c^H$. This contradicts that there is demand at such a date. Thus, high quality is not supplied in a CE. Consequently, low quality sellers capture the entire surplus, i.e., the price is $u^L$, as low quality sellers are the short side of the market.

Recall from propositions 3 and 4 that when frictions are small the surplus realized in a decentralized market is greater than the static competitive surplus, i.e., $S^{DE} > \bar{S}$. By Proposition 8, a dynamic competitive market that opens over a finite horizon generates the static competitive surplus, i.e., $S^{CE} = \bar{S}$. Thus, decentralized markets perform better than centralized markets when the horizon is short or, equivalently, the traders are sufficiently patient.

Proposition 9 below establishes that in a centralized market that opens over a sufficiently long horizon there are dynamic competitive *separating* equilibria in which all low quality units trade immediately and all high quality units trade with delay. Specifically, the horizon $T$ must be at least $\widetilde{T}$, which is defined by the inequality

$$\delta^{\widetilde{T}-2}(u^H - c^L) > u^L - c^L \geq \delta^{\widetilde{T}-1}(u^H - c^L).$$

Since $u^H > c^H$, then $\widetilde{T} \geq \overline{T}$.

**Proposition 9.** *If $T \geq \widetilde{T}$, then there are CE in which all low quality units trade at date 1 and all high quality units trade at date $\widetilde{T}$. Such CE yield a surplus of*

$$S^{CE} = m^L(u^L - c^L) + m^H \delta^{\widetilde{T}-1}(u^H - c^H) > \bar{S}.$$

By Proposition 9, a centralized market that opens over a sufficiently long horizon eventually *recovers* from adverse selection. Such markets have equilibria in which high quality trades that yield more surplus than the static competitive surplus. Consequently, when the horizon is infinite, centralized markets outperform decentralized markets (which by Proposition 6 yield the static competitive surplus).[8]

In the separating CE of Proposition 9, high quality trades with increasingly long delay as $\delta$ approaches one, i.e., $\widetilde{T}$ increases with $\delta$. We show (see the proof of Proposition 10) that

$$\lim_{\delta \to 1} \delta^{\widetilde{T}(\delta)-1} = \frac{u^L - c^L}{u^H - c^L},$$

and therefore that the surplus realized from trading high quality approaches

$$m^H \frac{u^L - c^L}{u^H - c^L}(u^H - c^H) = m^H (1 - \hat{q}) (u^L - c^L).$$

Thus, as $\delta$ approaches one the surplus in these separating CE approaches $\tilde{S}$, the surplus realized in a decentralized market open over a finite horizon as $\delta$ approaches one. This result is established in Proposition 10.

**Proposition 10.** *If $T = \infty$, then the surplus realized in a CE in which all low quality units trade at date 1 and all high quality units trade at $\widetilde{T}$ approaches $\tilde{S}$ as $\delta$ approaches one, i.e.,*

$$\lim_{\delta \to 1} S^{CE} = \tilde{S}.$$

Proposition 10 reveals that the same incentive constraints are at play in both centralized and decentralized markets: In a separating CE high quality trades with a sufficiently long delay that low quality sellers prefer trading immediately at a low price to waiting and trading at a high price. Likewise, in a decentralized market equilibrium high price offers are made with sufficiently low probability that low quality sellers accept a low price offer.

In summary, a centralized market that opens over a finite horizon does not perform well when $\delta$ is close to one: in every CE only low quality trades and the surplus equals the static competitive surplus. Likewise, decentralized markets that open over an infinite horizon do not perform well since they generate only the static competitive surplus.

[8] When $\overline{T} \leq T < \widetilde{T}$ there are no separating CE, but there are *partially pooling* CE in which high quality trades. The most efficient of these CE, in which some low quality trades at date 1 while the remaining low quality and all the high quality trade at date $T$, yields a surplus greater than $\bar{S}$.

Market Liquidity

We discuss the liquidity of high quality in centralized markets for lemons, and illustrate our conclusions for the market in which $u^H = 1$, $c^H = .6$, $u^L = .4$, $c^L = .2$, $m^H = .2$, and $m^L = .8$, whose decentralized market equilibrium is described in Table 1. We assume that $\delta = .95$ and, when the market is decentralized, that $\alpha = .95$. In this market $\overline{T} = 15$ and $\widetilde{T} = 29$.

Assume that $T$ is finite. Then for $\delta$ sufficiently large, we have $T < \overline{T}$, and by Proposition 8 in a centralized market high quality units do not trade. In contrast, in a decentralized market high quality units trade. Thus, high quality is more liquid when the market is decentralized. Consider the market for lemons described above, and assume that $T = 10$. When the market is centralized, since $T < \overline{T}$, then no high quality units trade. In contrast, using the results in Table 1, one can show that a measure .09583 of high quality units trade (48% of the total measure of high quality units in the market) when the market is decentralized.

When $T$ is infinite, in a centralized market there are CE in which all high quality units trade (by Proposition 9), although the *earliest* date at which high quality trades in any separating CE is $\widetilde{T}$. In a decentralized market all units eventually trade. The expected date at which a high quality unit trades is

$$\sum_{t=1}^{\infty} t\alpha\hat{\rho}_t^H \prod_{k=1}^{t-1} \left(1 - \alpha\hat{\rho}_k^H\right) = 1 + \left(\frac{1-\delta}{\delta}\frac{u^L - c^L}{c^H - u^L}\right)^{-1},$$

where $\hat{\rho}^H$ is given in Proposition 6. In our example, the expected date at which high quality trades is 20, while it trades at date $\widetilde{T} = 29$ in the separating CE. In this sense high quality is more liquid when the market is decentralized than when it is centralized.

In contrast, whether the market opens over a finite or an infinite horizon, in a separating CE low quality trades immediately, but trades with delay in a DE. In our example, a measure .79268 of low quality sellers trade (99% of a total measure of .8) in the DE. The expected date at which a low quality unit trades is

$$\sum_{t=1}^{\infty} t\alpha\hat{\rho}_t^L \prod_{k=1}^{t-1} \left(1 - \alpha\hat{\rho}_k^L\right) = 1 + \left(1 - \frac{\bar{q} - q^H}{\bar{q}\,(1 - q^H)}\right)\left(\frac{1-\delta}{\delta}\frac{u^L - c^L}{c^H - u^L}\right)^{-1},$$

which is 10.5 in our example. However, the entire measure .8 of low quality sellers trade immediately in the separating CE. Hence low quality is less liquid in a decentralized market.

Policy Intervention

As noted earlier, a subsidy or tax on low (high) quality changes the value of $u^L$ ($u^H$). In a centralized market marginal changes in the parameter values do not affect the value of $\overline{T}$ or $\widetilde{T}$ generically, and therefore have no effect on the net surplus. Moreover, any subsidy that changes the value of $\overline{T}$ but leaves it above $T$ has no effect on the net surplus since by Proposition 8 only low quality trades. In particular, a subsidy on high quality increases the value of $\overline{T}$, and therefore has not effect when $T < \overline{T}$.

When $T \geq \widetilde{T}$, a subsidy on low quality or a tax on high quality that reduces the value of $\widetilde{T}$ increases the net surplus realized in the separating CE of Proposition 9 as high quality trades sooner. Thus, subsidies or taxes (large enough to affect $\widetilde{T}$) tend to have analogous effects in centralized markets as in decentralized ones – recall that a subsidy on low quality or a tax on high quality raises net surplus in a DE (see remarks 3-6).

As $\delta$ approaches one, then $\overline{T}$ approaches infinity, and therefore in a centralized market open over a finite horizon $T$ only low quality trades (because $\overline{T}$ eventually exceeds $T$), and hence a subsidy or a tax has no effect. In a decentralized market, however, a subsidy on low quality or a tax on high quality raises the net surplus as $\delta$ and $\alpha$ approach one (see remarks 3 and 4).

In a centralized market open over an infinite horizon, however, the effect on the net surplus realized in the separating CE of Proposition 9 of a subsidy on the $\tau$ quality good is given by the derivative of $\tilde{S}$ with respect to $u^\tau$. Thus, a subsidy on low quality or a tax on high quality increases the net surplus. (Recall that in a decentralized market open over an infinite horizon a subsidy to low quality or a tax on high quality also increase the net surplus – see remarks 5 and 6.)

# 5 Appendix: Proofs

We begin by establishing a number of lemmas.

**Lemma 1.** *Assume that $1 < T < \infty$, and let $(\lambda, r^H, r^L)$ be a DE. Then for each $t \in \{1,\ldots,T\}$:*

(L1.1) $\lambda_t(\max\{r_t^H, r_t^L\}) = 1$.

(L1.2) $m_t^\tau > 0$ *and* $q_t^\tau > 0$ *for* $\tau \in \{H, L\}$.

(L1.3) $r_t^H = c^H > r_t^L$, $V_t^H = 0 < V_t^B$, *and* $V_t^L \leq c^H - c^L$.

(L1.4) $q_{t+1}^H \geq q_t^H$.

(L1.5) $\lambda_t(c^H) = 1$.

(L1.6) $\lambda_t(p) = \lambda_t(r_t^L)$ *for all* $p \in [r_t^L, c^H)$.

(L1.7) $\lambda_T^L = 1$.

(L1.8) *If* $\lambda_t^L = \lambda_t^H$, *then* $q_{t+1}^\tau = q_{t+1}^\tau$ for $\tau \in \{H, L\}$.

**Proof:** Let $t \in \{1,\ldots,T\}$. We prove $L1.1$. Write $\bar{p} = \max\{r_t^H, r_t^L\}$, and suppose that $\lambda_t(\bar{p}) < 1$. Then there is $\hat{p} > \bar{p}$ in the support of $\lambda_t$. Since $I(\bar{p}, r_t^\tau) = I(\hat{p}, r_t^\tau) = 1$ for $\tau \in \{H, L\}$, we have

$$
\begin{aligned}
V_t^B &\geq \alpha \sum_{\tau \in \{H,L\}} q_t^\tau I(\bar{p}, r_t^\tau)(u^\tau - \bar{p}) + \left[1 - \alpha \sum_{\tau \in \{H,L\}} q_t^\tau I(\bar{p}, r_t^\tau)\right] \delta V_{t+1}^B \\
&= \alpha \sum_{\tau \in \{H,L\}} q_t^\tau \left(u^\tau - \bar{p}\right) + (1-\alpha)\, \delta V_{t+1}^B \\
&> \alpha \sum_{\tau \in \{H,L\}} q_t^\tau \left(u^\tau - \hat{p}\right) + (1-\alpha)\, \delta V_{t+1}^B \\
&= \alpha \sum_{\tau \in \{H,L\}} q_t^\tau I(\hat{p}, r_t^\tau)(u^\tau - \hat{p}) + \left[1 - \alpha \sum_{\tau \in \{H,L\}} q_t^\tau I(\hat{p}, r_t^\tau)\right] \delta V_{t+1}^B,
\end{aligned}
$$

which contradicts $DE.B$.

Clearly $m_t^\tau > 0$ implies $q_t^\tau > 0$. We prove by induction that $m_t^\tau > 0$ for all $t$ and $\tau$. Let $\tau \in \{H, L\}$; we have $m_1^\tau = m^\tau > 0$. Assume that $m_k^\tau > 0$ for some $k \geq 1$; then

$\alpha \in (0,1)$ and $\lambda_k^\tau \in [0,1]$ imply $m_{k+1}^\tau = (1-\alpha\lambda_k^\tau)m_k^\tau > 0$. Hence $m_t^\tau > 0$ for all $t$ and $\tau$.

We prove $L1.3$ by induction. Because $V_{T+1}^\tau = 0$ for $\tau \in \{B,H,L\}$, then $DE.H$ and $DE.L$ imply

$$r_T^H = c^H + \delta V_{T+1}^H = c^H > c^L = r_T^L = c^L + \delta V_{T+1}^L.$$

Hence $\lambda_T(c^H) = 1$ by $L1.1$, and therefore $V_T^H = 0$ and $V_T^L \le c^H - c^L$. Moreover, $0 < q_T^L\left(u^L - c^L\right) \le V_T^B$. Let $k \le T$, and assume that $L1.3$ holds for $t \in \{k,...,T\}$; we show that it holds for $k-1$. Since $V_k^H = 0$, $DE.H$ implies $r_{k-1}^H = c^H + \delta V_k^H = c^H$. Since $V_k^L \le c^H - c^L$ and $\delta < 1$, then $DE.L$ implies $r_{k-1}^L = c^L + \delta V_k^L \le (1-\delta)c^L + \delta c^H < c^H$. Hence $\lambda_k(c^H) = 1$ by $L1.1$, and therefore $V_{k-1}^H = 0$. Also since $\lambda_t(c^H) = 1$ for $t \ge k-1$, then $V_{k-1}^L \le c^H - c^L$. Also $V_{k-1}^B \ge \delta V_k^B > 0$.

In order to prove $L1.4$, note that $L1.3$ implies $\lambda_t^H \le \lambda_t^L$. Hence

$$q_{t+1}^H = \frac{m_{t+1}^H}{m_{t+1}^H + m_{t+1}^L} = \frac{(1-\alpha\lambda_t^H)m_t^H}{(1-\alpha\lambda_t^H)m_t^H + (1-\alpha\lambda_t^L)m_t^L} \ge \frac{m_t^H}{m_t^H + m_t^L} = q_t^H.$$

As for $L1.5$, it is a direct implication of $L1.1$ and $L1.3$.

We prove $L1.6$. Suppose that $\lambda_t(p) > \lambda_t(r_t^L)$ for some $p \in (r_t^L, r_t^H)$. Then there is $\hat{p}$ in the support of $\lambda_t$ such that $r_t^L < \hat{p} < r_t^H$. Since $I(\hat{p}, r_t^L) = 1$ and $I(\hat{p}, r_t^H) = 0$, we have

$$\begin{aligned}
V_t^B &\ge \alpha \sum_{\tau\in\{H,L\}} q_t^\tau I(r_t^L, r_t^\tau)(u^\tau - r_t^L) + \left[1 - \alpha \sum_{\tau\in\{H,L\}} q_t^\tau I(r_t^L, r_t^\tau)\right] \delta V_{t+1}^B \\
&= \alpha q_t^L (u^L - r_t^L) + (1-\alpha q_t^L)\delta V_{t+1}^B \\
&> \alpha q_t^L (u^L - \hat{p}) + (1-\alpha q_t^L)\delta V_{t+1}^B \\
&= \alpha \sum_{\tau\in\{H,L\}} q_t^\tau I(\hat{p}, r_t^\tau)(u^\tau - \hat{p}) + \left[1 - \alpha \sum_{\tau\in\{H,L\}} q_t^\tau I(\hat{p}, r_t^\tau)\right] \delta V_{t+1}^B,
\end{aligned}$$

which contradicts $DE.B$.

We prove $\lambda_T^L = 1$. Suppose by way of contradiction that $\lambda_T^L < 1$. Then there is a $\hat{p} < c^L$ in the support of $\lambda_T$. Since $I(\hat{p}, r_t^H) = 0$ and $V_{T+1}^B = 0$, then the payoff to a buyer offering $\hat{p}$ is zero. Since $\hat{p}$ is the support of $\lambda_T$, then $DE.B$ implies $V_T^B = 0$, which contradicts $L1.3$.

In order to prove $L1.8$, simply note that $\lambda_t^L = \lambda_t^H$ implies

$$q_{t+1}^{\tau} = \frac{m_{t+1}^H}{m_{t+1}^H + m_{t+1}^L} = \frac{(1-\alpha\lambda_t^{\tau})\, m_t^{\tau}}{\left(1-\alpha\lambda_t^H\right) q_t^H + \left(1-\alpha\lambda_t^L\right) q_t^L} = \frac{m_t^{\tau}}{m_t^H + m_t^L} = q_t^{\tau}. \ \square$$

**Proof of Proposition 1.** Proposition (1.1) follows from lemmas $L1.3$ and $L1.4$, and Proposition (1.2) follows from $L1.5$ and $L1.6$. $\square$

**Proof of Proposition 2.** Since $\rho_T^H + \rho_T^L = \lambda_T^L$, then (2.3) follows from $L1.7$.

We prove (2.1). Suppose contrary to (2.1) that there is $k$ such that $\rho_k^H + \rho_k^L = 0$. By (2.3), $k < T$. Let $k$ be the largest such date. Then $\rho_{k+1}^H + \rho_{k+1}^L > 0$ and $q_{k+1}^{\tau} = q_k^{\tau}$ for $\tau \in \{H, L\}$. If $\rho_{k+1}^H > 0$, i.e., offering $r_{k+1}^H$ is optimal, then

$$V_{k+1}^B = \alpha(q_{k+1}^H u^H + q_{k+1}^L u^L - c^H) + (1-\alpha)\, \delta V_{k+2}^B.$$

Since $V_{k+1}^B \geq \delta V_{k+2}^B$ (because the payoff to offering a negligible price is $\delta V_{k+2}^B$), then

$$q_{k+1}^H u^H + q_{k+1}^L u^L - c^H \geq V_{k+1}^B.$$

And since $q_{k+1}^{\tau} = q_k^{\tau}$ for $\tau \in \{H, L\}$, $V_{k+1}^B > 0$ (by $L1.3$) and $\delta < 1$, then

$$q_k^H u^H + q_k^L u^L - c^H = q_{k+1}^H u^H + q_{k+1}^L u^L - c^H \geq V_{k+1}^B > \delta V_{k+1}^B.$$

Therefore a negligible price offer at $k$ is not optimal, which contradicts that $\rho_k^H + \rho_k^L = 0$. Hence $\rho_{k+1}^H = 0$ and thus $\rho_{k+1}^L > 0$.

Since $\rho_k^H = 0$, then

$$V_{k+1}^L = \alpha\rho_{k+1}^L \left(r_{k+1}^L - c^L\right) + \left(1-\alpha\rho_{k+1}^L\right) \delta V_{k+2}^L = \delta V_{k+2}^L.$$

Then

$$r_k^L = c^L + \delta V_{k+1}^L \leq c^L + V_{k+1}^L = c^L + \delta V_{k+2}^L = r_{k+1}^L.$$

Since $\rho_{k+1}^L > 0$, i.e., price offers of $r_{k+1}^L$ are optimal at date $k+1$, we have

$$q_{k+1}^L(u^L - r_{k+1}^L) + \left(1 - q_{k+1}^L\right) \delta V_{k+2}^B \geq \delta V_{k+2}^B.$$

Hence

$$\delta V_{k+2}^B \leq u^L - r_{k+1}^L$$

and

$$V^B_{k+1} = \alpha q^L_{k+1}(u^L - r^L_{k+1}) + \left(1 - \alpha q^L_{k+1}\right) \delta V^B_{k+2} \leq u^L - r^L_{k+1}.$$

Since $\rho^H_k + \rho^L_k = 0$, then the payoff to a negligible offer at date $k$ is greater or equal to the payoff to a low price offer at date $k$, i.e.,

$$\delta V^B_{k+1} \geq \alpha q^L_k \left(u^L - r^L_k\right) + \left(1 - \alpha q^L_k\right) \delta V^B_{k+1}.$$

Thus $u^L - r^L_k \leq \delta V^B_{k+1}$. Since $V^B_{k+1} > 0$ (by $L1.3$) and $\delta < 1$, then

$$u^L - r^L_k \leq \delta V^B_{k+1} < V^B_{k+1} \leq u^L - r^L_{k+1},$$

i.e., $r^L_{k+1} < r^L_k$, which is a contradiction. Hence $\rho^H_k + \rho^L_k > 0$ for all $k$, which establishes (2.1).

We prove (2.2). Since $q^H_1 = q^H < \bar{q}$ by assumption and $V^B_2 > 0$ by $L1.3$, then

$$q^H_1 u^H + q^L_1 u^L - c^H < 0 < \delta V^B_2.$$

Hence offering $c^H$ at date 1 is not optimal; i.e., $\rho^H_1 = 0$. Therefore $\rho^L_1 > 0$ by (2.1).□

**Proof of Proposition 3.** We show that properties (3.1), (3.2) and (3.3) together with the equilibrium conditions provide a system of equations that uniquely determine a DE; i.e., we show that this system has a unique solution, which we calculate. This solution provides the strategy distribution, $(\rho^H, \rho^L, r^H, r^L)$, as well as the sequences of traders' expected utilities, and the sequences of stocks and fractions of sellers of each type. We then calculate payoff and the surplus.

Let $(\rho^H, \rho^L, r^H, r^L)$ be a DE satisfying properties (3.1), (3.2) and (3.3). We show that such a DE exists and is unique. Since $\rho^H_T > 0$ and $\rho^L_T > 0$ by (3.3), and since $r^H_T = c^H$ and $r^L_T = c^L$ by $DE.H$ and $DE.L$, then

$$q^H_T u^H + q^L_T u^L - c^H = q^L_T(u^L - c^L),$$

by $DE.B$, i.e.,

$$q^H_T u^H + (1 - q^H_T) u^L - c^H = (1 - q^H_T)(u^L - c^L).$$

Hence

$$q^H_T = \frac{c^H - c^L}{u^H - c^L} = \hat{q}.$$

Thus, a buyer's expected utility at $T$ is

$$V_T^B = \alpha(1-\hat{q})(u^L - c^L).$$

Since $1-\rho_t^H-\rho_t^L>0$ for all $t<T$ by (3.1) and (3.2), then $V_t^B=\delta V_{t+1}^B$ for $t<T$ by $DE.B$, and therefore for all $t$ we have

$$V_t^B = \alpha\delta^{T-t}\left(1-\hat{q}\right)\left(u^L - c^L\right). \tag{4}$$

Moreover, since $\rho_t^H>0$ and $1-\rho_t^H-\rho_t^L>0$ for $1<t<T$ by (3.2), then

$$q_t^H(u^H-c^H)+(1-q_t^H)(u^L-c^H)=\delta V_{t+1}^B$$

by $DE.B$. Hence for $1<t<T$ we have

$$q_t^H = \frac{c^H-u^L+\alpha\delta^{T-t}(1-\hat{q})(u^L-c^L)}{u^H-u^L}. \tag{5}$$

Since $\rho_t^L>0$ and $1-\rho_t^H-\rho_t^L>0$ for $t<T$ by (3.1) and (3.2), then

$$\alpha q_t^L\left(u^L-r_t^L\right)+(1-\alpha q_t^L)\delta V_{t+1}^B=\delta V_{t+1}^B,$$

by $DE.B$, i.e.,

$$\delta V_{t+1}^B = u^L - r_t^L. \tag{6}$$

Hence for $t<T$ we have

$$r_t^L = u^L - \alpha\delta^{T-t}(1-\hat{q})(u^L-c^L). \tag{7}$$

Since $r_t^L-c^L=\delta V_{t+1}^L$ for all $t$ by $DE.L$, then

$$u^L-c^L-\alpha\delta^{T-t}(1-\hat{q})(u^L-c^L)=\delta V_{t+1}^L.$$

Reindexing we get

$$V_t^L = \frac{u^L-c^L}{\delta}-\alpha\delta^{T-t}(1-\hat{q})(u^L-c^L), \tag{8}$$

for $t>1$. And since $\rho_1^H=0$ by Proposition (2.2), then

$$V_1^L=\delta V_2^L=\left(1-\alpha\delta^{T-1}\left(1-\hat{q}\right)\right)\left(u^L-c^L\right). \tag{9}$$

Since $r_t^L - c^L = \delta V_{t+1}^L$ for all $t$ by $DE.L$, we can write the expected utility of a low quality seller as

$$V_t^L = \alpha\rho_t^H(c^H - c^L) + (1 - \alpha\rho_t^H)\delta V_{t+1}^L,$$

i.e.,

$$V_t^L - \delta V_{t+1}^L = \alpha\rho_t^H(c^H - c^L - \delta V_{t+1}^L).$$

Using equation (8), then for $1 < t < T$ we have

$$V_t^L - \delta V_{t+1}^L = \left(\frac{1-\delta}{\delta}\right)\left(u^L - c^L\right).$$

Hence

$$\left(\frac{1-\delta}{\delta}\right)\left(u^L - c^L\right) = \alpha\rho_t^H(c^H - c^L - \delta V_{t+1}^L).$$

Using again equation (8) and solving for $\rho_t^H$ yields

$$\rho_t^H = \frac{1-\delta}{\alpha\delta}\frac{u^L - c^L}{c^H - u^L + \alpha\delta^{T-t}(1-\hat{q})(u^L - c^L)} \tag{10}$$

for $1 < t < T$. Clearly $\rho_t^H > 0$. Further, since $\alpha\delta\left(c^H - u^L\right) > (1-\delta)\left(u^L - c^L\right)$ by assumption, then

$$\rho_t^H < \frac{1-\delta}{\alpha\delta}\frac{u^L - c^L}{c^H - u^L} < 1.$$

Since $r_T = c^L$ by $DE.L$, then

$$V_T^L = \alpha\rho_T^H(c^H - c^L).$$

Hence using (8) for $t = T$ we have

$$\frac{u^L - c^L}{\delta} - \alpha(1-\hat{q})(u^L - c^L) = \alpha\rho_T^H(c^H - c^L).$$

Solving for $\rho_T^H$ yields

$$\rho_T^H = (1 - \alpha\delta(1-\hat{q}))\frac{u^L - c^L}{\alpha\delta\left(c^H - c^L\right)}. \tag{11}$$

Since $0 < 1 - \alpha\delta\left(1 - \hat{q}\right) < 1$ and $\alpha\delta\left(c^H - c^L\right) > u^L - c^L$, then $0 < \rho_T^H < 1$.

Finally, we compute $\rho^L$. For each $t$ we have

$$q_{t+1}^H = \frac{(1-\alpha\rho_t^H)q_t^H}{(1-\alpha\rho_t^H)q_t^H + (1-\alpha(\rho_t^L+\rho_t^H))q_t^L}.$$

Solving for $\rho_t^L$ we obtain

$$\rho_t^L = (1-\alpha\rho_t^H)\frac{q_{t+1}^H - q_t^H}{\alpha q_{t+1}^H\left(1-q_t^H\right)} \tag{12}$$

for $t < T$. Since $1 \geq q_{t+1}^H \geq q_t^H > 0$ and $\rho_t^H < 1$, then $\rho_t^L > 0$. Since $c^H - u^L > (1-\delta)(u^L - c^L)$ by assumption, using (5) we have

$$\begin{aligned}
\frac{q_{t+1}^H - q_t^H}{\alpha q_{t+1}^H\left(1-q_t^H\right)} &= \frac{(1-\delta)\delta^{T-t-1}(1-\hat{q})(u^L-c^L)}{c^H-u^L+\alpha\delta^{T-t-1}(1-\hat{q})(u^L-c^L)}\left(\frac{u^H-u^L}{u^H-c^H-\alpha\delta^{T-t}(1-\hat{q})(u^L-c^L)}\right)\\
&< (1-\delta)\frac{(1-\hat{q})(u^L-c^L)}{c^H-u^L}\left(\frac{u^H-u^L}{u^H-c^H-(1-\hat{q})(u^L-c^L)}\right)\\
&= (1-\delta)\frac{u^L-c^L}{c^H-u^L}\\
&< 1
\end{aligned}$$

for $t > 1$. Hence $\rho_t^L < 1$. And since $\alpha\left(1-q^H\right)\hat{q} > 1-q^H$ by assumption, using (12) and noticing that $\rho_1^H = 0$ by Proposition (2.2), and $q_2^H \leq \hat{q}$ as shown above we have

$$\rho_1^L = \frac{q_2^H - q^H}{q_2^H\alpha\left(1-q^H\right)} < \frac{q_2^H - q^H}{q_2^H\left(1-\frac{q^H}{\hat{q}}\right)} = \frac{q_2^H - q^H}{q_2^H - q^H\frac{q_2^H}{\hat{q}}} < 1.$$

Finally, $\rho_T^L + \rho_T^H = 1$ implies

$$\rho_T^L = 1 - \rho_T^H = 1 - (1-\alpha\delta(1-\hat{q}))\frac{u^L-c^L}{\alpha\delta\left(c^H-u^L\right)}. \tag{13}$$

Since $0 < \rho_T^H < 1$ as shown above, we have $0 < \rho_T^L < 1$.

Using equations (4) and (9), and noticing that $q^H + q^L = 1$, we can calculate the surplus as

$$S^{DE} = \left[m^L + \delta^{T-1}\alpha m^H\left(1-\hat{q}\right)\right]\left(u^L-c^L\right). \square \tag{14}$$

Lemmas 2 and 3 establish properties of DE when frictions are small. These results are used in the proof of Proposition 4. Recall that our assumptions imply that $q^H < \bar{q} < \hat{q} < 1$.

**Lemma 2.** *Assume* $\alpha\delta^{T-1}(c^H - c^L) > u^L - c^L$ *and* $\delta\left[\alpha\left(1-q^H\right)\hat{q} - \hat{q} + q^H\right](c^H - c^L) > q^H(1-\hat{q})(u^L - c^L)$*, and let* $(\rho^H, \rho^L, r^H, r^L)$ *be a DE. Then for all* $t \in \{1,\ldots,T\}$*:*

($L2.1$) $\rho_t^H < 1$.

($L2.2$) $\rho_t^L < 1$.

($L2.3$) $\rho_T^H > 0$, $\rho_T^L > 0$, *and* $q_T^H = \hat{q}$.

($L2.4$) $V_t^L > 0$.

($L2.5$) $\rho_t^L > 0$.

($L2.6$) $\rho_t^H < \bar{\rho}^H := \dfrac{u^L - c^L}{\alpha\delta\left(c^H - c^L\right)}$.

**Proof:** We prove $L2.1$.

Since $\rho_1^H = 0$ by Proposition 2.2, then assume by way of contradiction that $\rho_t^H = 1$ for some $t > 1$. Since $V_{t+1}^L \geq 0$, then

$$V_t^L = \alpha(c^H - c^L) + (1-\alpha)\,\delta V_{t+1}^L \geq \alpha(c^H - c^L).$$

Since $\delta^{t-1} \geq \delta^{T-1}$ and since $\alpha\delta^{T-1}\left(c^H - c^L\right) > u^L - c^L$ by assumption, and $V_t^L \geq \delta V_{t+1}^L$ for all $t$ (because a low quality seller obtains a payoff of $\delta V_{t+1}^L$ by choosing a reservation price $r_t^L > c^H$), the inequality above implies

$$r_1^L = c^L + \delta V_2^L \geq c^L + \delta^{t-1} V_t^L \geq c^L + \delta^{T-1}\alpha\left(c^H - c^L\right) > c^L + u^L - c^L = u^L.$$

Therefore offering $r_1^L$ at date 1 is suboptimal, i.e., $\rho_1^L = 0$. Since $\rho_1^H = 0$, then $\rho_1^H + \rho_1^L = 0$, which contradicts Proposition 2.1. Hence $\rho_t^H < 1$.

We prove $L2.2$. We first show that $\rho_t^L < 1$ for $t < T$. Assume that $\rho_t^L = 1$ for some $t < T$. By assumption

$$\delta\left[\alpha\left(1-q^H\right)\hat{q} - \hat{q} + q^H\right](c^H - c^L) > q^H(1-\hat{q})(u^L - c^L);$$

which we may write as

$$1 - \alpha\left(1-q^H\right) < \left(1 - (1-\hat{q})\frac{(u^L - c^L)}{\delta(c^H - c^L)}\right)\frac{q^H}{\hat{q}}.$$

Since $\delta\left(c^H - c^L\right) > \alpha\delta\left(c^H - c^L\right) > u^L - c^L$, then $\hat{q} < 1$ implies

$$0 < 1 - (1-\hat{q})\,\frac{u^L - c^L}{\delta\left(c^H - u^L\right)} < 1.$$

Hence

$$1 - \alpha(1-q^H) < \frac{q^H}{\hat{q}},$$

and therefore

$$q^H + (1-\alpha)(1-q^H) = 1 - \alpha(1-q^H) < \frac{q^H}{\hat{q}}.$$

Since $\rho_t^L = 1$

$$q_{t+1}^H = \frac{m_t^H}{m_t^H + (1-\alpha)\, m_t^L} = \frac{q_t^H}{q_t^H + (1-\alpha)(1-q_t^H)}.$$

Since $x/[x + (1-\alpha)(1-x)]$ is increasing in $x$ and $q_T^H \geq q_{t+1}^H \geq q_1^H = q^H$ by $L1.4$, then the above inequality implies

$$q_T^H \geq q_{t+1}^H \geq \frac{q^H}{q^H + (1-\alpha)(1-q^H)} > \hat{q}.$$

Hence

$$\begin{aligned} q_T^H u^H + q_T^L u^L - c^H &> \hat{q}u^H + (1-\hat{q})\,u^L - c^H \\ &= (1-\hat{q})\,(u^L - c^L) \\ &> q_T^L(u^L - c^L), \end{aligned}$$

i.e., offering $r_T^L = c^L$ at date $T$ is not optimal. Hence $\rho_T^L = 0$, and therefore $\rho_T^H = 1$ by Proposition 2.3, which contradicts $L2.1$. Hence $\rho_t^L < 1$ for all $t < T$.

We show that $\rho_T^L < 1$. Assume that $\rho_T^L = 1$. Then $q_T^H \leq \hat{q}$ (since otherwise an offer of $r_T^L$ is suboptimal), $V_T^L = 0$ and $V_T^B = \alpha q_T^L\left(u^L - c^L\right)$. Hence $r_{T-1}^L = c^L$ by $DE.L$, and

$$\begin{aligned} q_{T-1}^L(u^L - r_{T-1}^L) + q_{T-1}^H \delta V_T^B &= q_{T-1}^L(u^L - c^L) + (1 - q_{T-1}^L)\delta V_T^B \\ &> q_{T-1}^L \delta V_T^B + (1 - q_{T-1}^L)\delta V_T^B \\ &= \delta V_T^B, \end{aligned}$$

i.e., the payoff to offering $r^L_{T-1}$ at date $T-1$ is greater than that of offering a negligible price. Therefore $\rho^L_{T-1} + \rho^H_{T-1} = 1$. Since $q^H_{T-1} \leq q^H_T$ by $L1.4$ and $q^H_T \leq \hat{q}$, then the payoff to offering $r^H_{T-1} = c^H$ at $T-1$ is

$$\begin{aligned} q^H_{T-1}u^H + q^L_{T-1}u^L - c^H &\leq q^H_T u^H + q^L_T u^L - c^H \\ &\leq q^L_T(u^L - c^L) \\ &\leq q^L_{T-1}(u^L - c^L) \\ &< q^L_{T-1}(u^L - c^L) + q^H_{T-1}\delta V^B_T, \end{aligned}$$

where the last term is the payoff to offering $r^L_{T-1} = c^L$ at $T-1$. Hence $\rho^H_{T-1} = 0$, and therefore $\rho^L_{T-1} = 1$, which contradicts that $\rho^L_t < 1$ for all $t < T$ as shown above. Hence $\rho^L_T < 1$.

We prove $L2.3$. Proposition 2.3, $L2.1$ and $L2.2$ imply that $\rho^H_T > 0$ and $\rho^L_T > 0$. Since both high price offers and low price offers are optimal at date $T$, and reservation prices are $r^H_T = c^H$ and $r^L_T = c^L$, we have

$$q^H_T u^H + q^L_T u^L - c^H = q^L_T(u^L - c^L).$$

Thus, using $q^L_T = 1 - q^H_T$ and solving for $q^H_t$ yields

$$q^H_T = \frac{c^H - c^L}{u^H - c^L} = \hat{q}.$$

We prove $L2.4$ by induction. By $L2.3$, $V^L_T = \alpha\rho^H_T\left(c^H - c^L\right) > 0$. Since $V^L_t \geq \delta V^L_{t+1}$ for all $t \leq T$, then $V^L_t \geq \delta^{T-t} V^L_T > 0$.

We prove $L2.5$. Suppose by way of contradiction that $\rho^L_t = 0$ for some $t$. Since $\rho^L_T > 0$ by $L2.3$, then $t < T$. Also $\rho^L_t = 0$ implies $\rho^H_t > 0$ by Proposition 2.1. Since $\rho^H_t < 1$ by $L2.1$, then buyers are indifferent at date $t$ between offering $c^H$ or a negligible price, i.e.,

$$q^H_t u^H + q^L_t u^L - c^H = \delta V^B_{t+1}.$$

We show that $\rho^H_{t+1} = 0$. Suppose that $\rho^H_{t+1} > 0$; then

$$V^B_{t+1} = \alpha\left(q^H_{t+1}u^H + q^L_{t+1}u^L - c^H\right) + (1-\alpha)\delta V^B_{t+2}.$$

Hence $\delta < 1$ and $V^B_{t+1} > 0$ by $L1.3$ imply

$$q^H_t u^H + q^L_t u^L - c^H = \delta V^B_{t+1} < V^B_{t+1} = \alpha \left( q^H_{t+1} u^H + q^L_{t+1} u^L - c^H \right) + (1-\alpha)\delta V^B_{t+2},$$

But $\rho^L_t = 0$ implies that $q^H_{t+1} = q^H_t$, and therefore

$$q^H_{t+1} u^H + q^L_{t+1} u^L - c^H < \delta V^B_{t+2},$$

i.e., offering $c^H$ at date $t+1$ yields a payoff smaller than offering a negligible price, which contradicts that $\rho^H_{t+1} > 0$.

Since $\rho^H_{t+1} = 0$, then $DE.L$ implies

$$V^L_{t+1} = \alpha \rho^L_{t+1} \left( r^L_{t+1} - c^L \right) + \left( 1 - \alpha \rho^L_{t+1} \right) \delta V^L_{t+2} = \delta V^L_{t+2}.$$

Since $V^L_{t+1} > 0$ by $L2.4$, then $V^L_{t+2} > 0$, and therefore $DE.L$ and $\delta < 1$ imply

$$r^L_t = c^L + \delta V^L_{t+1} = c^L + \delta^2 V^L_{t+2} < c^L + \delta V^L_{t+2} = r^L_{t+1}.$$

i.e., $r^L_t < r^L_{t+1}$. We show that this inequality cannot hold, which leads to a contradiction.

Since $\rho^H_t < 1$ by $L2.1$, then $\rho^L_t = 0$ implies $1 - \rho^H_t - \rho^L_t > 0$; i.e., negligible price offers are optimal at date $t$. Hence at date $t$ the payoff to offering $r^L_t$ must be less than or equal to the payoff to offering a negligible price, i.e.,

$$q^H_t \delta V^B_{t+1} + q^L_t (u^L - r^L_t) \leq \delta V^B_{t+1}.$$

Using $q^H_t = 1 - q^L_t$ we may write this inequality as

$$u^L - r^L_t \leq \delta V^B_{t+1}.$$

Likewise, $\rho^H_{t+1} = 0$ implies $0 < \rho^L_{t+1} < 1$ by Proposition 2.1 and $L2.2$, and therefore $1 - \rho^H_{t+1} - \rho^L_{t+1} > 0$. Hence low and negligible price offers are both optimal at date $t+1$, and therefore

$$V^B_{t+1} = \alpha q^L_{t+1} (u^L - r^L_{t+1}) + \left( 1 - \alpha q^L_{t+1} \right) \delta V^B_{t+2} = \delta V^B_{t+2}.$$

Hence

$$V^B_{t+1} = u^L - r^L_{t+1}.$$

Thus, $\delta < 1$ and $V_{t+1}^{B} > 0$ by $L1.3$ imply

$$u^L - r_t^L \leq \delta V_{t+1}^B < V_{t+1}^B = u^L - r_{t+1}^L.$$

Therefore $r_t^L > r_{t+1}^L$, which contradicts $r_t^L < r_{t+1}^L$.

We prove $L2.6$. For $t \in \{1, ..., T\}$, since $V_t^L \geq 0$, and $r_t^L - c^L = \delta V_{t+1}^L$ by $DE.L$, we have

$$\begin{aligned} V_t^L &= \alpha \left(\rho_t^H \left(c^H - c^L\right) + \rho_t^L \left(r_t^L - c^L\right)\right) + \left(1 - \alpha \left(\rho_t^H + \rho_t^L\right)\right) \delta V_{t+1}^L \\ &\geq \alpha \rho_t^H \left(c^H - c^L\right). \end{aligned}$$

By Proposition 2.2, we have $\rho_1^H = 0 < \bar{\rho}^H$. For $1 < t \leq T$, since $\rho_{t-1}^L > 0$ by $L2.5$ (i.e., low price offers are optimal at date $t-1$) and $V_{t-1}^B > 0$ by $L1.3$, then $u^L > r_{t-1}^L$. Hence

$$u^L - c^L > r_{t-1}^L - c^L = \delta V_t^L \geq \alpha \delta \rho_t^H \left(c^H - c^L\right),$$

and therefore

$$\rho_t^H < \frac{u^L - c^L}{\alpha \delta \left(c^H - c^L\right)} = \bar{\rho}^H. \ \square$$

**Lemma 3.** *Assume* $\alpha \delta^{T-1}(c^H - c^L) > u^L - c^L$ *and* $\delta \left[\alpha \left(1 - q^H\right) \hat{q} - \hat{q} + q^H\right] (c^H - c^L) > q^H (1 - \hat{q}) (u^L - c^L)$, *and let* $(\rho^H, \rho^L, r^H, r^L)$ *be a DE. Then* $\rho_t^L + \rho_t^H < 1$ *and* $\rho_{t+1}^H > 0$ *for all* $t \in \{1, \ldots, T-1\}$.

**Proof:** Let $t \in \{1, \ldots, T-1\}$. We proceed by showing that (i) $\rho_t^H > 0$ implies $\rho_t^H + \rho_t^L < 1$, and (ii) $\rho_t^H + \rho_t^L < 1$ implies $\rho_{t+1}^H > 0$. Then Lemma 3 follows by induction: Since $\rho_1^H = 0$ by $L2.3$ and $\rho_1^L < 1$ by $L2.2$, then $\rho_1^H + \rho_1^L < 1$, and therefore $\rho_2^H > 0$ by (ii). Assume that $\rho_k^H + \rho_k^L < 1$ and $\rho_{k+1}^H > 0$ holds for some $1 \leq k < T-1$; we show that $\rho_{k+1}^H + \rho_{k+1}^L < 1$ and $\rho_{k+2}^H > 0$. Since $\rho_{k+1}^H > 0$, then $\rho_{k+1}^H + \rho_{k+1}^L < 1$ by (i), and therefore $\rho_{k+2}^H > 0$ by (ii).

We establish (i), i.e., $\rho_t^H > 0$ implies $\rho_t^H + \rho_t^L < 1$. Suppose not; let $t < T$ be the first date such that $\rho_t^H > 0$ and $\rho_t^H + \rho_t^L = 1$. Since $\rho_t^H + \rho_t^L = 1$ (i.e., all low quality sellers matched at date $t$ trade) and $q_t^L = 1 - q_t^H$, we may write $q_{t+1}^H = \phi(q_t^H, \rho_t^H)$, where

$$\phi(x, y) := \frac{(1 - \alpha y) x}{x + (1 - \alpha y)(1 - x)}.$$

Since $\phi$ is increasing in $x$ and decreasing in $y$, $q_t^H \geq q_1^H = q^H$ by $L1.4$, and $\rho_t^H < \bar{\rho}^H$ by $L2.6$, then

$$q_{t+1}^H = \phi(q_t^H, \rho_t^H) > \phi(q^H, \bar{\rho}^H) = \frac{(1-\alpha\bar{\rho}^H)q^H}{(1-\alpha\bar{\rho}^H)q^H + (1-\alpha)(1-q^H)}.$$

Since by assumption

$$\delta\left[\alpha\left(1-q^H\right)\hat{q} - \hat{q} + q^H\right](c^H - c^L) > q^H(1-\hat{q})(u^L - c^L),$$

using $\alpha\bar{\rho}^H = (u^L - c^L)/[\delta(c^H - c^L)]$ yields

$$\alpha\left(1-q^H\right) > 1 - \left(1-(1-\hat{q})\,\alpha\bar{\rho}^H\right)\frac{q^H}{\hat{q}}.$$

Then

$$\begin{aligned}(1-\alpha\bar{\rho}^H)q^H + (1-\alpha)(1-q^H) &= -\alpha\bar{\rho}^H q^H + 1 - \alpha(1-q^H) \\ &< -\alpha\bar{\rho}^H q^H + 1 - \left(1 - \left(1-(1-\hat{q})\,\alpha\bar{\rho}^H\right)\frac{q^H}{\hat{q}}\right) \\ &= (1-\alpha\bar{\rho}^H)\frac{q^H}{\hat{q}}.\end{aligned}$$

Hence

$$q_{t+1}^H > \frac{(1-\alpha\bar{\rho}^H)q^H}{(1-\alpha\bar{\rho}^H)\dfrac{q^H}{\hat{q}}} = \hat{q} = q_T^H,$$

which contradicts $L1.4$.

Next we prove (ii), i.e., $\rho_t^H + \rho_t^L < 1$ implies $\rho_{t+1}^H > 0$. Suppose by way of contradiction that $\rho_t^H + \rho_t^L < 1$ and $\rho_{t+1}^H = 0$ for some $t < T$. Since $\rho_t^L > 0$ by $L2.5$, then low and negligible offers are optimal at date $t$. Hence

$$u^L - r_t^L = \delta V_{t+1}^B.$$

Since $\rho_{t+1}^H = 0$, then

$$V_{t+1}^L = \delta V_{t+2}^L.$$

Since $V_{t+1}^L > 0$ by $L2.4$ and $\delta < 1$,we have

$$r_{t+1}^L = c^L + \delta V_{t+2}^L = c^L + V_{t+1}^L > c^L + \delta V_{t+1}^L = r_t^L.$$

Since $0 < \rho^L_{t+1} < 1$ by $L2.2$ and $L2.5$ and $\rho^H_{t+1} = 0$, then $1 - \rho^H_{t+1} - \rho^L_{t+1} > 0$; i.e., low and negligible offers are optimal at $t+1$. Therefore

$$u^L - r^L_{t+1} = \delta V^B_{t+2}.$$

Thus, $\delta < 1$ and $V^B_{t+1} > 0$ by $L1.3$ imply

$$u^L - r^L_t = \delta V^B_{t+1} < V^B_{t+1} = \delta V^B_{t+2} = u^L - r^L_{t+1},$$

i.e.,

$$r^L_t > r^L_{t+1},$$

which contradicts the inequality above. □

**Proof of propositions 4.** We show that conditions (3.1), (3.2), and (3.3) are satisfied, and therefore that the equilibrium identified in Proposition 3 is the unique DE. Now, (3.1) follows from (2.1) and $L2.2$. Also, (3.2) follows from $L2.5$ and Lemma 3. And (3.3) follows from $L2.3$. □

**Proof of Proposition 5.** We have $\tilde{\rho}^H_1 = \lim_{\alpha,\delta\to 1} \rho^H_1 = 0$, and for $1 < t < T$, using (10) above we have

$$\tilde{\rho}^H_t = \lim_{\alpha,\delta\to 1} \rho^H_t = \lim_{\alpha,\delta\to 1} \frac{1-\delta}{\alpha\delta} \frac{u^L - c^L}{c^H - u^L + \alpha\delta^{T-t}(1-\hat{q})(u^L - c^L)} = 0.$$

Also (11) yields

$$\tilde{\rho}^H_T = \lim_{\alpha,\delta\to 1} \rho^H_T = \frac{u^L - c^L}{c^H - c^L}\hat{q}.$$

From (5) we have for $1 < t < T$

$$\lim_{\alpha,\delta\to 1} q^H_t = \lim_{\alpha,\delta\to 1} \frac{c^H - u^L + \alpha\delta^{T-t}(1-\hat{q})(u^L - c^L)}{u^H - u^L} = \hat{q}.$$

Then (12) yields for $1 \le t < T$

$$\tilde{\rho}^L_t = \lim_{\alpha,\delta\to 1} \rho^L_t = \lim_{\alpha,\delta\to 1} (1 - \alpha\rho^H_t) \frac{q^H_{t+1} - q^H_t}{\alpha q^H_{t+1}(1 - q^H_t)}.$$

Hence

$$\tilde{\rho}_1^L = \frac{\hat{q} - q^H}{\hat{q}\,(1 - q^H)},$$

and $\tilde{\rho}_t^L = 0$ for $1 < t < T$. Also (13) yields

$$\tilde{\rho}_T^L = \lim_{\alpha,\delta\to 1} \rho_T^L = 1 - \frac{u^L - c^L}{c^H - u^L}\hat{q}.$$

Note that the limiting values $(\tilde{\rho}^H, \tilde{\rho}^L)$ satisfy $\tilde{\rho}_t^H \geq 0, \tilde{\rho}_t^L \geq 0$ and $\tilde{\rho}_t^H + \tilde{\rho}_t^L \leq 1$ for all $t \in \{1,\ldots,T\}$.

As for the traders' expected utilities, (4) implies

$$\tilde{V}_1^B = \lim_{\alpha,\delta\to 1} V_1^B = \lim_{\alpha,\delta\to 1} \alpha\delta^{T-1}\,(1-\hat{q})\,(u^L - c^L) = (1-\hat{q})(u^L - c^L),$$

and (9) implies

$$\tilde{V}_1^L = \lim_{\alpha,\delta\to 1} V_1^L = \lim_{\alpha,\delta\to 1} \left(1 - \alpha\delta^{T-1}\,(1-\hat{q})\right)(u^L - c^L) = \hat{q}(u^L - c^L),$$

and $\tilde{V}_t^H = \lim_{\alpha,\delta\to 1} V_t^H = 0$.

Finally, using (14) we get

$$\lim_{\alpha,\delta\to 1} S^{DE} = \lim_{\alpha,\delta\to 1} \left[m^L(u^L - c^L) + m^H\delta^{T-1}\alpha(1-\hat{q})(u^L - c^L)\right] = \tilde{S}. \square$$

**Proof of Proposition 6.** Assume that $T = \infty$ and frictions are small. We show that the strategy distribution $(\hat{\rho}^H, \hat{\rho}^L, \hat{r}^H, \hat{r}^L)$ given by $\hat{r}_t^H = c^H$, $\hat{r}_t^L = u^L$ for all $t$, $\hat{\rho}_1^H = 0$,

$$\hat{\rho}_1^L = \frac{\bar{q} - q^H}{\alpha(1-q^H)\bar{q}},$$

and $\hat{\rho}_t^L = 0$,

$$\hat{\rho}_t^H = (1-\delta)\frac{u^L - c^L}{\alpha\delta(c^H - u^L)}$$

for $t > 1$ forms a DE. Since $\alpha(1-q^H)\bar{q} > \bar{q} - q^H$, then $0 < \hat{\rho}_1^L < 1$, and since $\alpha\delta(c^H - u^L) > (1-\delta)(u^L - c^L)$ by assumption, then $0 < \hat{\rho}_t^H < 1$ for all $t > 1$.

Since $\hat{r}_t^H = c^H$ and $\hat{r}_t^L = u^L$, then the (maximum) expected utility of high quality sellers is $V_t^H = 0$ for all $t$. Hence $\hat{r}_t^H = c^H$ for all $t$ satisfies $DE.H$. For $t > 1$ the expected utility of low quality sellers is

$$V_t^L = \frac{u^L - c^L}{\delta}.$$

Hence $\hat{r}_t^L = u^L$ satisfies $DE.L$ for $t > 1$. For $t = 1$ we have $\hat{r}_1^L = c^L + \delta V_2^L = u^L$. Hence $\hat{r}_1^L = u^L$ satisfies $DE.L$. Also

$$V_1^L = \alpha\hat{\rho}_1^L(u^L - c^L) + (1 - \alpha\hat{\rho}_1^L)\delta V_2^L = u^L - c^L.$$

Using $\hat{\rho}_1^H$ and $\hat{\rho}_1^L$ we have

$$q_2^H = \frac{q^H}{q^H + (1 - \alpha\hat{\rho}_1^L)(1 - q^H)} = \bar{q}.$$

And since $\hat{\rho}_t^L = 0$ for $t > 1$, then $q_t^H = q_2^H = \bar{q}$. Hence

$$q_t^H(u^H - c^H) + (1 - q_t^H)(u^L - c^H) = 0$$

for $t > 1$, and therefore offering the high price ($c^H$) leads to zero instantaneous payoff for all $t > 1$. Since $q_1^H < \bar{q}$ by assumption, then offering the high price ($c^H$) at $t = 1$ leads to a negative instantaneous payoff. Also since $\hat{r}_t^L = u^L$ for all $t$, then offering the low price ($u^L$) yields a zero instantaneous payoff. Thus, the buyers maximum expected utility is zero at all dates, i.e., $V_t^B = 0$ for all $t$. Hence the sequence of price offers $(\hat{\rho}^H, \hat{\rho}_t^L)$ defined above satisfies $DE.B$.

Therefore the given strategy distribution is a DE. □

**Proof of Remark 3.** We calculate the cost of a subsidy $\mu > 0$ on low quality, and show that it approaches $\mu m^L$ from below as $\delta$ approaches 1. For $\delta < 1$ and $\mu > 0$, denote this cost by $C(\delta)$. Then

$$C(\delta) = \mu\alpha\rho_1^L m_1^L + \sum_{t=2}^{\infty} \delta^{t-1}\mu\alpha\rho_t^H m_t^L.$$

Since $\rho_t^H$ is independent of $t$ for $t > 1$ by Proposition 6.3, denote $\rho_t^H = \rho^H$. Also, we have $m_1^L = m^L$, and $m_t^L = (1 - \alpha\rho_1^L)(1 - \alpha\rho^H)^{t-2}m^L$ for $t > 1$. Hence

$$\begin{aligned} C(\delta) &= \mu m^L \left( \alpha\rho_1^L + \alpha\rho^H(1 - \alpha\rho_1^L) \sum_{t=2}^{\infty} \delta^{t-1}(1 - \alpha\rho^H)^{t-2} \right) \\ &= \mu m^L \left( \alpha\rho_1^L + \alpha\rho^H(1 - \alpha\rho_1^L) \sum_{t=1}^{\infty} \delta^{t}(1 - \alpha\rho^H)^{t-1} \right). \end{aligned}$$

Since

$$\sum_{t=1}^{\infty} \delta^t (1-\alpha\rho^H)^{t-1} = \frac{1}{(1-\alpha\rho^H)} \sum_{t=1}^{\infty} \left(\delta(1-\alpha\rho^H)\right)^t$$
$$= \frac{1}{(1-\alpha\rho^H)} \frac{\delta(1-\alpha\rho^H)}{1-\delta(1-\alpha\rho^H)}$$
$$= \frac{\delta}{1-\delta(1-\alpha\rho^H)},$$

then

$$C(\delta) = \mu m^L c(\delta),$$

where

$$c(\delta) := \alpha\rho_1^L + (1-\alpha\rho_1^L)\frac{\alpha\delta\rho^H}{\alpha\delta\rho^H + (1-\delta)}.$$

Since $0 < \alpha\rho_1^L < 1$ and $\delta < 1$, then $c(\delta)$ is a convex combination of 1 and a number less than 1. Therefore $c(\delta) < 1$ and $C(\delta) < \mu m^L$. Further, since $\lim_{\delta\to 1} c(\delta) = 1$, then $\lim_{\delta\to 1} C(\delta) = \mu m^L$. □

In lemmas 4 and 5 we establish some basic properties of dynamic competitive equilibria.

**Lemma 4.** *In every CE,* $(p, u, s^H, s^L, d)$*, we have* $\sum_{\{t|s_t^H>0\}} s_t^L < m^L$.

**Proof.** Let $(p, u, s^H, s^L, d)$ be a CE. For all $t$ such that $s_t^H > 0$ we have

$$\delta^{t-1}(p_t - c^H) = v^H(p) \geq 0$$

by $(S.2)$. Hence $p_t \geq c^H$. Also $d_t > 0$ by $CE.1$, and therefore

$$v^B(p) = \delta^{t-1}(u_t - p_t) \geq 0$$

implies $0 \leq u_t - p_t \leq u_t - c^H$, i.e., $u_t \geq c^H = u(\bar{q})$. Thus

$$\frac{s_t^H}{s_t^H + s_t^L} \geq \bar{q},$$

i.e.,

$$(1-\bar{q}) \sum_{\{t|s_t^H>0\}} s_t^H \geq \bar{q} \sum_{\{t|s_t^H>0\}} s_t^L.$$

Since $\sum_{\{t|s_t^H>0\}} s_t^H \leq m^H$, then

$$(1-\bar{q})m^H \geq (1-\bar{q}) \sum_{\{t|s_t^H>0\}} s_t^H \geq \bar{q} \sum_{\{t|s_t^H>0\}} s_t^L.$$

Since $q^H = m^H/(m^H + m^L) < \bar{q}$ by assumption, then

$$\sum_{\{t|s_t^H>0\}} s_t^L \leq \frac{1-\bar{q}}{\bar{q}} m^H < \frac{1-q^H}{q^H} m^H = \frac{\frac{m^L}{m^H+m^L}}{\frac{m^H}{m^H+m^L}} m^H = m^L. \ \square$$

Lemma 5 shows that low quality must trade before high quality.

**Lemma 5.** *Let $(p, u, s^H, s^L, d)$ be a CE. If $s_t^H > 0$ for some $t$, then there is $t' < t$ such that $s_{t'}^L > 0 = s_{t'}^H$ and $\delta^{t'-1}(u^L - c^L) \geq \delta^{t-1}(c^H - c^L)$.*

**Proof.** Let $(p, u, s^H, s^L, d)$ be a CE, and assume that $s_t^H > 0$. Then $\delta^{t-1}(p_t - c^H) = v^H(p) \geq 0$ by $S.2$, and therefore $p_t \geq c^H$. Hence $v^L(p) \geq \delta^{t-1}(p_t - c^L) \geq \delta^{t-1}(c^H - c^L) > 0$, and therefore $\sum_{k=1}^T s_k^L = m^L$ by $S.3$. Since

$$\sum_{\{k|s_k^H>0\}} s_k^L < m^L$$

by Lemma 4, then there is $t'$ such that $s_{t'}^L > 0 = s_{t'}^H$. Hence $d_{t'} > 0$ by $CE.1$, which implies $u_{t'} = u^L$ by $CE.2$, and $p_{t'} \leq u^L$ by $D.2$. Also $s_{t'}^L > 0$ implies $v^L(p) = \delta^{t'-1}(p_{t'} - c^L) \geq \delta^{t-1}(p_t - c^L)$ by $S.2$. Thus

$$\delta^{t'-1}(u^L - c^L) \geq \delta^{t'-1}(p_{t'} - c^L) \geq \delta^{t-1}(p_t - c^L) \geq \delta^{t-1}(c^H - c^L).$$

Since $u^L < c^H$ this inequality implies $t' < t$. $\square$

**Proof of Proposition 8.** The profile in Proposition 8 is clearly a CE.

Assume that $u^L - c^L < \delta^{T-1}(c^H - c^L)$. We show that every CE, $(p, u, s^H, s^L, d)$, satisfies $p_1 = u_1 = u^L$, $s_1^L = d_1 = m^L$ and $s_1^H = s_t^H = s_t^L = d_t = 0$ for $t > 1$.

We first show that $s_t^H = 0$ for all $t \in \{1, \ldots, T\}$. Suppose that $s_t^H > 0$ for some $t$. Then Lemma 5 implies that there is $t' < t$ such that

$$u^L - c^L \geq \delta^{t'-1}(u^L - c^L) \geq \delta^{t-1}(c^H - c^L) \geq \delta^{T-1}(c^H - c^L),$$

which is a contradiction.

We show that $p_t \geq u^L$ for all $t$. If $p_t < u^L$ for some $t$, then

$$v^B(p,u) = \max_{t\in\{1,\ldots,T\}}\{0, \delta^{t-1}(u_t - p_t)\} > 0,$$

and therefore $\sum_{t=1}^T d_t = m^B = m^H + m^L$. However, $s_t^H = 0$ for all $t$ implies

$$\sum_{t=1}^T (s_t^H + s_t^L) \leq m^L < m^L + m^H = \sum_{t=1}^T d_t,$$

which contradicts $CE.1$.

Since $p_t \geq u^L$ for all $t$, then

$$v^L(p) = \max_{t\in\{1,\ldots,T\}}\{0, \delta^{t-1}(p_t - c^L)\} > 0,$$

and therefore $\sum_{t=1}^T s_t^L = m^L$ by $S.3$.

We now show that $p_1 = u^L$ and $s_1^L = d_1 = m^L$ and $s_t^L = 0$ for $t > 1$. Let $t$ be such that $s_t^L > 0$. Then $s_t^H = 0$ implies $u_t = u^L$. By $CE.1$ we have $d_t = s_t^L > 0$ and thus

$$\delta^{t-1}(u_t - p_t) = \delta^{t-1}(u^L - p_t) \geq 0$$

by $D.2$. This inequality and $p_t \geq u^L$ imply that $p_t = u^L$. Hence for all $t$ such that $s_t^L > 0$ we have $p_t = u^L$.

Let $t > 1$ and assume that $s_t^L > 0$. Then $p_t = u^L$. Since $\delta < 1$ and as shown above $p_1 \geq u^L$, then

$$p_1 - c^L > \delta^{t-1}(u^L - c^L) = \delta^{t-1}(p_t - c^L),$$

which contradicts $S.2$. Hence $s_t^L = 0$ for $t > 1$, and therefore $\sum_{t=1}^T s_t^L = m^L$ implies $s_1^L = d_1 = m^L > 0$, and $p_1 = u^L$. $\square$

**Proof of Proposition 9.** Assume that $T \geq \widetilde{T}$. We show that the profile $(p, u, s^H, s^L, d)$ given by $p_t = u_t = u^L$ for $t < \widetilde{T}$, and $p_t = u_t = u^H$ for $t \geq \widetilde{T}$, $s_1^H = 0$, $s_1^L = m^L = d_1$, $s_{\widetilde{T}}^L = 0$, $s_{\widetilde{T}}^H = d_{\widetilde{T}} = m^H$, and $s_t^H = s_t^L = d_t = 0$ for $t \notin \{1, \widetilde{T}\}$ is a CE.

Since $p_{\widetilde{T}} = u^H > c^H$, then $v^H(p) \geq \delta^{\widetilde{T}-1}(p_{\widetilde{T}} - c^H) > 0$. Further, since $\delta < 1$ then

$$\delta^{\widetilde{T}-1}(p_{\widetilde{T}} - c^H) = \delta^{\widetilde{T}-1}(u^H - c^H) > \delta^{t-1}(p_t - c^H)$$

for $t \neq \widetilde{T}$. Hence $s^H \in S^H(p)$. For low quality sellers, $\delta < 1$ and $u^L - c^L \geq \delta^{\widetilde{T}-1}(u^H - c^H)$ imply

$$v^L(p) = p_1 - c^L = u^L - c^L \geq \delta^{t-1}(p_t - c^H)$$

for $t > 1$. Hence $s^L \in S^L(p)$. For buyers,

$$v^B(p, u) = \delta^{t-1}(u_t - p_t) = 0$$

for all $t$. Hence $d \in D(p, u)$. Finally, $s_t^L + s_t^H = d_t$ for all $t$, and therefore $CE.1$ is satisfied, and $u_1 = u^L$ and $u_{\widetilde{T}} = u^H$ satisfy $CE.2$. Thus, the profile defined is a CE.

The surplus in this CE is

$$S^{CE} = m^L(u^L - c^L) + m^H \delta^{\widetilde{T}-1}(u^H - c^H). \ \square$$

**Proof of Proposition 10.** Assume that $T = \infty$. Let $\delta < 1$. The surplus at the CE defined in Proposition 9 is

$$S^{CE}(\delta) = q^L(u^L - c^L) + q^H \delta^{\widetilde{T}(\delta)-1}(u^H - c^H).$$

By definition $\widetilde{T}(\delta)$ satisfies

$$\delta^{\widetilde{T}(\delta)-1}(u^H - c^L) \leq u^L - c^L < \delta^{\widetilde{T}(\delta)-2}(u^H - c^L).$$

i.e.,

$$\delta < \frac{u^H - c^L}{u^L - c^L} \delta^{\widetilde{T}(\delta)-1} \leq 1$$

Hence

$$\lim_{\delta \to 1} \delta = \frac{u^H - c^L}{u^L - c^L} \lim_{\delta \to 1} \delta^{\widetilde{T}(\delta)-1} = 1,$$

i.e.,

$$\lim_{\delta \to 1} \delta^{\widetilde{T}(\delta)-1} = \frac{u^L - c^L}{u^H - c^L} = (1 - \hat{q}) \frac{u^L - c^L}{u^H - c^H}.$$

Substituting, we have

$$\lim_{\delta \to 1} \hat{S}^{CE}(\delta) = \left[m^L + m^H(1 - \hat{q})\right](u^L - c^L) = \tilde{S}. \ \square$$

# References

[1] Akerlof, G., The Market for 'Lemons': Quality Uncertainty and the Market Mechanism, *Quarterly Journal of Economics* (1970) **84**, 488-500.

[2] Binmore, K. and M. Herrero (1988), Matching and Bargaining in Dynamic Markets, *Review of Economic Studies* (1988) **55**, 17-31.

[3] Bilancini, E., and L. Boncinelli, Dynamic adverse selection and the size of the informed side of the market, manuscript (2011).

[4] Blouin, M., Equilibrium in a Decentralized Market with Adverse Selection, *Economic Theory* (2003) **22**, 245-262.

[5] Blouin, M., and R. Serrano, A Decentralized Market with Common Values Uncertainty: Non-Steady States," *Review of Economic Studies* (2001) **68**, 323-346.

[6] Camargo, B., and B. Lester, Trading dynamics in decentralized markets with adverse selection, manuscript (2011).

[7] Gale, D.,Limit Theorems for Markets with Sequential Bargaining, *Journal of Economic Theory* (1987) **43**, 20-54.

[8] Gale, D., Equilibria and Pareto Optima of Markets with Adverse Selection, *Economic Theory* **7**, (1996) 207-235.

[9] Jackson, M., and T. Palfrey, Efficiency and voluntary implementation in markets with repeated pairwise bargaining," *Econometrica* (1999) **66**, 1353-1388.

[10] Janssen, M., and S. Roy, Dynamic Trading in a Durable Good Market with Asymmetric information," *International Economic Theory* (2002) **43**, 257 - 282.

[11] Kim, K., Information about sellers' past behavior in the market for lemons, manuscript. 2011.

[12] Moreno, D., and J. Wooders, Prices, Delay and the Dynamics of Trade, *Journal of Economic Theory* (2002) **104**, 304-339.

[13] Moreno, D., and J. Wooders, Decentralized trade mitigates the lemons problem, *International Economic Review* (2010) **51**: 383-399.

[14] Moreno, D., and J. Wooders, The Efficiency of Centralized and Decentralized Markets for Lemons, University of Arizona Working Paper (2001), 01-03 – http://econ.arizona.edu/docs/Working_Papers/Misc%20Years/quality_y2.pdf.

[15] Morris, S., and H.S. Shin, Contagious adverse selection, *The American Economic Journal - Macroeconomics,* (2012) **4**, 1-21.

[16] Osborne, M., and A. Rubinstein, *Bargaining and Markets*, Academic Press, New York, 1990.

[17] Rubinstein, A., and A. Wolinsky, Equilibrium in a Market with Sequential Bargaining," *Econometrica* (1985) **53**, 1133-1150.

[18] Rubinstein, A., and A. Wolinsky, Decentralized Trading, Strategic Behavior and the Walrasian Outcome, *Review of Economic Studies* (1990) **57**, 63-78.

[19] Serrano, R., Decentralized Information and the Walrasian Outcome: a Pairwise Meetings Market with Private Values, *Journal of Mathematical Economics* (2002) **38**, 65-89.

[20] Wilson, C., The Nature of Equilibrium in Markets with Adverse Selection, *The Bell Journal of Economics* (1980) **11**, 108-130.

[21] Wolinsky, A., Information Revelation in a Market with Pairwise Meetings, *Econometrica* (1990) **58**, 1-23.

[22] Wooders, J., Walrasian Equilibrium in Matching Models, *Mathematical Social Sciences* (1998) **35**, 245-259.

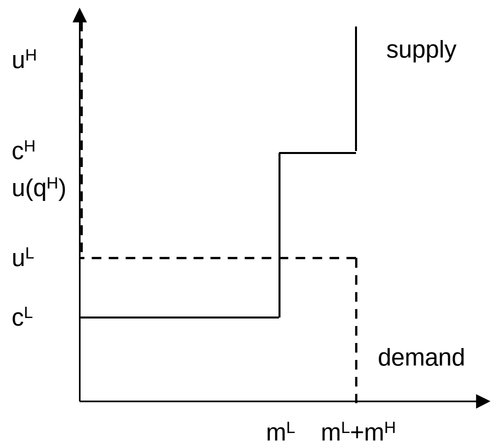


Figure 1: $c^H>u(q^H)>u^L$

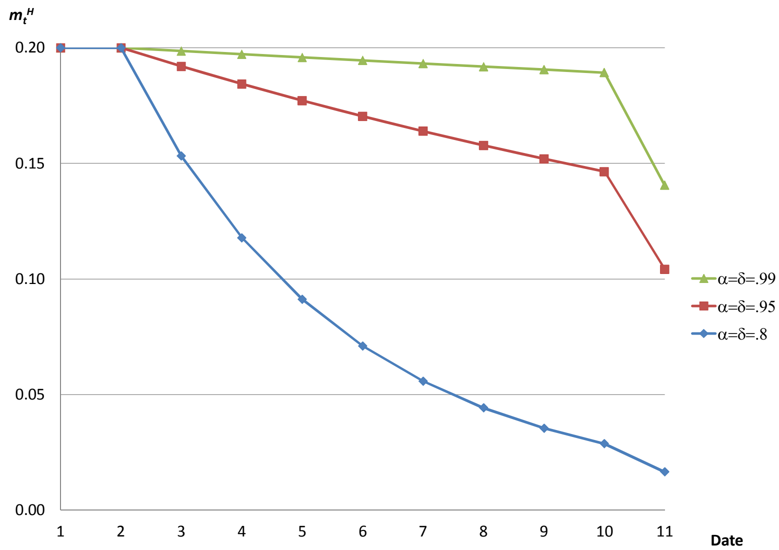


**Figure 2(a): High Quality Stocks**

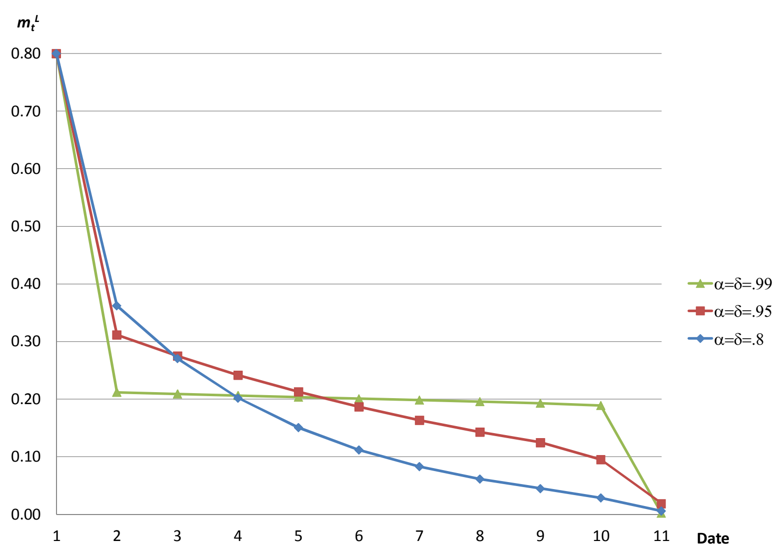


**Figure 2(b): Low Quality Stocks**

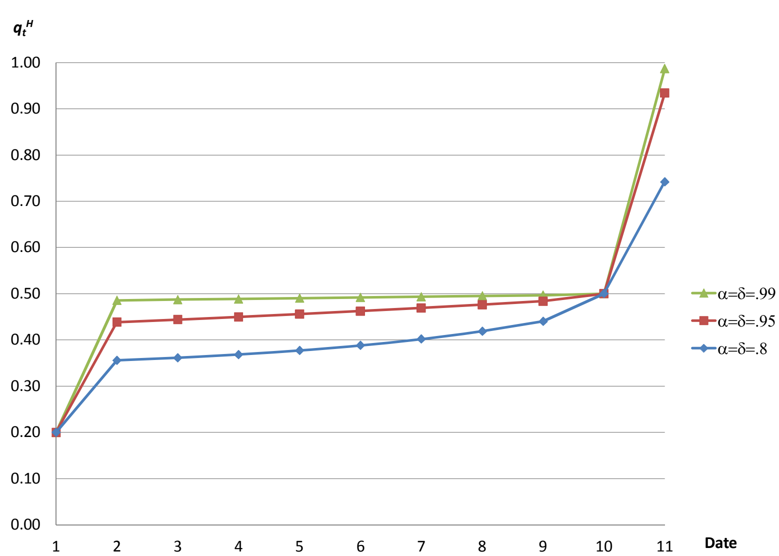


**Figure 2(c): Proportion of High Quality in the Market**